Scattering Amplitudes in $N = 4$ SYM

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Introduction to $N = 4$ SYM

Problems with Earlier methods

Recent Developments

Analytic Results in Reduced Kinematics

$N = 4$ SYM

- SU(N) 4d Gauge Theory
- 1. Gauge Field
  2. 6 Massless Scalars (adjoint)
  3. 4 Massless Fermions (adjoint)
- Conformal Theory

Many hidden symmetries, structures for both amplitudes and correlators

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- "Can either study the wrong calculations in the right theory or the right calculations in the wrong theory."

Feynman Diagrams: Naive

- Too many diagrams - Individual diagrams not gauge invariant
- Too many terms in each diagram - Non-Abelian gauge boson self-interactions complex
- Too many kinematic variables - Allowing construction of arbitrarily complex expressions

$\Rightarrow$ Intermediate results vastly more complex than final results!

Example: 5-gluon Tree-Level

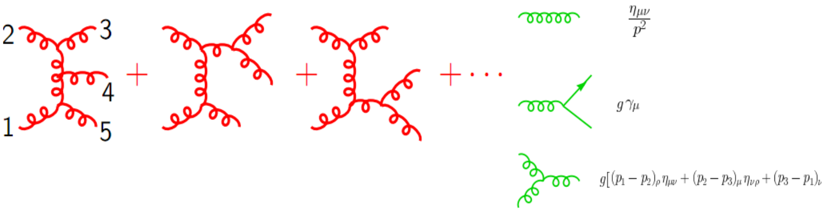

Figure: Tree-Level Feynman Rules

5-Point Tree-Level Expression

Result of a brute force calculation (actually only a small part of it):

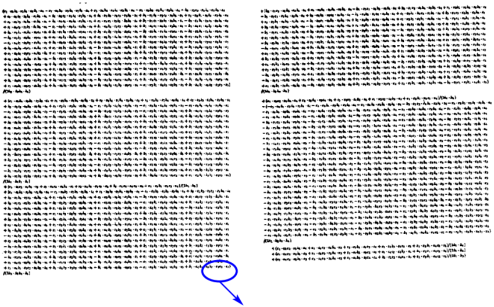

Figure: 5-Point Tree-Level Expression taken from "Quantum Field Theory in a Nutshell" A.Zee 2$^{\text{nd}}$ Edition

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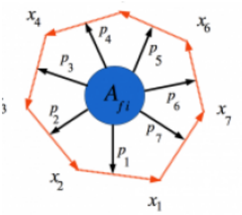
  Colour stripped gluon amplitudes
- Use variables which enforce stronger symmetries:
  Spinor Helicity $\rightsquigarrow$ Twistors
- Supersymmetric Ward identities:
  All amplitudes with less than two of each helicity gluons go to zero

MHV case study



- $\rightsquigarrow$ n-point Tree-level
- $\rightsquigarrow$ Planar
- $\rightsquigarrow$ Colour Stripped
- $\rightsquigarrow$ MHV = Maximally Helicity Violating (only two $-^{\text{ve}}$ helicity gluons)

$$A_{\text{n}}^{\text{tree}}(i,j) = \frac{\langle i,j\rangle}{\langle 1,2\rangle \cdots \langle n,1\rangle}$$

(Parke Taylor 1986)

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- Dual Superconformal Symmetry
  [Drummond Henn Korchemsky Sokatchev, Brandhuber Travaglini Heslop]
  $\rightsquigarrow$ Unrelated to standard conformal invariance: Whole new structure lurking in theory - became evident from ...

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- Wilson loop / Amplitude duality (Super)
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- Symbol
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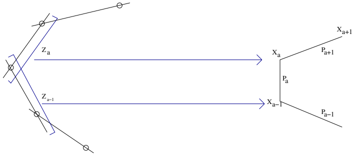

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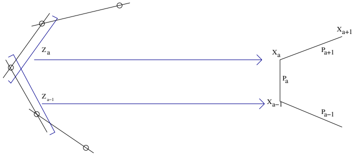

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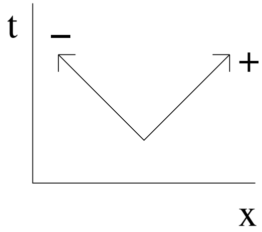
- Powerful new variables, symmetries constraints built-in!

Restriction

- Two choices for null vector and need to distinguish particle i from particle $i + 1 \Rightarrow$ if particle i has 4-momentum $(1, 0, 0, -1)$ particle $i + 1$ needs to have 4-momentum $(1, 0, 0, 1)$.



