Periodic Monopoles - Dynamics and the Dual Picture

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CPT Student Seminar

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[based on RM 1212.4481 and RM & Ward 130x.xxxx]
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SDYM

- Yang-Mills action

\[ S = \int_{\mathbb{R}^4} \text{tr}(F \wedge *F) \]

- Stationary points from Bogomolny argument \( \Rightarrow F \) (anti)-self-dual

\[ F = *F \quad \implies \quad F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \]

- Independence from \( x^4 \) gives Bogomolny (monopole) equations,

\[ F = *D\Phi \quad \implies \quad F_{ij} = \epsilon_{ijk} (\partial_k \Phi + [A_k, \Phi]) \ldots \]

- Dimensionally reduce again for Hitchin equations,

\[ D_{\bar{s}} \Phi = 0 \quad \text{and} \quad F_{s\bar{s}} = -\frac{1}{4} [\Phi, \Phi^\dagger] \ldots \]

- And again for Nahm equations,

\[ \partial_s A_i = \frac{1}{2} \epsilon_{ijk} [A_j, A_k]. \]
Periodic Monopole

- $\hat{A}, \hat{\Phi} \in su(2)$ - i.e. $2 \times 2$ traceless anti-Hermitian matrices.

Topology
- Magnetic charge given by first Chern class,
  \[
  k = \lim_{R \to \infty} \int_{\rho=R} \frac{\text{tr}(\hat{F} \hat{\Phi})}{4\pi \|\hat{\Phi}\|}
  \]

Boundary conditions
- Asymptotically Abelian, $\hat{A} \sim \hat{A}_\infty \sigma_3$, $\hat{\Phi} \sim \hat{\Phi}_\infty \sigma_3$
- Resembles Dirac chain, $\nabla^2 \hat{\Phi}_\infty = 0$, $\hat{\Phi}_\infty \sim \log(\rho)$
  \[
  (\hat{A}_z + i\hat{\Phi})_\infty \sim \ell \log(\zeta) + \nu + O(\zeta^{-1})
  \]
  - charge + s.b.  
  - size  
  - moduli
- For monopoles in $\mathbb{R}^3$, charge is given by sub-leading term - interesting implications for SU(3).
Nahm Transform

- Powerful tool to solve the Bogomolny equations, it is an adaptation of the ADHM construction for instantons. [Corrigan & Goddard '84]

- Nahm equations are easier to solve than Bogomolny - but inverse transform hard: look for approximate or numerical solutions.

- Bijection between two systems satisfying the SDYM equations.

- Roughly, swap rank of gauge group and ‘soliton number’.

- The manifolds we’re interested in can be related by the example of SDYM on a 4-torus, which is self-reciprocal under Nahm transform. [Braam & van Baal '89]
Nahm Transform

Arrange instantons or monopoles in a lattice. The Nahm transform allows us to consider the self-duality equations on the reciprocal lattice. [Jardim ’04]

<table>
<thead>
<tr>
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<th>‘physical space’</th>
<th>‘Nahm space’</th>
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</thead>
<tbody>
<tr>
<td>instanton</td>
<td>$\mathbb{R}^4$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>caloron (periodic instanton)</td>
<td>$\mathbb{R}^3 \times \mathbb{S}^1$</td>
<td>$\mathbb{S}^1$</td>
</tr>
<tr>
<td>doubly periodic instanton</td>
<td>$\mathbb{R}^2 \times \hat{T}^2$</td>
<td>$\mathbb{T}^2$</td>
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<tr>
<td>monopole</td>
<td>$\mathbb{R}^3$</td>
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<td>periodic monopole</td>
<td>$\mathbb{R}^2 \times \mathbb{S}^1$</td>
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</tbody>
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(N.B. the Nahm transform has only been proved for some of these!)
Inverse Nahm for Periodic Monopole

For an SU(2) charge $k$ periodic monopole, solve rank $k$ Hitchin equations on $\mathbb{R} \times S^1$,

$$F_{s\bar{s}} = -\frac{1}{4}[\Phi, \Phi^\dagger] \quad \quad D_{\bar{s}}\Phi = \partial_{\bar{s}}\Phi + [A_{\bar{s}}, \Phi] = 0,$$

where $\det(\Phi)$ is given by the spectral curve.

