Collective coordinate approximation to the scattering of solitons in modified (1+1) NLS and sine-Gordon models.

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November 19, 2014

Outline

- Quasi-integrability
- Collective coordinate approximation
- NLS model
- Modified NLS
- Modified sine-Gordon

Integrable systems

EoM can be rewritten in terms of a zero curvature condition:

$$\partial_t A_x - \partial_x A_t + [A_x, A_t] = 0$$

Abelianization procedure...

$$\frac{dQ^{(n)}}{dt} = 0$$

IST $\rightarrow$ exact soliton solutions

Quasi-integrable systems

EoM can be rewritten in terms of an anomalous zero curvature condition:

$$\partial_t A_x - \partial_x A_t + [A_x, A_t] = X T_3^0$$

Abelianization procedure...

$$\partial_t a_x^{(3,-n)} - \partial_x a_t^{(3,-n)} = X \alpha^{(3,-n)} \qquad n = 0, 1, 2, \ldots$$

$$\Rightarrow \frac{dQ^{(n)}}{dt} = \beta_n; \qquad \text{with} \qquad Q^{(n)} = \int_{-\infty}^{\infty} dx\, a_x^{(3,-n)}$$

$$\text{where} \qquad \beta_n = \int_{-\infty}^{\infty} dx\, X\, \alpha^{(3,-n)}.$$

Quasi-integrable systems

When fields are eigenstates of the space-time parity

$$P: \qquad (\tilde{x}, \tilde{t}) \rightarrow (-\tilde{x}, -\tilde{t}) \qquad\qquad \tilde{x} = x - x_{\Delta} \;\; \textit{and} \;\; \tilde{t} = t - t_{\Delta},$$

then the anomaly integrates over time to zero and the charges are asymptotically conserved

$$Q^{(n)}(t \rightarrow \infty) - Q^{(n)}(t \rightarrow -\infty) = \int_{-\infty}^{\infty} dt\, \beta_n = 0$$

Collective coordinate approximation

Start with static solution $\psi(x, q_1, ..., q_n)$ and allow the parameters to vary with time $q_i = q_i(t)$.

- Reduces the infinite-dimensional problem to a coupled set of ODEs **but**
- Requires a sensible choice of coordinates
- Does not account for deformations or energy radiation

NLS

(1+1) dim NLS eq:

$$i\partial_t \psi = -\partial_x^2 \psi + \frac{\delta V}{\delta |\psi|^2}\psi$$

$$V_{\text{NLS}} = \eta \, |\psi|^4$$

1-soliton solution ($\eta = -1$):

$$\psi = \frac{b}{\cosh\left[b\left(x - vt - x_0\right)\right]} e^{i\left[\left(b^2 - \frac{v^2}{4}\right)t + \frac{v}{2}x + \delta\right]}$$

$$\psi = \psi_1 + \psi_2 = \varphi_1 e^{i\theta_1} + \varphi_2 e^{i\theta_2}$$

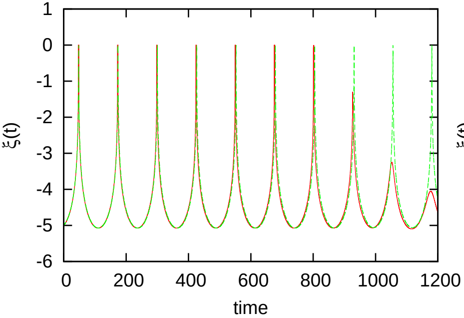
$$\begin{aligned} \varphi_1 &= \frac{a}{\cosh\left[b\,(x+\xi)\right]}, \quad \theta_1 = -\mu\,(x+\xi) + b^2\,t + \lambda + \delta_1, \\ \varphi_2 &= \frac{a}{\cosh\left[b\,(x-\xi)\right]}, \quad \theta_2 = \mu\,(x-\xi) + b^2\,t + \lambda + \delta_2, \end{aligned}$$

Take our collective coordinates:
$a(t) \sim$ height, $\xi(t) \sim$ position, $\mu(t) \sim$ velocity, $\lambda(t) \sim$ phase