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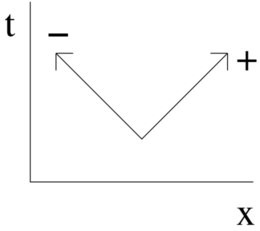
- Need even number of particles to allow distinguishing of 'n' and '1'
  (we will often label particles by their cyclic position)



Useful choice for restriction namely (t, z)-plane, allows us to split the theory as: even particles + odd particles

$$\lambda_{\text{even}} = \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix} \qquad \lambda_{\text{odd}} = \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix} \tag{1}$$

$\rightarrow \epsilon_{abcd}\lambda_i^a\lambda_j^b\lambda_k^c\lambda_l^d = 0$ unless i,j,k,l involve two odd and two even indices.

(1+1)-dimensions (MHV)

- Simplification in number of variables answer is expressed in, only $u_{ij} \rightsquigarrow$ conformal cross-ratios
- number of u's $= (n - 6)$ rather than $(3n - 15)$

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- 2 loop, n-point [Khoze Heslop]
- 3-loop, 8-point [Khoze Heslop]

n-point 2-loop example

$$\mathcal{R}_n^{(2)} = -\frac{1}{2}\sum_{\mathcal{S}} \log(u_{i_1 i_5})\log(u_{i_2 i_6})\log(u_{i_3 i_7})\log(u_{i_4 i_8}) - \frac{\pi^4}{72}(n-4)$$

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- Shuffle Algebra
  $\mathcal{S}(\log(a)\log(b)) = a \otimes b + b \otimes a$
- $\mathcal{S}(\log(a)\log(b)\log(c)) =$
  $a \otimes b \otimes c + a \otimes c \otimes b + c \otimes a \otimes b + b \otimes a \otimes c + b \otimes c \otimes a + c \otimes b \otimes a$

Using the Symbol

How would we use this to write down the solution seen above?
Answer use symmetries

- Length given by loop-level e.g. 2-loop $\Rightarrow$ transcendentality 4 etc.

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- Collinear Limits (triple)

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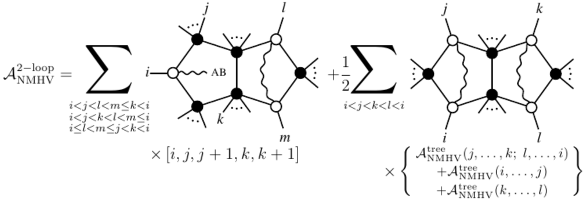
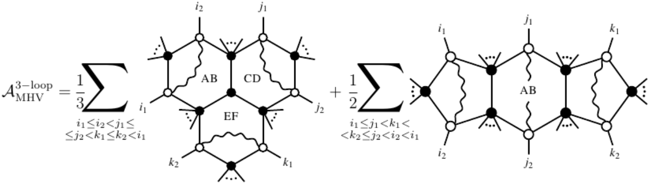
Local Planar Integrals



1.pdf

2.pdf



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- Problem: Coefficients (R-invariants) blow-up term-by-term.

R-Invariants

$$[i, j, j+1, k, k+1] = \frac{\delta^{0|4}(i\langle j, j+1, k, k+1\rangle + \text{cyclic})}{\langle i, j, j+1, k\rangle\langle j, j+1, k, k+1\rangle\langle j+1, k, k+1, i\rangle\langle k, k+1, i, j\rangle\langle k+1, i, j, j+1\rangle}$$

- Real poles $\langle j, j+1, k, k+1\rangle$ collinear pairs
- $\mathrm{SU}(4) \rightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)$: term-by-term
- Spurious poles cancel in sum!

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- $\Rightarrow$ two 'even' χ's and two 'odd' χ's

Results

Left with a result like the MHV case:

For component of superamplitude to be non-zero need helicity to be split equally between even indexed and odd indexed external particles

Have n-point equation at tree-level:

$$\Gamma^{\mathrm{NMHV}}_{n,\mathrm{tree}} = \sum_{j,k\epsilon A} \tilde{R}(1,j,k)(-1)^{j+k+1}$$

$A = \{j, k \text{ such that } |1-j| > 1, |1-k| > 1, |j-k| > 1, j < k, \mathrm{mod(n)}\}$

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- 2-loop?
- N^2MHV $\rightsquigarrow$ Grasmannian
- Symbol $\longrightarrow$ result purely from symmetry if we input some constraints (cyclic, collinear limits)
- Symmetries we're not yet using to mould equation