Solve the equation

$$\Delta \psi = \begin{pmatrix} 1_k \otimes (2\partial_{\bar{s}} - z) + 2A_{\bar{s}} & 1_k \otimes \bar{\zeta} - \Phi^\dagger \\ 1_k \otimes \zeta - \Phi & 1_k \otimes (2\partial_s + z) + 2A_s \end{pmatrix} \psi = 0.$$

Construct the monopole fields from normalised solutions $\int \int \psi^\dagger \psi = 1_2$,

$$\hat{\Phi} = i \int_{-\infty}^{\infty} dr \int_{-\pi/\beta}^{\pi/\beta} dt \left(r\psi^\dagger \psi\right) \quad \quad \hat{A}_i = \int_{-\infty}^{\infty} dr \int_{-\pi/\beta}^{\pi/\beta} dt \left(\psi^\dagger \partial_i \psi\right).$$

Gauge transformations act as $\psi \mapsto U(s)^{-1} \psi \hat{g}(\zeta, z)$ with $U = h \otimes g$. 
Spectral Curve

As for the monopole in $\mathbb{R}^3$, it is useful to consider solutions of

$$(\partial_z + \hat{A}_z + i\hat{\Phi})V(\zeta, z) = 0 \quad \text{with} \quad V(\zeta, 0) = 1_2.$$ 

$V(\zeta, \beta)$ defines the holonomy. Its characteristic equation is a polynomial in $w = e^{\beta s}$ [Cherkis & Kapustin '01, '03]

$$w^2 + P_k(\zeta)w + 1 = 0.$$ 

This defines a curve in $\mathbb{C} \times \mathbb{C}^*$. The same curve can be written

$$\zeta^k - \text{tr}(\Phi)\zeta^{k-1} + \ldots + (-1)^k\det(\Phi) = 0.$$ 

$P_k(\zeta)$ has $2k + 2$ coefficients: 2 parameters (b.c.s), 2 for centre of mass, and $2k - 2$ moduli. This is half the number of moduli for a charge $k$ monopole in $\mathbb{R}^3$. 
The spectral curves are

\[ w^2 - 2\zeta w/C + 1 = 0 \quad \zeta = \det(\Phi) \]

Hitchin data is smooth and Abelian [Ward '05]

\[ A = 0 \quad \Phi = C \cosh(\beta s). \]

The Nahm equation

\[ \Delta \Psi = \left( \begin{array}{cc} 2\partial_s - z & \zeta - \Phi \\ \bar{\zeta} - \Phi^\dagger & 2\partial_s + z \end{array} \right) \Psi = 0 \]

has two distinct solutions if \( \zeta \) remains away from \( \zeta = \pm C \). In this region, the approximate monopole fields are

\[ \hat{A}_\zeta = \hat{A}_{\bar{\zeta}} \approx 0 \quad -(\hat{A}_z + i\hat{\Phi}) \approx \frac{1}{\beta} \cosh^{-1} \left( \frac{\zeta}{C} \right) \sigma_3 \]

\( \hat{A}_\zeta \) has off-diagonal terms which decay exponentially away from \( \zeta = \pm C \). Note \( \hat{\Phi} \) can be read off by solving the ‘w’ spectral curve for \( s(\zeta) \).
Energy density is given by $\mathcal{E} = \nabla^2 \| \hat{\Phi} \|^2$. On a cross section of the chain,

- Energy peaks are located at $\zeta = \pm C$.
- Total energy diverges as $\log(\rho)$, but can still consider relative moduli space. [Cherkis & Kapustin '02]
Spectral Approximation

Results suggest a general pattern:

• We assume monopole fields can be read off from the spectral curve.