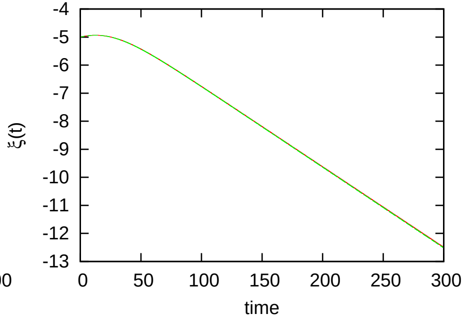
Initial phase difference:

$$\boxed{\delta \equiv \delta_1 - \delta_2}$$

F. Zou and J. Yan, *Chin. Phys. Lett.* **11**, 265 (1994)

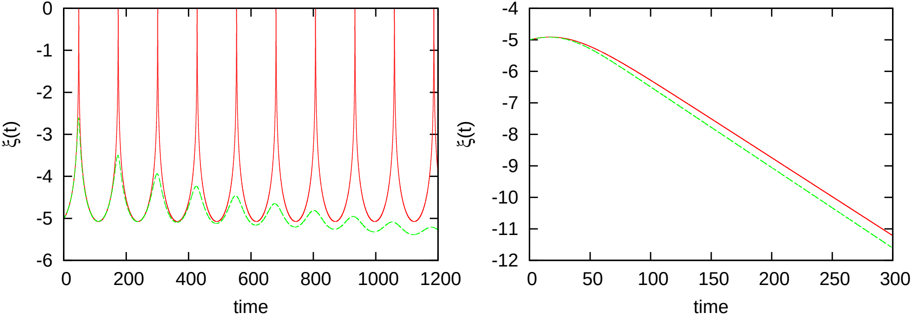


$\delta = 0$



$\delta = \pi$

red: full simulation, green: approximation



$\delta = \frac{\pi}{32}$ $\qquad$ $\delta = \frac{3\pi}{4}$

red: full simulation, green: approximation

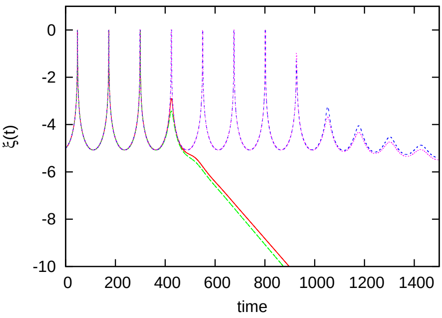
$$\psi = \psi_1 + \psi_2 = \varphi_1 e^{i\theta_1} + \varphi_2 e^{i\theta_2}$$

$$\begin{aligned} \varphi_1 &= \frac{a_1}{\cosh\left[a_1\left(x+\xi_1\right)\right]}, \quad \theta_1 = -\mu_1\left(x + \frac{\xi_1}{2}\right) + a_1^2\, t + \lambda_1 \\ \varphi_2 &= \frac{a_2}{\cosh\left[a_2\left(x+\xi_2\right)\right]}, \quad \theta_2 = -\mu_2\left(x + \frac{\xi_2}{2}\right) + a_2^2\, t + \lambda_2 \end{aligned}$$

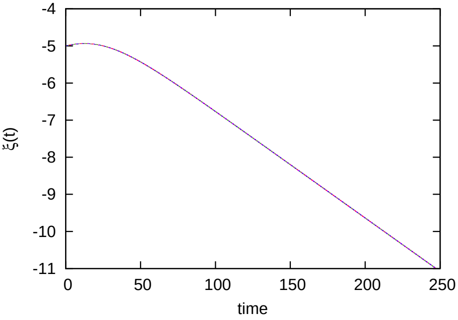
New collective coordinates:
$a_{1,2}(t) \sim$ heights, $\xi_{1,2}(t) \sim$ positions, $\mu_{1,2}(t) \sim$ velocities,
$\lambda_{1,2}(t) \sim$ phases

Phase difference:

$$\boxed{\delta \equiv \lambda_1 - \lambda_2}$$

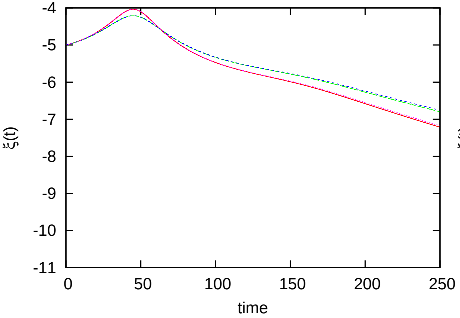


$\delta = 0$

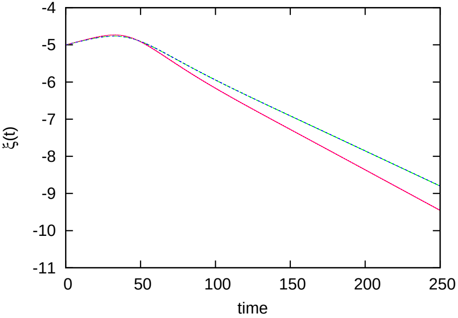


$\delta = \pi$

red/green: LH/RH approximation, pink/blue: LH/RH full simulation



$\delta = \frac{\pi}{4}$



$\delta = \frac{\pi}{2}$

red/green: LH/RH approximation, pink/blue: LH/RH full simulation

Modified NLS

(1+1) dim NLS eq:

$$i\partial_t \psi = -\partial_x^2 \psi + \frac{\delta V}{\delta |\psi|^2} \psi$$
$$V = \frac{2}{2+\epsilon} \eta \, \left(|\psi|^2 \right)^{2+\epsilon}$$

