Adiabatic Hydrodynamics and the Eightfold Way to Dissipation

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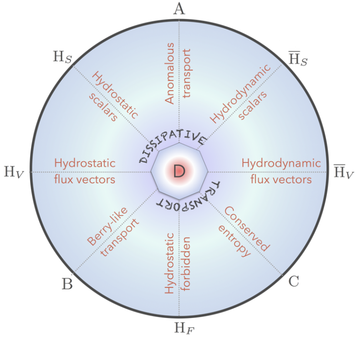

9 February 2015

FH, R. Loganayagam, M. Rangamani [1502.00636], [1412.1090]

The hydrodynamic gradient expansion

- Hydrodynamics: low-energy, near-equilibrium eff. field theory for generic Gibbsian density matrix

microscopic theory

$$\downarrow L \gg \ell_{\mathsf{mfp}}$$

macroscopic fluid variables:	$u^\mu(x),\ T(x),\ \mu(x)$	$(u^2=-1)$
background sources:	$g_{\mu\nu}(x),\ A_\mu(x)$	

$$\downarrow$$

Constitutive relations:	Dynamics:
$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \ldots$	$\nabla_\nu T^{\mu\nu} \simeq F^{\mu\nu} J_\nu$
$J^\alpha = J^\alpha_{(0)} + J^\alpha_{(1)} + \ldots$	$\nabla_\alpha J^\alpha \simeq 0$

- E.g. (charged) ideal fluid: $\quad T^{\mu\nu}_{(0)} = \varepsilon\, u^\mu u^\nu + p\, P^{\mu\nu}\,, \qquad J^\alpha_{(0)} = q\, u^\alpha$

The hydrodynamic gradient expansion

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macroscopic fluid variables: $u^\mu(x),\ T(x),\ \mu(x)$ $\quad (u^2 = -1)$
background sources: $g_{\mu\nu}(x),\ A_\mu(x)$

$$\downarrow$$

Constitutive relations:

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \dots$$

$$J^\alpha = J^\alpha_{(0)} + J^\alpha_{(1)} + \dots$$

Dynamics:

$$\nabla_\nu T^{\mu\nu} \simeq F^{\mu\nu} J_\nu + \boxed{\mathrm{T}_H^{\perp\nu}}$$

$$\nabla_\alpha J^\alpha \simeq \boxed{\mathrm{J}_H^{\perp}} \quad \text{(cov. anomalies)}$$

- E.g. (charged) ideal fluid: $\quad T^{\mu\nu}_{(0)} = \varepsilon\, u^\mu u^\nu + p\, P^{\mu\nu}\,, \qquad J^\alpha_{(0)} = q\, u^\alpha$

The hydrodynamic gradient expansion

- **"Current Algebra"** approach:
  - Provide most general symmetry-allowed constitutive relations order by order in ∇_μ
  - Transport coefficients of any particular fluid are determined by microscopics
- On top of all this: **Second Law constraint**

$$\exists\ J_S^\alpha = s\, u^\alpha + J_{S,(1)}^\alpha + \ldots \qquad \text{with} \qquad D_\alpha J_S^\alpha \gtrsim 0 \qquad \text{(on-shell)}$$

  - Gives quite non-trivial constraints on physically allowed constitutive relations, e.g.:
    - ★ Neutral 1st order: viscosities $\eta, \zeta \geq 0$
    - ★ Neutral 2nd order: 5 relations among 15 a-priori independent transport coefficients
    - ★ Anomaly induced transport completely fixed

Bhattacharyya '12

Son-Surowka '09

Jensen-Loganayagam-Yarom '13

Outline

- ✓ Review of hydrodynamics
- → So what's the problem?
- ◦ Adiabaticity and dissipation
- ◦ Classification of adiabatic transport
- ◦ An example: neutral fluid
- ◦ The adiabatic master Lagrangian
- ◦ Conclusion

So what's the problem?

- This phenomenological framework doesn't make very much sense from point of view of **Wilsonian field theory**
  - Instead of just currents: would like **effective action**
  - Would like to associate conservation laws with **equations of motion** of the effective action
  - J_S^μ **is particularly strange from microscopic perspective:** is not associated to any underlying symmetry principle
  - In Wilsonian picture, how does the constraint $D_\mu J_S^\mu \gtrsim 0$ arise from microscopic theory?

So what's the problem?

- Some progress: **Non-dissipative effective actions** $(D_\mu J^\mu_S = 0)$
  - Goldstone modes of spntaneously broken symmetries as fluid degrees of freedom *Dubovsky-Hui-Nicolis-Son '11*
- This approach can't be the whole story:
  - Empirically only gives a subset of non-dissipative hydro *Bhattacharya-Bhattacharyya-Rangamani '12*
  - Dynamics in general involves **dissipation** *FH-Rangamani '13*
  - Hydro states are **mixed**. Wilsonian picture for mixed states should involve something like **Schwinger-Keldysh doubling**
    - ⋆ A lot about Schwinger-Keldysh is not well understood (arbitrariness of influence functionals, violation of microscopic KMS condition, ...)

Goals and phantasies

- **Understand** most general constitutive relations allowed by second law:
  - ▷ **Classify** hydrodynamic transport in a physically useful way
- Suggest a **unifying framework** for adiabatic transport:
  - ▷ Hydrodynamics as proper effective field theory
  - ▷ **New symmetry principle** that explains the 2nd law constraint
- Use hydrodynamics as a tractable starting point to learn basic lessons about some important problems across physics:
  - ▷ Wilsonian picture for **systems out of equilibrium**
  - ▷ Wilsonian picture for **noisy/dissipative systems**
  - ▷ Via AdS/CFT: **gravity with horizons**

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Off-shell entropy production and adiabaticity