• Peaks in energy density are then found at the values of $\zeta$ where the eigenvalues of $V(\zeta, \beta)$ coincide.

• There are $2k$ such *spectral points*, which come in pairs.

• This provides a way of studying higher charge chains, or larger gauge groups, simply from the spectral curves.

• Approximation improves for monopole size $C \gg$ period $\beta$, i.e. in the limit of $z$ independence.
The spectral curves are

\[ w^2 - (2 \zeta^2 - K)w/C + 1 = 0 \]

\[ \zeta^2 = -\det(\Phi) \]

with spectral points at

\[ \zeta = \pm \sqrt{K/2 \pm C}. \]

\( K \) is a complex modulus.

Impose symmetries on the spectral curve, e.g. \((w, \zeta; K) \mapsto (\bar{w}, \bar{\zeta}; \bar{K})\) and \((w, \zeta; K) \mapsto (-w, i\zeta; -K)\) show that \( K \in \mathbb{R} \) is a one-parameter family where the spectral points undergo right angled scattering.

Similarly \((w, \zeta; K) \mapsto (-i\bar{w}, e^{i\pi/4}\bar{\zeta}; i\bar{K})\) shows that \( K \in i\mathbb{R} \) is another one-parameter family.

Note \( K = 0 \) has enhanced symmetry, while if \( K = \pm 2C \) two spectral points coincide.
Charge 2 - Spectral Approximation
The Hitchin field has

\[-\det(\Phi) = C \cosh(\beta s) + K/2.\]

The two zeroes of this function also undergo scattering as \(K\) is varied:

\(K \in \mathbb{R}\)
The Hitchin field has

\[-\det(\Phi) = C \cosh(\beta s) + K/2.\]

The two zeroes of this function also undergo scattering as $K$ is varied:

$K \in i\mathbb{R}$
The monopole field $\hat{\phi} = \hat{A}_z + i\hat{\Phi}$ is known explicitly in the spectral limit. A metric on the moduli space can be obtained by varying the fields with respect to the modulus $K$, 

$$g \sim \dot{K} \dot{\bar{K}} \int_{\mathbb{R}^2} \text{tr} \left( \partial_K \hat{\phi} \partial_{\bar{K}} \hat{\phi}^\dagger \right) \rho \, d\rho \, d\theta$$

The gauge condition that variations are orthogonal to gauge orbits is automatically satisfied (dim. red. of $D_\mu (\partial_K \hat{A}_\mu) = 0$).

In terms of the spectral points $\zeta_i = \pm \sqrt{K/2} \pm C$, 

$$g \sim \dot{K} \dot{\bar{K}} \int_{\mathbb{R}^2} \frac{1}{|\zeta - \zeta_1||\zeta - \zeta_2||\zeta - \zeta_3||\zeta - \zeta_4|} \rho \, d\rho \, d\theta$$

Perform integral for $K = 0, \pm 2C$, otherwise numerically...
Conformal factor as function of $K$:

- $K = 0$ and $K = \pm 2C$ are special!
- $K \in \mathbb{R}$ and $K \in i\mathbb{R}$ are indeed geodesics.
- Otherwise, numerically evolve geodesic equations...
Note different constituent behaviour according to whether or not the geodesic crosses the line segment $-2C \leq K \leq 2C$. 
Charge 2 - Moduli Space

Note different constituent behaviour according to whether or not the geodesic crosses the line segment $-2C \leq K \leq 2C$. 
Recall we are to solve rank 2 Hitchin equations
\[
F_{s\bar{s}} = -\frac{1}{4} [\Phi, \Phi^\dagger] \quad D_{\bar{s}} \Phi = \partial_{\bar{s}} \Phi + [A_{\bar{s}}, \Phi] = 0,
\]
with \(-\det(\Phi) = C \cosh(\beta s) + K/2\).

