Noncommutative U(2) Instantons

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Outline

M-Theory as 'unifying theory' of the five superstring theories.

M-branes: fundamental objects of M-theory. M5-brane has no Lagrangian description; related to 6d $(2, 0)$ SCFTs.

Successess with M2-brane analysis (e.g. J. Bagger & N. Lambert; A. Gustavsson; O. Aharony, O. Bergman, D. L. Jafferis & J. Maldacena).

M-branes display N^3 scaling degrees of freedom. No natural interpretation.

M-Theory

It is believed that 5d SYM (N D4-brane stack) can contain all information about the non-Abelian $(2, 0)$ theory. H. Kim, S. Kim, E. Koh, K. Lee, S. Lee

Kaluza-Klein states from the S^1 compactification appear as instantons of the 5d theory. Index theorems agree for both theories with inclusion of instantons. D. Bak & A. Gustavsson

Instantons as "master solutions" for solitons: can obtain vortices, monopoles, skyrmions under certain dimensional reductions.

Configurations of M5-M2 branes $\leftrightarrow$ configurations of D-branes in Type IIB theory.

$\downarrow$

Low-energy dynamics of D-branes $\leftrightarrow$ Yang-Mills theory.

$\downarrow$

Static solutions to Yang-Mills theory $\leftrightarrow$ instantons.

$\Downarrow$

Can we learn about the behaviour of M-branes from studying instantons?

Solitons

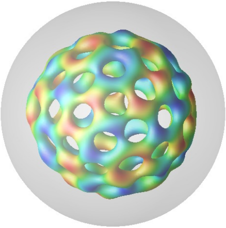
Definition

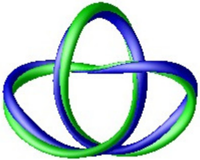
A topological soliton is any solution of a set of partial differential equations that is stable against decay to the "trivial" (vacuum) solution. Wikipedia...

Solitons

Definition

A topological soliton is a mathematically rigorous way of generating pretty pictures.





Instantons - A Summary

Instantons: Static, self-dual, solutions to Yang-Mills field theory.

Can build moduli space of solutions; geodesics on the moduli space represent 'slowly moving' instanton configurations.

Commutative space gives singularities $\Leftrightarrow$ 'small' zero-size instantons.

Can be resolved by considering noncommutative underlying space (i.e. $\mathbb{R}^2_{NC} \times \mathbb{R}^2_{NC}$).

Set-up

Bosonic part of 5d Yang-Mills theory

$$S \sim \int d^5x \, \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu X^I D^\mu X^I \right)$$

Consider two D4-branes, so gauge group of theory is $SU(2)$. Can also take one non-zero scalar $X^I \equiv \phi$ which will induce a Higgs potential. Bogomolny procedure gives the conditions for instantons

$$\begin{aligned} F_{ij} &= \pm \star F_{ij} \equiv \pm \frac{1}{2} \epsilon_{ijkl} F_{kl}, \\ F_{i0} &= D_i \phi, \\ D_0 \phi &= 0. \end{aligned}$$

Example: Pure Yang-Mills

$$S = -\frac{1}{8}\int \mathrm{d}^4x \,\mathrm{Tr}\,(F_{ij}F_{ij})$$
$$= -\frac{1}{8}\int \mathrm{d}^4x \,\mathrm{Tr}\,\left((F_{ij} \mp \star F_{ij})^2 \pm 2F_{ij} \star F_{ij} - \star F_{ij} \star F_{ij}\right).$$

First term positive-definite, so obtain a bound on energy

$$S \geq \pi^2 |k|,$$

$$k = -\frac{1}{8\pi^2}\int \mathrm{d}^4x \,\mathrm{Tr}(F_{ij} \star F_{ij}) \in \mathbb{Z}.$$

Equality when square term vanishes: $F_{ij} = \pm \star F_{ij}$.

Classifying Instantons

Integer k related to boundary conditions. Non-compact space requires the field strength F_{ij} vanishes at infinity. A_i must be *pure gauge* at infinity:

$$A_i = -\partial_i g^\infty (g^\infty)^{-1}.$$

where $g^\infty : S^3_\infty \mapsto SU(2)$. k is the degree of g: 'winding number' or 'instanton number'.
Groups of solutions with the same instanton number can be smoothly deformed into one another, and furnish a moduli space for each integer k.