- Inequality constraint $D_\mu J_S^\mu \gtrsim 0$ is much more conveniently incorporated if we don't have to simplify it using equations of motion.
- Use Lagrange multipliers $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$ and consider **off-shell statement**:

$$\Delta \equiv \nabla_\mu J_S^\mu + \boldsymbol{\beta}_\mu \left\{ \nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu} - \mathrm{T}_H^{\mu\perp} \right\} \\ + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda) \cdot \left\{ D_\nu J^\nu - \mathrm{J}_H^\perp \right\} \geq 0$$

- Natural Lagrange multipliers:
  - $\boldsymbol{\beta}^\mu = \frac{1}{T}\, u^\mu$ (thermal vector along 'local thermal circle')
  - $(\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda) = \frac{\mu}{T}$ (chemical potential in thermal units)
- Task: solve for $\{J_S^\mu, T^{\mu\nu}, J^\nu\}$ as functionals of $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}, g_{\mu\nu}, A_\mu\}$
- Marginal case $\Delta = 0$: **'adiabaticity equation'**
  - Particularly rich structure! $\Rightarrow$ study separately

Off-shell entropy production and adiabaticity

$$\Delta \equiv \nabla_\mu J_S^\mu + \boldsymbol{\beta}_\mu \left\{ \nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu} - \mathrm{T}_H^{\mu\perp} \right\} + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda) \cdot \left\{ D_\nu J^\nu - \mathrm{J}_H^\perp \right\} \geq 0$$

- Can switch from microcanonical to grand-canonical ensemble and talk about **free energy current** $\mathcal{G}^\sigma$ instead of J_S^σ:

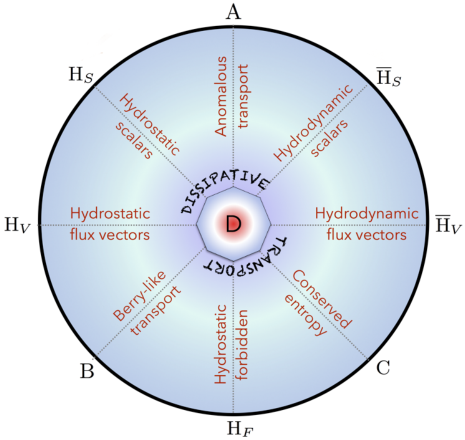
$$-\frac{\mathcal{G}^\sigma}{T} \equiv J_S^\sigma - (J_S^\sigma)_{canonical} = J_S^\sigma + \left[\boldsymbol{\beta}_\nu T^{\nu\sigma} + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\nu A_\nu) \cdot J^\sigma \right]$$

- Grand-canonical version:

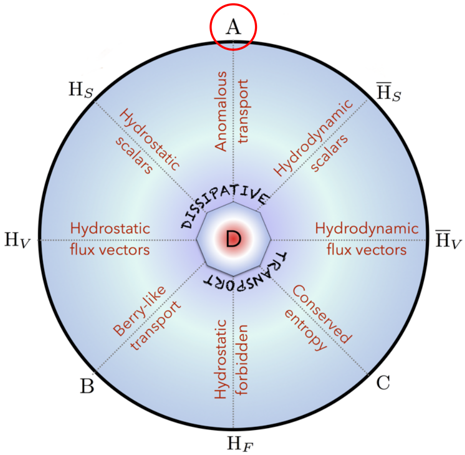
$$-\left[\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T} \right) - \frac{\mathcal{G}_H^\perp}{T} \right] = \frac{1}{2} T^{\mu\nu} \, \delta_{\mathscr{B}} \, g_{\mu\nu} + J^\mu \cdot \delta_{\mathscr{B}} \, A_\mu + \Delta$$

  - $\delta_{\mathscr{B}} \, g_{\mu\nu} \equiv \pounds_{\boldsymbol{\beta}} g_{\mu\nu} = \nabla_\mu \boldsymbol{\beta}_\nu + \nabla_\nu \boldsymbol{\beta}_\mu$
  - $\delta_{\mathscr{B}} \, A_\mu \equiv \pounds_{\boldsymbol{\beta}} A_\mu + \partial_\mu \Lambda_{\boldsymbol{\beta}} + [A_\mu , \Lambda_{\boldsymbol{\beta}}]$

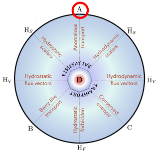
Classification of hydrodynamic transport



Classification of hydrodynamic transport



Anomaly induced transport (Class A)



$$-\left[\nabla_\sigma\left(\frac{\mathcal{G}^\sigma}{T}\right)-\frac{\mathcal{G}_H^\perp}{T}\right]=\frac{1}{2}\,T^{\mu\nu}\,\delta_{\mathscr{B}}\,g_{\mu\nu}+J^\mu\cdot\delta_{\mathscr{B}}\,A_\mu+\Delta$$

- Will now discuss various classes of solutions
- First of all: let's get rid of anomalies $\mathcal{G}_H^\perp=-\left[u_\nu \mathrm{T}_H^{\nu\perp}+\mu\cdot \mathrm{J}_H^\perp\right]$
  - Can always split off from a solution $\{\mathcal{G}^\sigma, T^{\mu\nu}, J^\nu\}$ a **particular solution** $\{(\mathcal{G}^\sigma)_A,(T^{\mu\nu})_A,(J^\nu)_A\}$ that takes care of anomalies with $(\Delta)_A=0$:

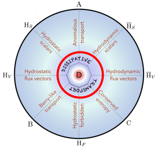
$$-\left[\nabla_\sigma\left(\frac{(\mathcal{G}^\sigma)_A}{T}\right)-\frac{\mathcal{G}_H^\perp}{T}\right]=\tfrac{1}{2}\,(T^{\mu\nu})_A\,\delta_{\mathscr{B}}\,g_{\mu\nu}+(J^\nu)_A\cdot\delta_{\mathscr{B}}\,A_\mu$$

Loganayagam '11

Jensen-Loganayagam-Yarom '13

  - Anomalous transport coefficients fixed in terms of anomaly polynomial $\Rightarrow$ finite class

Dissipative transport (Class D)