Up to a gauge, we can write [Harland & Ward '09]
\[
\Phi = \begin{pmatrix} 0 & \mu e^{\psi/2} \\ \mu^{-1} e^{-\psi/2} & 0 \end{pmatrix} \quad A_{\bar{s}} = a\sigma_3 + \alpha \Phi
\]
\(\alpha\) encodes the other moduli (z-offset and relative phase).

Now, look at symmetries of the Hitchin equations and the Nahm operator. Imposing invariance under \((\zeta, z) \mapsto (\zeta, -z)\) shows \(\alpha = 0\) is a geodesic submanifold. Justifies looking for geodesics on \(K\) plane!
Ansatz for Hitchin fields:

\[
\Phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix} \Rightarrow \mu_+ \mu_- = C \cosh(\beta s) + K/2
\]

Note \(\det(\Phi)\) has two zeroes. This gives two distinct smooth solutions for \(\Phi\) according to their allocation between the non-zero components.

- ‘zeroes together’ [Harland]

  \[
  \mu_+ = C \cosh(\beta s) + K/2 \quad \mu_- = 1
  \]

- ‘zeroes apart’ [Harland & Ward ’09]

  \[
  \mu_{\pm} = \sqrt{C/2} \left( e^{\beta s/2} + \lambda^{\pm 1} e^{-\beta s/2} \right) \quad 2C \lambda^{\pm 1} = K \pm \sqrt{K^2 - 4C^2}
  \]

Note that both solutions have the same spectral limit.
On the $\alpha = 0$ geodesic, solve Hitchin equations numerically. Plot $|F|$, 

- ‘zeroes together’

- ‘zeroes apart’

size/period ratio now determined by $1/C$
Charge 2 - Limits of $C$

$C \gg 1$
- monopole size $\gg$ period ($z$ independence)
- spectral approximation holds
- sharply localised lumps on cylinder
- lumps closely track zeroes of $\det(\Phi)$

$C \ll 1$
- chain of small monopoles
- wide peaks on cylinder, approach Nahm data
  (singularities develop at finite $r$)

$\alpha \neq 0$
- $z$ offset $\to$ $t$-holonomy in central region
- relative phase $\to$ relative phase on lumps
Charge 2 - Metric from Cylinder - $C \gg 1$

- Simple solution to Hitchin equations.
- Gauge condition can be solved explicitly, $\Phi \mapsto \Phi' = \sqrt{\det(\Phi)} \sigma_1$.
- Metric depends only on $\det(\Phi)$
  \(\Rightarrow\) same for ‘zeroes together’ and ‘zeroes apart’

\[ g \sim \dot{K} \dot{\bar{K}} \int_{\mathbb{R} \times S^1} \frac{|\partial_K \det(\Phi')|^2}{|\det(\Phi')|} \sim \dot{K} \dot{\bar{K}} \int_{\mathbb{R} \times S^1} \frac{dr \, dt}{|C \cosh(\beta s) + K/2|} \]

- Conformal factor agrees with spectral approximation!
- Asymptotically $\log(K)/K$
  [agreement with Cherkis & Kapustin '02]
- Peaks \(\sim \log(|K \pm 2C|)\)
So far, only numerically.

Approach rotational symmetry.

Asymptotically $\log(K)/K$.

'zeroes together'

smooth scatter in plane
A-H cone?

'zeroes apart'

peak $\sim (K - 2C)^{-1}$
double scattering along $z$
A-H trumpet?
Summary & Outlook

- Bogomolny eqs on $\mathbb{R}^2 \times \hat{S}^1 \xrightarrow{\text{Nahm transform}}$ Hitchin eqs on $\mathbb{R} \times S^1$.
- Chain of large monopoles $\rightarrow z$ independence $\rightarrow$ all information contained in holonomy $\rightarrow$ spectral approximation.
- Nahm transform $\rightarrow$ motion of lumps on dual space.
- Moduli space: two solutions which coincide in spectral limit.
- Spectral approximation can be applied to e.g. charge 3 or SU(3).
- Can the constituents be studied in their own right?
END