ADHM Procedure

Bogomolny procedure reduces second to first order differential equations.
ADHM procedure reduces these to purely **algebraic** constraints.
For k $SU(N)$ instantons, build the ADHM data: an $(N+2k)\times 2k$ x-dependent matrix Δ. Then self-dual instanton solutions come from the constraints

$$\Delta^\dagger \Delta = f^{-1}(x) \otimes \mathrm{I}_2.$$

For $SU(2)$ instantons, can consider Δ as quaternion-valued.
ADHM data contains $8k$ free parameters.

ADHM Procedure

Can write ADHM data as

$$\Delta = \begin{pmatrix} L \\ M \end{pmatrix} - x \begin{pmatrix} 0 & 0 \\ \mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{pmatrix}$$

where all x-dependence is within second term. Then constraints imply

$$L^\dagger L + M^\dagger M + \bar{x}x - \left(M^\dagger x + \bar{x} M \right) = f^{-1}(x) \otimes \mathrm{I}_2.$$

Final term requires $M^\dagger = M$, and $\bar{x}x = x_i^2 \mathrm{I}_2 \equiv |x|^2$, $i = 1, \ldots, 4$. Hence non-trivial constraints given by

$$L^\dagger L + M^\dagger M = \mu \otimes \mathrm{I}_2$$

for some x-independent 2×2 matrix μ.

ADHM Data

Explicitly, we may write

$$\begin{pmatrix} L \\ M \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ \tau & \sigma \\ \sigma & -\tau \end{pmatrix}$$

where v_1, v_2, τ and σ are quaternionic. σ is given by off-diagonal constraints

$$\sigma = \frac{\tau}{4|\tau|^2} \left(\bar{v}_2 v_1 - \bar{v}_1 v_2 \right).$$

Degrees of freedom: 4 CoM (suppressed) $+ 4\ \tau + 8\ v_i = 16$ parameters.

Given solutions to the ADHM constraints, we can produce a self-dual field via the unit null vectors U of Δ:

$$\Delta^\dagger U = 0, \qquad U^\dagger U = \mathrm{I}_2.$$

Then a self-dual field strength arises from the gauge field A_i

$$\begin{aligned} A_i &= U^\dagger \partial_i U, \\ F_{ij} &= \partial_{[i} A_{j]} - i[A_i, A_j]. \end{aligned}$$

The Moduli Space

The $8k$ parameters in Δ can be seen as forming a space of instanton solutions. Can induce a metric on this space.

Explictly,

$$S = \frac{1}{2} \int \mathrm{d}t\, g_{rs} \dot{z}^r \dot{z}^s$$

where

$$g_{rs} = \int \mathrm{d}^4 x \, \mathrm{Tr} \left(\delta_r A_i \delta_s A_i \right),$$

z^r, $r = 1, \ldots, 8k$ collective coordinates for moduli space, and $\delta_r A_i$ zero-modes (i.e., modes orthogonal to gauge transformations of Δ).

Calculating the Metric

The metric can be found using an identity due to Osborn:

$$\mathrm{Tr}\left(\delta_r A_i \delta_s A_i\right) = -\frac{1}{2}\partial^2 \mathrm{Tr}\left(C_r^\dagger P C_s f + f C_r^\dagger C_s\right).$$

C_r related to gauge trafos, P to projection of ADHM data. Then

$$\begin{aligned}\mathrm{d}s^2 = & \, 2\,\mathrm{Tr}\left(\mathrm{d}\bar{v}_1 \mathrm{d}v_1 + \mathrm{d}\bar{v}_2 \mathrm{d}v_2 + \mathrm{d}\bar{\tau}\mathrm{d}\tau + \mathrm{d}\bar{\sigma}\mathrm{d}\sigma\right) \\ & - \frac{4}{N_A}\mathrm{Tr}\left(\left(\bar{v}_1 \mathrm{d}v_2 - \bar{v}_2 \mathrm{d}v_1 + 2\left(\bar{\tau}\mathrm{d}\sigma - \bar{\sigma}\mathrm{d}\tau\right)\right)^2\right).\end{aligned}$$

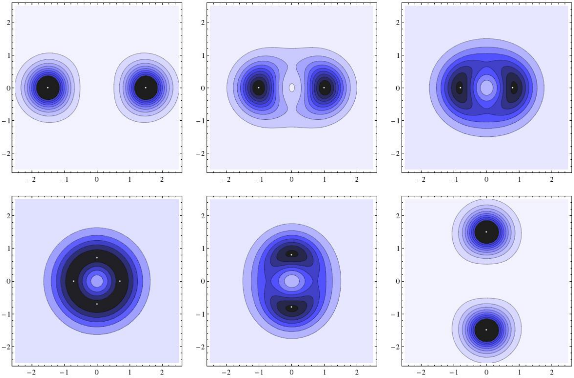
up to "suitable" defintions of $\mathrm{d}v_i^2 \equiv \mathrm{d}\bar{v}_i \mathrm{d}v_i$ and N_A.