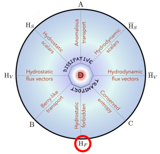


$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{1}{2} T^{\mu\nu}\, \delta_{_\mathscr{B}} g_{\mu\nu} - J^\mu \cdot \delta_{_\mathscr{B}} A_\mu \geq 0$$

- Now consider transport which does generically produce entropy ($\Delta > 0$)
- Such terms appear in three varieties:
  1. Sign-definite terms (inequalities from 2nd law) *Bhattacharyya '11 '13 '14*
     $\rightarrow$ **These only show up at leading order!**
  2. Sign-indefinite terms which are dominated by sign-definite terms (no constraints from 2nd law)
  3. Sign-indefinite terms which are dominant in derivative expansion (forbidden by 2nd law)
- **Example:** $T^{\mu\nu}_{(1)} = -\zeta\, \Theta\, P^{\mu\nu}$ $\quad (\Theta \equiv \nabla_\mu u^\mu)$
  gives $\Delta = \zeta\, \frac{1}{T} \Theta^2 \quad \Rightarrow \quad \zeta \geq 0$ (type 1)

  $\Rightarrow$ At $\mathcal{O}(\partial^{k\geq 2})$:
  any $T^{\mu\nu}_{(k)} = \gamma\, [\mathcal{O}(\partial^k)]^{\mu\nu}$ s.t. $\Delta = \gamma\, \Theta^2\, [\mathcal{O}(\partial^{k-1})]$
  will be subdominant, hence unconstrained (type 2)

Hydrostatically forbidden terms (Class H_F)



$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{1}{2} T^{\mu\nu}\, \delta_{_\mathscr{B}} g_{\mu\nu} - J^\mu \cdot \delta_{_\mathscr{B}} A_\mu \geq 0$$

- Type ③: sign-indefinite terms at dominant order in ∂
  - Need to be zero for consistency with 2nd law!
  - **Example:** Ideal fluid

$$T^{\mu\nu}_{(0)} = \varepsilon\, u^\mu u^\nu + p\, P^{\mu\nu}\,, \qquad J^\mu_{S,(0)} = s\, u^\mu$$

$$\Rightarrow \quad \Delta \simeq \underbrace{(Ts - \varepsilon - p)}_{=0\ (!)}\, \frac{\Theta}{T} + \underbrace{\left(T\frac{ds}{dT} - \frac{d\varepsilon}{dT}\right)}_{=0\ (!)}\, \frac{(u\nabla)T}{T}$$

  - A-priori: 3 parameters
  - But second law enforces: 2 relations
- This is **Class** H_F**:** combinations forbidden by 2nd law (or equivalently by existence of equilibrium)

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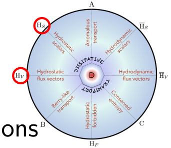
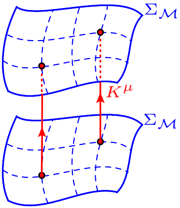
Hydrostatics (Class H)



- Hydrostatic transport: time-independent equilibrium configurations
  - $\exists$ timelike Killing vector and gauge transformation $\mathfrak{K} = \{K^\mu, \Lambda_K\}$:

    $$\delta_{\mathfrak{K}} g_{\mu\nu} = \delta_{\mathfrak{K}} A_\mu = 0$$

  - Spacetime manifold $\mathcal{M}$: Euclidean fibre bundle $\Sigma_{\mathcal{M}} \times S^1$

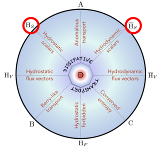


- Transport captured by **Euclidean path integral/partition function:**

$$W_{\text{Hydrostatic}} = -[\text{total free energy}] = -\left[\int_{\Sigma_{\mathcal{M}}} \left(\frac{\mathcal{G}^\sigma}{T}\right) d^{d-1}S_\sigma\right]_{\text{Hydrostatic}}$$

  - Decompose: $\mathcal{G}^\sigma = \mathcal{S}\,\boldsymbol{\beta}^\sigma + \mathcal{V}^\sigma$
  - This splits Class H into two subclasses: $\mathrm{H} = \mathrm{H}_S \cup \mathrm{H}_V$
  - Variation w.r.t. $\{g_{\mu\nu}, A_\mu\}$ gives all hydrostatic $\{T^{\mu\nu}, J^\mu\}$

Lagrangian solutions (Class L)



- Consider effective actions with obvious symmetries:

$$S = \int \sqrt{-g}\, \mathcal{L}[\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}, g_{\mu\nu}, A_\mu]$$

- Basic variation defines hydrodynamic currents:

$$\delta S = \int \sqrt{-g}\, \left[\tfrac{1}{2}\, T^{\mu\nu}\, \delta g_{\mu\nu} + J^\mu \cdot \delta A_\mu + T\, \mathfrak{h}_\sigma\, \delta \boldsymbol{\beta}^\sigma + T\, \mathfrak{n} \cdot (\delta \Lambda_{\boldsymbol{\beta}} + A_\sigma\, \delta\boldsymbol{\beta}^\sigma) + \nabla_\mu (\not{\delta}\,\Theta_{\mathsf{PS}})^\mu \right]$$

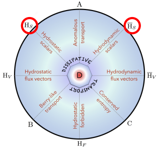
- Demand invariance under diffeos & flavour $\mathscr{X} = \{\xi^\mu, \Lambda\}$: $\quad \delta_{\mathscr{X}} S = 0\ (!)$
  - This gives **Bianchi identities:**

$$\nabla_\nu T^{\mu\nu} = J_\nu \cdot F^{\mu\nu} + \frac{g^{\mu\nu}}{\sqrt{-g}} \delta_{\mathscr{B}} \left(\sqrt{-g}\, T\, \mathfrak{h}_\nu\right) + g^{\mu\nu}\, T\, \mathfrak{n} \cdot \delta_{\mathscr{B}} A_\nu$$

$$D_\sigma J^\sigma = \frac{1}{\sqrt{-g}} \delta_{\mathscr{B}} \left(\sqrt{-g}\, T\, \mathfrak{n}\right)$$

  - Together with $\mathcal{G}^\sigma \equiv -\mathcal{L}\, u^\sigma + T(\not{\delta}_{\mathscr{B}}\Theta_{\mathsf{PS}})^\sigma$ one can show that these **imply adiabaticity equation!**

Lagrangian solutions (Class L)



- So far: have effectively treated $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$ as non-dynamical
- To get hydrodynamic equations, consider **constrained variational principle** "$\mathbb{\delta}$":
  - Vary $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$ along Lie orbits while holding $\{g_{\mu\nu}, A_\mu\}$ fixed:

    $$\mathbb{\delta}: \quad \mathbb{\delta}\boldsymbol{\beta}^\mu = \delta_{\mathfrak{X}}\boldsymbol{\beta}^\mu\,, \qquad \mathbb{\delta}\Lambda_{\boldsymbol{\beta}} = \delta_{\mathfrak{X}}\Lambda_{\boldsymbol{\beta}}\,, \qquad \mathbb{\delta}g_{\mu\nu} = \mathbb{\delta}A_\mu = 0\,.$$

  - These variations give equations of motion:

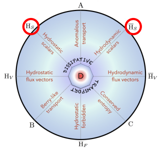
    $$\frac{g^{\mu\nu}}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}\;T\,\mathfrak{h}_\nu\right) + g^{\mu\nu}\,T\,\mathfrak{n}\cdot\delta_{\mathscr{B}}A_\nu \simeq 0$$

    $$\frac{1}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}\;T\,\mathfrak{n}\right) \simeq 0$$

  - Together with Bianchi identities, get hydro equations:

    $$\nabla_\nu T^{\mu\nu} \simeq J_\nu\cdot F^{\mu\nu}$$
    $$D_\sigma J^\sigma \simeq 0$$

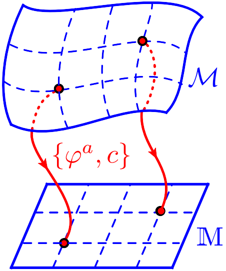
Lagrangian solutions (Class L)



- Alternative picture for constrained variational principle:
  - Physical fields are pullbacks of a **reference configuration**:

$$g_{\mu\nu} = \frac{\partial\varphi^a}{\partial x^\mu}\frac{\partial\varphi^b}{\partial x^\nu}\,\mathbb{g}_{ab}[\varphi(x)]\,, \qquad \boldsymbol{\beta}^\mu = \frac{\partial x^\mu}{\partial\varphi^a}\,\mathbb{\beta}^a[\varphi(x)]$$

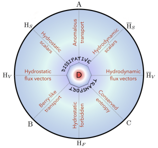
(and similarly $\{A_\mu, \Lambda_{\boldsymbol{\beta}}\} \xrightarrow{\{\varphi^a, c\}} \{\mathbb{A}_a, \Lambda_{\mathbb{\beta}}\}$)



  - Dynamics now encoded in $\{\varphi^a, c\}$
  - Can get hydrodynamic conservation equations by varying pullback fields $\{\varphi^a, c\}$, while holding the reference configuration fixed

- Aside: **non-dissipative effective actions** are a special case of this
  - Pullback fields $\{\varphi^a, c\}$ correspond to Goldstones $\{\phi^a, \mathrm{c}\}$ of broken symmetries (after Legendre transform and gauge fixing)

What do we have so far?

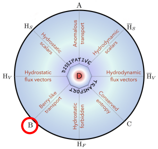


- **Class A:** anomalies can be dealt with once and forever
- **Class D:** can genuinely produce entropy $(\Delta \geq 0)$
- **Class** H_F**:** Constitutive relations inconsistent with existence of equilibrium

$$\mathcal{G}^\sigma = \mathcal{S}\boldsymbol{\beta}^\sigma + \mathcal{V}^\sigma$$

- **Class H** $= \mathrm{H}_S \cup \mathrm{H}_V$**:** Hydrostatic response to free energy density and flux
- **Class L:** Wilsonian action giving currents consistent with second law
  - But Lagrangians being scalars, we only get: Class L $= \mathrm{H}_S \cup \overline{\mathrm{H}}_S$
- Some more situations that we're missing so far:
  - Free energy current $\mathcal{G}^\sigma$ could be **zero** or **topological**
  - **Non-hydrostatic** free energy flux vectors

Berry-curvature type solutions (Class B)



$$\Delta \equiv -\nabla_\sigma \left(\frac{\mathcal{G}^\sigma}{T}\right) - \frac{1}{2} T^{\mu\nu}\, \delta_{\mathscr{B}}\, g_{\mu\nu} - J^\mu \cdot \delta_{\mathscr{B}}\, A_\mu = 0$$

- Consider the following currents:

$$(T^{\mu\nu})_{\mathsf{B}} \sim \left(\mathcal{N}^{(\mu\nu)(\alpha\beta)} - \mathcal{N}^{(\alpha\beta)(\mu\nu)}\right)\, \delta_{\mathscr{B}}\, g_{\alpha\beta}$$

$$(J^\alpha)_{\mathsf{B}} \sim \mathcal{S}^{[\alpha\beta]} \cdot \delta_{\mathscr{B}}\, A_\beta$$

$$(\mathcal{G}^\sigma)_{\mathsf{B}} = 0$$

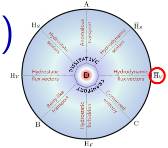
(and cross terms $\mathcal{X}$)

  - Trivially solve adiabaticity equation
  - Manifestly **non-hydrostatic** ($\delta_{\mathscr{B}} = 0$ in hydrostatics)
  - Seemingly not captured by Lagrangians (Class L)

- Easy task at any order in ∇:
  find all tensor structures $\{\mathcal{N}, \mathcal{X}, \mathcal{S}\}$ built out of $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}, g_{\mu\nu}, A_\mu\}$

- Examples in $d = 2+1$: Hall conductivity, Hall viscosity

Transverse non-hydrostatic free energy (Class $\overline{\mathrm{H}}_V$)