Given a metric on the moduli space, can consider geodesics. A geodesic on the space $\Leftrightarrow$ smoothly varying between instanton solutions of same instanton number, k. *Slowly* varying along geodesics is a good approximation to instanton evolution in time.

Using the moduli space approximation, we can consider scattering of two or more instantons.

This procedure only holds when the speed of collision (and for dyonic instantons, the strength of the potential) is small compared to instanton size.

Two SU(2) instantons scatter at right angles, passing through the "zero-size" singularity J. Allen, D. J. Smith



Commutative Results

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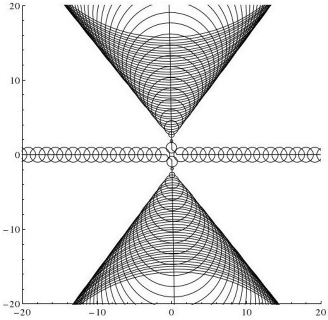
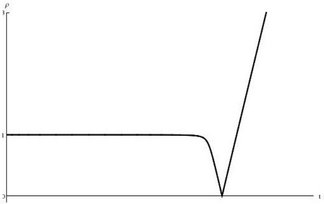




How can we resolve the singularity at the point of collision?

Noncommutativity

Consider a noncommutative space: for spatial coordinates x_μ, $\mu = 1, \ldots, 4$, have non-trivial commutation relations

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}.$$

θ is an antisymmetric, real matrix. Choose

$$\theta = \begin{pmatrix} 0 & -\frac{\zeta}{2} & 0 & 0 \\ \frac{\zeta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\frac{\zeta}{2} \\ 0 & 0 & \pm\frac{\zeta}{2} & 0 \end{pmatrix}.$$

Picking out a “preferred complex structure” on $\mathbb{R}^4$.
Noncommutativity for a single instanton: metric becomes Eguchi-Hanson, which smoothes out the singularity at the origin

K. Lee, D. Tong & S. Yi

Noncommutative Effects

From the point of view of Yang-Mills, SU(2) $\to$ U(2); multiplication is replaced by the Moyal $\star$-product: for functions f, g valued in $\mathbb{R}^4_{NC}$,

$$f \star g = e^{i\theta_{\mu\nu}\partial_\mu \partial'_\nu} f(x) g(x')|_{x=x'}$$

Not tractable in practice. Instead, use an identification between Moyal $\star$ on function space and Hilbert space of operators: modification to ADHM constraints via

$$|x|^2 = x_i^2 + \begin{pmatrix} 2\zeta & 0 \\ 0 & -2\zeta \end{pmatrix}.$$

Then we must solve the modified ADHM constraints.

A noncommutative deformation

The NC modified constraints are soluble: introduce a deformation to the commutative solutions of the form

$$v_i \to \frac{1}{\sqrt{|v_i|^2}} \begin{pmatrix} \sqrt{|v_i|^2 + \alpha\zeta} & 0 \\ 0 & \sqrt{|v_i|^2 - \alpha\zeta} \end{pmatrix} v_i$$

where

$$\alpha = \frac{32|\tau|^2|v_1|^2|v_2|^2}{16|\tau|^2|v_1|^2|v_2|^2 + |\bar{v}_2 v_1 - \bar{v}_1 v_2|^2(|v_1|^2 + |v_2|^2)}.$$

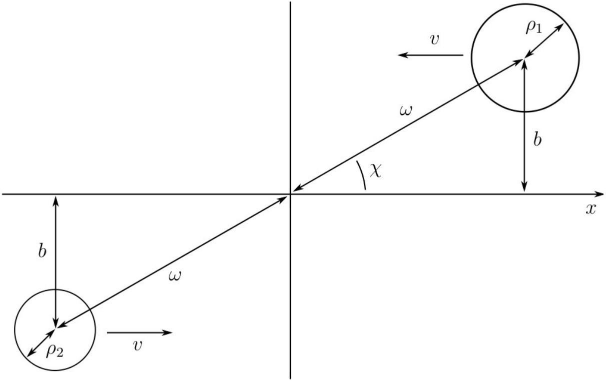
Complexification

It is not computationally easy to calculate the full 16-dimensional metric. Instead, choose a geodesic submanifold of the data:

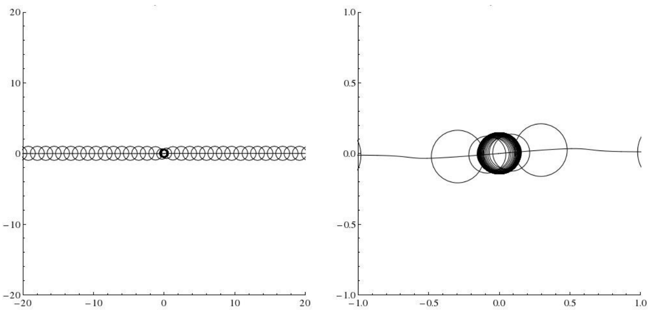
$$v_i = \mathrm{diag}(\rho_i e^{i\theta_i}, \rho_i e^{-i\theta_i}),$$
$$\tau = \mathrm{diag}(\omega e^{i\chi}, \omega e^{-i\chi}).$$

This gives a 6-dimensional subspace of the full moduli space. Can consider scattering in this context.

Identify ω as separation, χ as scattering angle, ρ_i as instanton sizes and θ_i as internal gauge orientations.

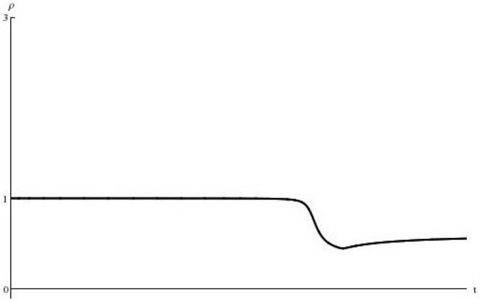


Head-on scattering



Very different scattering behaviour: instantons coalesce stably, without approaching zero-size. The zero-size resolution is expected: what about the scattering?

Head-on scattering



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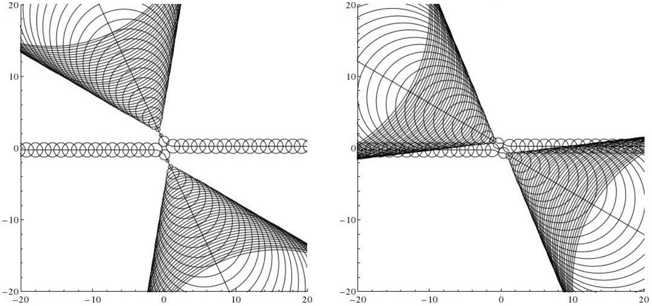
Near head-on Scattering

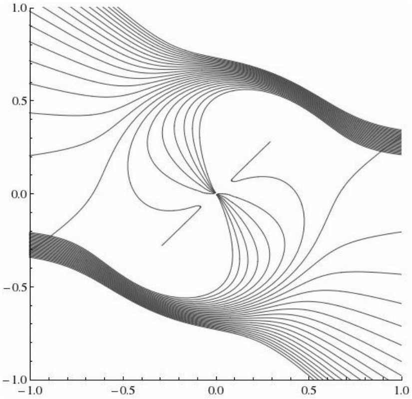
Moving slightly away from head-on scattering, we observe commutative-like behaviour.



Suggestive of a transition point between scattering and coalescence.

Scattering Criteria

Varying impact parameter and ζ and observing the scattering angle demonstrates the transition.



For ζ large enough, or impact parameter small enough, NC effect overcomes repulsive force of instantons.

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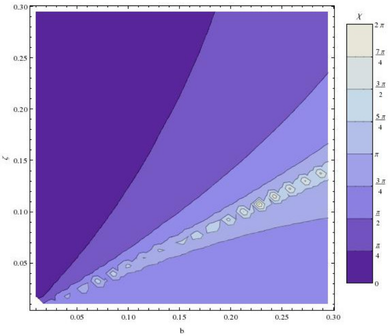

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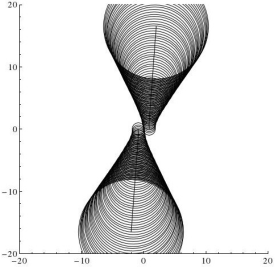
Noncommutativity as a potential

Commutative results with potential term display attractive behaviour. Can the ζ-effect be viewed as potential-like?



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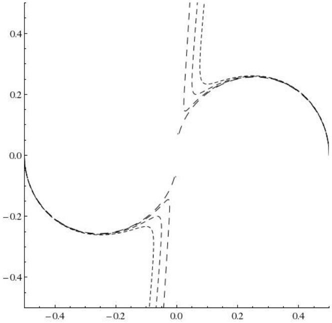

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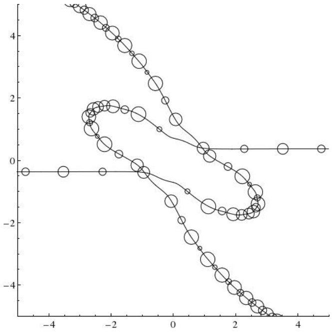
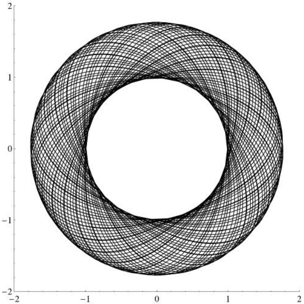


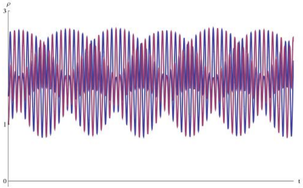
Maybe! More analysis needed.

If NC causes a potential-like force, the strength is beyond that allowed in the geodesic approximation for a "by-hand" potential term.

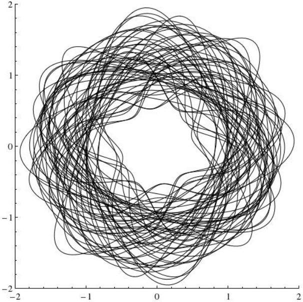
Dyonic noncommutative instantons

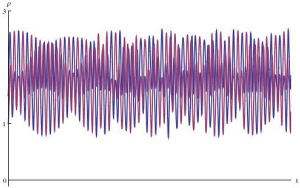
Also consider the noncommutative effect and a by-hand potential. The commutative case allowed for stable orbits:



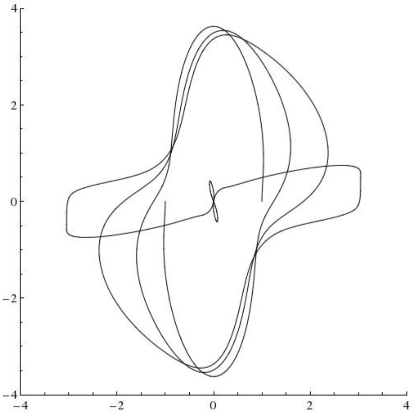


Also consider the noncommutative effect and a by-hand potential. In the noncommutative case, a similar picture appears...





Stability is not guaranteed. Coalescence (or escape) may occur in finite time.



Attraction

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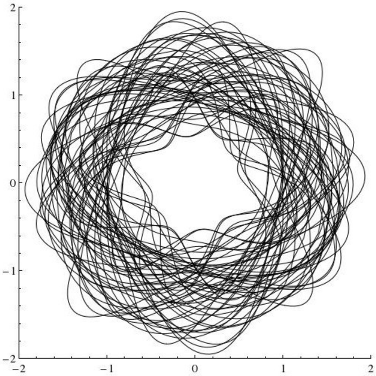

Stability

Dyonic noncommutative instantons

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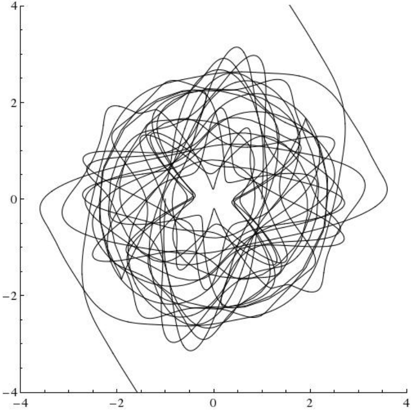

Escape

Summary

- Instantons provide an alternative to solving the Yang-Mills field equations via the ADHM construction.
- One can build a moduli space of allowed configurations and "evolve" to consider scattering.
- Singularities occur at zero-size: resolution comes from considering a noncommutative space $\mathbb{R}^4$.
- NC dynamics is markedly different from the commutative case: coalescence can be seen when the NC effect dominates.
- Dyonic NC instantons have much more chaotic behaviour than their commutative counterparts.

Outlook

- Can we extend this to higher gauge groups? Particularly U($2n$), $n \in \mathbb{Z}$.
- Would be nice to replicate NC attraction as a potential in the moduli space metric.
- Connections to other solitons (vortices, hyperbolic monopoles...) via dimensional reduction.
- Examine the string picture: index calculation for induced D-branes like that for the single instanton D. Bak, A. Gustavsson.
- Noncommutative monopoles? Vortices?
- More pretty pictures! Preferably with colour.

Thanks!

Thank you for listening!