- Remember splitting: $\mathcal{G}^\sigma = \mathcal{S}\,\boldsymbol{\beta}^\sigma + \mathcal{V}^\sigma$ with $\boldsymbol{\beta}_\sigma \mathcal{V}^\sigma = 0$
- Consider solutions to adiabaticity equation with non-trivial and **non-hydrostatic** $\mathcal{V}^\sigma$
  - Transport genuinely due to free energy flux
- These are in general parameterized as

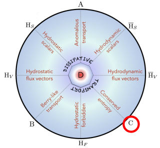
$$(T^{\mu\nu})_{\overline{\mathrm{H}}_V} \sim D_\rho \mathfrak{C}_{\mathcal{N}}^{\rho(\mu\nu)(\alpha\beta)}\,\delta_{\mathscr{B}}\, g_{\alpha\beta} + 2\,\mathfrak{C}_{\mathcal{N}}^{\rho(\mu\nu)(\alpha\beta)}\,D_\rho \delta_{\mathscr{B}}\, g_{\alpha\beta}$$

$$(J^{\alpha})_{\overline{\mathrm{H}}_V} \sim D_\rho \mathfrak{C}_{\mathcal{S}}^{\rho(\alpha\beta)} \cdot \delta_{\mathscr{B}}\, A_\beta + 2\,\mathfrak{C}_{\mathcal{S}}^{\rho(\alpha\beta)} \cdot D_\rho \delta_{\mathscr{B}}\, A_\beta$$

(and cross terms $\mathfrak{C}_{\mathcal{X}}$)

  - Easy task at any order in ∇:
    find all tensor structures $\{\mathfrak{C}_{\mathcal{N}}, \mathfrak{C}_{\mathcal{X}}, \mathfrak{C}_{\mathcal{S}}\}$ built out of $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}, g_{\mu\nu}, A_\mu\}$

Conserved entropy current (Class C)



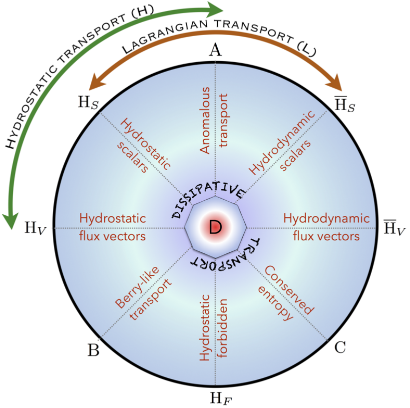
- Another trivial solution to adiabaticity equation: **exactly conserved entropy current**

$$(J_S^\mu)_C = \mathsf{J}^\mu \quad \text{with} \quad D_\mu \mathsf{J}^\mu \equiv 0\,, \qquad (T^{\mu\nu})_C = 0\,, \qquad (J^\mu)_C = 0$$

- If cohomologically non-trivial: describes **topological states** in the fluid (no energy/charge transport)
  - Example: Euler current $\mathsf{J}^\sigma_{\text{Euler}}$ in $d = 2+1$ *Golkar-Roberts-Son '14*

$$D_\sigma \mathsf{J}^\sigma_{\text{Euler}} \equiv 0\,, \qquad \int_{\Sigma_\mathcal{M}} \sqrt{-\gamma}\,(\mathsf{J}^\sigma_{\text{Euler}}\, u_\sigma) \quad \propto \quad \chi(\Sigma_\mathcal{M})$$

Summary of eight classes of transport



Theorem: The eightfold way of hydrodynamic transport

These eight classes describe all of hydrodynamic transport consistent with the second law: every second law-compatible transport coefficient falls into one of these classes.

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Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Most general 2[nd] order (neutral, Weyl-invariant) stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{(2)} = {} & (\lambda_1 - \kappa)\, \sigma^{<\mu\alpha} {\sigma_\alpha}^{\nu>} \\
& + (\lambda_2 + 2\tau - 2\kappa)\, \sigma^{<\mu\alpha} {\omega_\alpha}^{\nu>} \\
& + \tau \, \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha} {\omega_\alpha}^{\nu>} \right) \\
& + \lambda_3 \, \omega^{<\mu\alpha} {\omega_\alpha}^{\nu>} \\
& + \kappa \, \left(C^{\mu\alpha\nu\beta} u_\alpha u_\beta + \sigma^{<\mu\alpha} {\sigma_\alpha}^{\nu>} + 2\sigma^{<\mu\alpha} {\omega_\alpha}^{\nu>} \right)
\end{aligned}
$$

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

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$$
\begin{aligned}
T^{\mu\nu}_{(2)} = {} & (\lambda_1 - \kappa)\, \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} \\
& + (\lambda_2 + 2\tau - 2\kappa)\, \sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>} \\
& + \tau\, \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \overline{\mathrm{H}}_S \\
& + \lambda_3\, \omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \to \text{Class } \mathrm{H}_S \\
& + \kappa\, \left(C^{\mu\alpha\nu\beta} u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \mathrm{H}_S
\end{aligned}
$$

τ, λ_3, κ

Are all derivable from a Lagrangian (Class L)

$$
\mathcal{L}_2^{\mathcal{W}} = \frac{1}{4}\left[-\frac{2\,\kappa}{(d-2)}({}^{\mathcal{W}}R) + 2\,(\kappa - \tau)\,\sigma^2 + (\lambda_3 - \kappa)\,\omega^2\right]
$$

Note: λ_3 and κ are hydrostatic, τ is genuinely hydrodynamic

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Most general 2nd order (neutral, Weyl-invariant) stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{(2)} = {} & (\lambda_1 - \kappa)\,\sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} && \to \text{Class D} \\
& + (\lambda_2 + 2\tau - 2\kappa)\,\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \\
& + \tau\,\left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \overline{\mathrm{H}}_S \\
& + \lambda_3\,\omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \to \text{Class } \mathrm{H}_S \\
& + \kappa\,\left(C^{\mu\alpha\nu\beta}u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \mathrm{H}_S
\end{aligned}
$$

$(\lambda_1 - \kappa)$

Leads to $\Delta \simeq -(\lambda_1 - \kappa)\,\frac{1}{T}\,\sigma^\mu{}_\nu\sigma^\nu{}_\rho\sigma^\rho{}_\mu$
$\Rightarrow$ Class D but unconstrained (subleading compared to $\Delta \sim \sigma^2$)

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Most general 2[nd] order (neutral, Weyl-invariant) stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{(2)} = {} & (\lambda_1 - \kappa)\, \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} && \to \text{Class D} \\
& + (\lambda_2 + 2\tau - 2\kappa)\, \sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \to \text{Class B} \\
& + \tau \left(u^\alpha \mathcal{D}^{\mathcal{W}}_\alpha \sigma^{\mu\nu} - 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \overline{\mathrm{H}}_S \\
& + \lambda_3\, \omega^{<\mu\alpha}\omega_\alpha{}^{\nu>} && \to \text{Class } \mathrm{H}_S \\
& + \kappa \left(C^{\mu\alpha\nu\beta} u_\alpha u_\beta + \sigma^{<\mu\alpha}\sigma_\alpha{}^{\nu>} + 2\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\right) && \to \text{Class } \mathrm{H}_S
\end{aligned}
$$

$(\lambda_2 + 2\tau - 2\kappa)$

Is of the form of a Class B constitutive relation

$$
\begin{aligned}
(T^{\mu\nu})_{\mathsf{B}} &\equiv -\frac{1}{4}\left(\mathcal{N}^{(\mu\nu)(\alpha\beta)} - \mathcal{N}^{(\alpha\beta)(\mu\nu)}\right) \delta_{\mathscr{B}}\, g_{\alpha\beta} \\
(\mathcal{G}^\sigma)_{\mathsf{B}} &= 0
\end{aligned}
$$

because of orthogonality: $\sigma^{<\mu\alpha}\omega_\alpha{}^{\nu>}\, \delta_{\mathscr{B}}\, g_{\mu\nu} = 0$

Example: neutral Weyl-invariant fluid at $\mathcal{O}(\partial^2)$

- Out of 5 transport coefficients, 3 come from a Lagrangian: τ, λ_3 and κ
- Within Class L, the other 2 combinations are zero:

$$(\lambda_1 - \kappa) = 0 \quad \text{and} \quad (\lambda_2 + 2\tau - 2\kappa) = 0$$

  - These relations have been **observed in holography**

    Haack-Yarom '08

  - First relation ensures **no entropy production at subleading order** (this is not required by second law!)
    $\rightarrow$ **"Principle of minimum dissipation"** in holography?

    FH-Loganayagam-Rangamani '14

Outline

- ✓ Review of hydrodynamics
- ✓ So what's the problem?
- ✓ Adiabaticity and dissipation
- ✓ Classification of adiabatic transport ($\Delta = 0$)
- ✓ An example: neutral fluid
- $\rightarrow$ The adiabatic master Lagrangian
- $\circ$ Conclusion

A new symmetry for hydrodynamics

- So far only a subset of our 8 classes were described by Lagrangians
- This was to be expected: non-equilibrium effective field theory should involve **Schwinger-Keldysh doubling**

> Just doubling everything gives too much freedom (violate second law by writing effective Lagrangians for forbidden Class H_F)

  - Important problem for understanding EFT for mixed states

- Introduce **new** $U(1)_{\mathsf{T}}$ **gauge symmetry** to keep this under control

> **Proposed field content:**
>
> - Hydrodynamic fields: $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$
> - Background sources: $\{g_{\mu\nu}, A_\mu\}$
> - SK-like partner sources: $\{\tilde{g}_{\mu\nu}, \tilde{A}_\mu\}$
> - $U(1)_{\mathsf{T}}$ photon and holonomy field: $\{\mathsf{A}^{(\mathsf{T})}{}_\mu, \Lambda^{(\mathsf{T})}_{\boldsymbol{\beta}}\}$

- Action of $U(1)_{\mathsf{T}}$ is twisted: **longitudinal diffeo** on all fields plus inhomogeneous thermal "Goldstone-like" shift on partner sources

The eightfold master Lagrangian (Class L_{T})

- **Any constitutive relations** $\{T^{\mu\nu}, J^\mu, \mathcal{G}^\sigma\}$ **which satisfy adiabaticity equation** can be obtained from a diffeo/flavour/$U(1)_{\mathsf{T}}$ invariant Lagrangian:

$$\boxed{\mathcal{L}_{\mathsf{T}} = \frac{1}{2} T^{\mu\nu} \tilde{g}_{\mu\nu} + J^\mu \cdot \tilde{A}_\mu - \frac{\mathcal{G}^\sigma}{T} \mathsf{A}^{(\mathsf{T})}{}_\sigma}$$

  - Bianchi identity for $U(1)_{\mathsf{T}}$ invariance reduces to adiabaticity equation
  - Equations of motion are:
    - For diffeo invariance: $D_\nu T^{\mu\nu} \simeq F^{\mu\nu} \cdot J_\nu$
    - For flavour gauge invariance: $D_\mu J^\mu \simeq 0$
    - For $U(1)_{\mathsf{T}}$ invariance: $D_\mu J_S^\mu \simeq 0$

- Conversely: any diffeo/flavour/$U(1)_{\mathsf{T}}$ invariant Lagrangian gives adiabatic constitutive relations

Heuristic picture for $U(1)_{\mathsf{T}}$ symmetry

- Field content and symmetries are such that we **get precisely the 7 adiabatic classes** and nothing more (Class H_F)
- Conserved entropy current is now associated to a symmetry
- General picture:
  - Non-equilibrium dynamics captured by effective action after Schwinger-Keldysh doubling
  - Influence functionals are constrained by requirement of $U(1)_{\mathsf{T}}$ invariance
  - $U(1)_{\mathsf{T}}$ invariance is the **macroscopic manifestation of KMS condition** and ensures consistency with the second law

FH-Loganayagam-Rangamani [w.i.p.]

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Summary

- 8 classes of constitutive relations consistent with the second law describe all of hydrodynamic transport at all orders
- The classification is not just about mathematical structure, but **seems to be cognizant of physical properties of fluids:**
  - Various fluid systems seem to know about this classification
  - A lot about holographic fluids seems to be described in Class L
  - Conjecture: long-wavelength **near-horizon AdS dynamics** can be usefully characterized using the eightfold way
- Computations become simpler in this framework (eightfold way often wants to pick "nice" basis)
- We suggest **a new $U(1)_{\mathsf{T}}$ symmetry principle** that unifies the 7 adiabatic classes and explains the entropy current constraint

Many questions...

- Much of this story should have a counterpart in **gravity**
  - Most obvious hints come from Class L. Can $\mathcal{L}$ be derived from holography?
- Investigate **“Minimum dissipation conjecture“:**
  Holographic fluids optimize entropy production (at all orders).
- Extend the formalism to write **effective actions for dissipative fluids**
- $U(1)_{\mathsf{T}}$ hints at profound consequences for the structure of non-equilibrium QFT in general
  - Understand microscopic origin (KMS) and **consequences of** $U(1)_{\mathsf{T}}$

Further Details

Anomaly-induced transport (Class A)

- **Inflow mechanism:** anomalous constitutive relations can be derived from Class L extended to $d+1$ dimensions ($\partial \mathcal{M}_{d+1} = \mathcal{M}$):

$$S_{anom} = \int_{\mathcal{M}_{d+1}} \boldsymbol{V_{\mathcal{P}}}[\boldsymbol{A}, \boldsymbol{\Gamma}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{\Gamma}}]\,, \qquad \boldsymbol{V_{\mathcal{P}}} \equiv \boldsymbol{I}_{CS}[\boldsymbol{A}, \boldsymbol{\Gamma}] - \boldsymbol{I}_{CS}[\hat{\boldsymbol{A}}, \hat{\boldsymbol{\Gamma}}] - \mathsf{d}\boldsymbol{B}_d[\boldsymbol{A}, \boldsymbol{\Gamma}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{\Gamma}}]$$

$$\text{with} \quad \hat{\boldsymbol{A}} = \boldsymbol{A} + \mu \boldsymbol{u} \quad \text{and} \quad \hat{\boldsymbol{\Gamma}}^m{}_n = \boldsymbol{\Gamma}^m{}_n + \Omega^m{}_n \boldsymbol{u}$$

$$(T^{\alpha\beta})_{\mathsf{A}} = \frac{2}{\sqrt{-g}} \frac{\delta S_{anom}}{\delta g_{\alpha\beta}}\bigg|_{boundary} \qquad (J^\alpha)_{\mathsf{A}} = \frac{1}{\sqrt{-g}} \frac{\delta S_{anom}}{\delta A_\alpha}\bigg|_{boundary}$$

  - These currents solve adiabaticity equation with $(J_S^\alpha)_{\mathsf{A}} = -\frac{1}{2}\boldsymbol{\beta}_\sigma\, \hat{\Sigma}_H^{\perp[\alpha\sigma]}$
  - Currents make sense, but e.o.m. are *not* anomalous hydrodynamics

- To get correct anomalous dynamics, perform **Schwinger-Keldysh** doubling with suggestive influence functional:

$$S_{anom}^{\mathsf{SK}} = \int_{\mathcal{M}_{d+1}} \boldsymbol{V_{\mathcal{P}}}[\boldsymbol{A}_{\mathsf{R}}, \boldsymbol{\Gamma}_{\mathsf{R}}, \hat{\boldsymbol{A}}_{\mathsf{R}}, \hat{\boldsymbol{\Gamma}}_{\mathsf{R}}] - \boldsymbol{V_{\mathcal{P}}}[\boldsymbol{A}_{\mathsf{L}}, \boldsymbol{\Gamma}_{\mathsf{L}}, \hat{\boldsymbol{A}}_{\mathsf{L}}, \hat{\boldsymbol{\Gamma}}_{\mathsf{L}}] + \boldsymbol{V_{\mathcal{P}}}[\hat{\boldsymbol{A}}_{\mathsf{R}}, \hat{\boldsymbol{\Gamma}}_{\mathsf{R}}, \hat{\boldsymbol{A}}_{\mathsf{L}}, \hat{\boldsymbol{\Gamma}}_{\mathsf{L}}]$$

FH-Loganayagam-Rangamani '13

Dissipative transport (Class D)

- Most general dissipative constitutive relations are of the following form:

$$\begin{pmatrix} T^{\mu\nu} \\ J^{\alpha} \end{pmatrix}_{\mathsf{D}} = -\begin{pmatrix} \Upsilon^{\dagger}_{\boldsymbol{\eta}_g} & \Upsilon^{\dagger}_{\boldsymbol{\sigma}_g} \\ \Upsilon^{\dagger}_{\boldsymbol{\eta}_A} & \Upsilon^{\dagger}_{\boldsymbol{\sigma}_A} \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \Upsilon_{\boldsymbol{\eta}_g} & \Upsilon_{\boldsymbol{\eta}_A} \\ \Upsilon_{\boldsymbol{\sigma}_g} & \Upsilon_{\boldsymbol{\sigma}_A} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\delta_{\mathscr{B}}\, g \\ \delta_{\mathscr{B}}\, A \end{pmatrix}$$

  - Υ are derivative operators, $\{\boldsymbol{\eta}, \boldsymbol{\sigma}\}$ are "intertwining" tensor fields

- Corresponding entropy production:

$$\begin{aligned} \Delta = & \left[\frac{1}{2}\Upsilon_{\boldsymbol{\eta}_g}\delta_{\mathscr{B}}\, g + \Upsilon_{\boldsymbol{\eta}_A}\delta_{\mathscr{B}}\, A\right] \boldsymbol{\eta} \left[\frac{1}{2}\Upsilon_{\boldsymbol{\eta}_g}\delta_{\mathscr{B}}\, g + \Upsilon_{\boldsymbol{\eta}_A}\delta_{\mathscr{B}}\, A\right] \\ & + \left[\frac{1}{2}\Upsilon_{\boldsymbol{\sigma}_g}\delta_{\mathscr{B}}\, g + \Upsilon_{\boldsymbol{\sigma}_A}\delta_{\mathscr{B}}\, A\right] \boldsymbol{\sigma} \left[\frac{1}{2}\Upsilon_{\boldsymbol{\sigma}_g}\delta_{\mathscr{B}}\, g + \Upsilon_{\boldsymbol{\sigma}_A}\delta_{\mathscr{B}}\, A\right] \end{aligned}$$

  - This is positive definite (i.e. allowed by second law) if $\{\boldsymbol{\eta}, \boldsymbol{\sigma}\}$ have eigenvalues ≥ 0

Relation between Class B and Class D

- Consider the following constitutive relations:

$$(T^{\mu\nu})_{\mathsf{B,D}} \equiv -\frac{1}{2}\,\mathcal{N}^{(\alpha\beta)(\mu\nu)}\,\delta_{_\mathscr{B}}\,g_{\alpha\beta}$$
$$(J^{\alpha})_{\mathsf{B,D}} \equiv -\mathcal{S}^{\alpha\beta}\cdot\delta_{_\mathscr{B}}\,A_{\beta}$$
$$(\mathcal{G}^{\sigma})_{\mathsf{B,D}} = 0\,,$$

  - Recall: this parameterizes (part of) **Class B** for:

    $$\mathcal{N}^{(\alpha\beta)(\mu\nu)} = -\mathcal{N}^{(\mu\nu)(\alpha\beta)}\,, \qquad \mathcal{S}^{\alpha\beta} = -\mathcal{S}^{\beta\alpha}$$

  - The same gives (a subset of) **Class D** for:

    $$\mathcal{N}^{(\alpha\beta)(\mu\nu)} = \mathcal{N}^{(\mu\nu)(\alpha\beta)}\,, \qquad \mathcal{S}^{\alpha\beta} = \mathcal{S}^{\beta\alpha}$$ (& both positive definite)

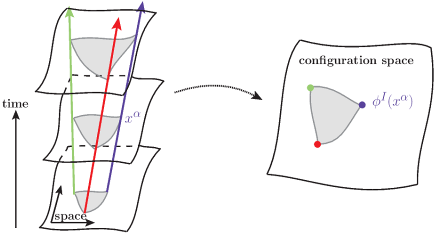
  - This can be seen from the corresponding entropy production being a quadratic form:

$$\Delta = \tfrac{1}{4}\,\mathcal{N}^{(\alpha\beta)(\mu\nu)}\,(\delta_{_\mathscr{B}}\,g_{\alpha\beta})\,(\delta_{_\mathscr{B}}\,g_{\mu\nu}) + \mathcal{S}^{\alpha\beta}\cdot(\delta_{_\mathscr{B}}\,A_{\alpha})\,(\delta_{_\mathscr{B}}\,A_{\beta})$$

Non-dissipative effective actions (Class ND)

- Fundamental fields (uncharged fluid):
  fluid element labels / Goldstone modes ϕ^I $\quad (I = 1, \ldots, d-1)$

Dubovsky-Hui-Nicolis-Son '11



- Reparametrization symmetry:

$$\phi^I \mapsto \xi^I(\phi)$$
$$\det(\partial \xi^I / \partial \phi^J) = 1$$

- Entropy current:

$$J_S^\sigma \propto \varepsilon^{\sigma\alpha_1\cdots\alpha_{d-1}} \varepsilon_{I_1\cdots I_{d-1}} \prod_{j=1}^{d-1} \partial_{\alpha_j} \phi^{I_j}$$
$$\equiv s\, u^\sigma$$

- The ϕ^I are nothing else than our pullback fields $\mathcal{M} \to \mathbb{M}$ after Legendre transformation to **entropic description** $(T \to T(s))$
- The symmetries (volume preserving diffeomorphisms) are obtained by fixing a particular gauge for the redundancy of the theory on $\mathbb{M}$

Euler current $d = 2n + 1$ dimensions (Class C)

- The exactly conserved Euler current in $d = 2n + 1$ dimensions:

$$\begin{aligned} \mathsf{J}^\sigma_{\text{Euler}} &\equiv -\frac{1}{2^n}\, c_{\text{Euler}}\; \varepsilon^{\sigma\alpha_1\alpha_2\ldots\alpha_{2n-1}\alpha_{2n}}\; u_\mu\; \varepsilon^{\mu\nu_1\nu_2\ldots\nu_{2n-1}\nu_{2n}} \\ &\qquad \times \left(\frac{1}{2} R_{\nu_1\nu_2\alpha_1\alpha_2} - \nabla_{\alpha_1} u_{\nu_1} \nabla_{\alpha_2} u_{\nu_2}\right) \\ &\qquad\quad \vdots \\ &\qquad \times \left(\frac{1}{2} R_{\nu_{2n-1}\nu_{2n}\alpha_{2n-1}\alpha_{2n}} - \nabla_{\alpha_{2n-1}} u_{\nu_{2n-1}} \nabla_{\alpha_{2n}} u_{\nu_{2n}}\right) \\ \nabla_\sigma \mathsf{J}^\sigma_{\text{Euler}} &= 0 \end{aligned}$$

  - E.g. in $d = 3$ with $\mathcal{M}_3 = \mathbb{R} \times \Sigma_{\mathcal{M}}$: measures topology of $\Sigma_{\mathcal{M}}$ via

$$\int_{\Sigma_{\mathcal{M}}} \sqrt{-\gamma}\, \left(\mathsf{J}^\sigma_{\text{Euler}}\, u_\sigma\right) \;\propto\; \chi(\Sigma_{\mathcal{M}})$$