1-soliton solution ($\eta < 0$):

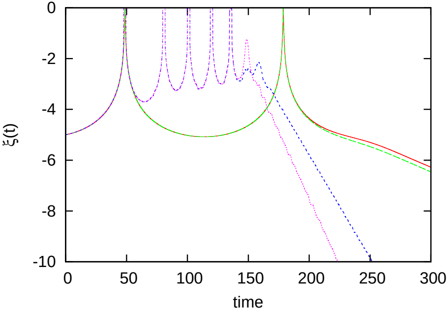
$$\psi = \left(\sqrt{\frac{2+\epsilon}{2\,|\eta|}} \frac{b}{\cosh \left[(1+\epsilon)\, b\, (x - vt - x_0) \right]} \right)^{\frac{1}{1+\epsilon}} e^{i \left[\left(b^2 - \frac{v^2}{4} \right) t + \frac{v}{2} x \right]}$$

Modified NLS

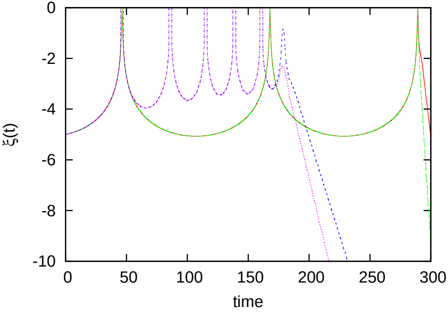
Collective coordinate approximation

$$\psi = \psi_1 + \psi_2 = \varphi_1 e^{i\theta_1} + \varphi_2 e^{i\theta_2}$$

$$\begin{aligned}
\varphi_1 &= \left(\sqrt{\frac{2+\epsilon}{2}}\frac{a_1}{\cosh\left[(1+\epsilon)\, a_1\,(x+\xi_1)\right]}\right)^{\frac{1}{1+\epsilon}} \\
\varphi_2 &= \left(\sqrt{\frac{2+\epsilon}{2}}\frac{a_2}{\cosh\left[(1+\epsilon)\, a_2\,(x+\xi_2)\right]}\right)^{\frac{1}{1+\epsilon}}
\end{aligned}$$

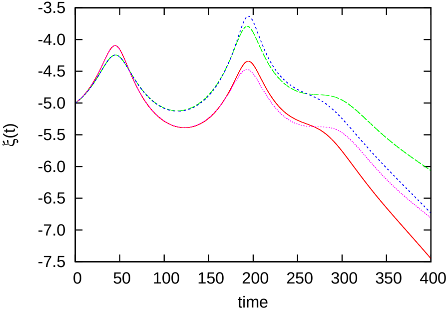


$\epsilon = 0.06$

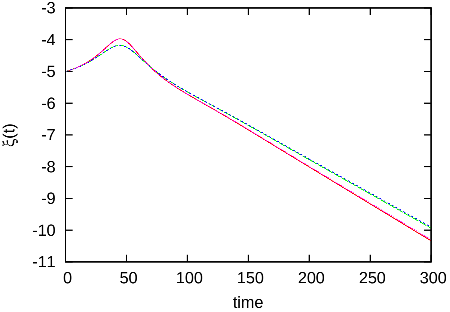


$\epsilon = -0.06$

red/green: LH/RH approximation, pink/blue: LH/RH full simulation

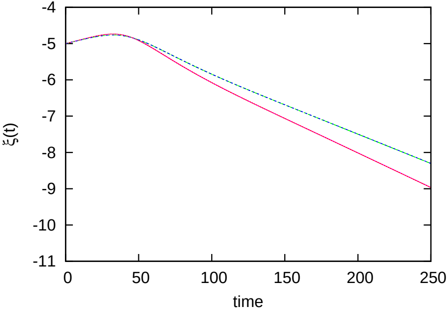


$\epsilon = 0.06$

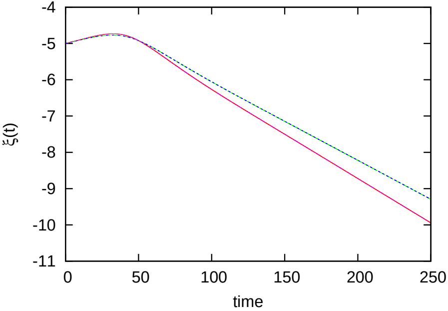


$\epsilon = -0.06$

red/green: LH/RH approximation, pink/blue: LH/RH full simulation

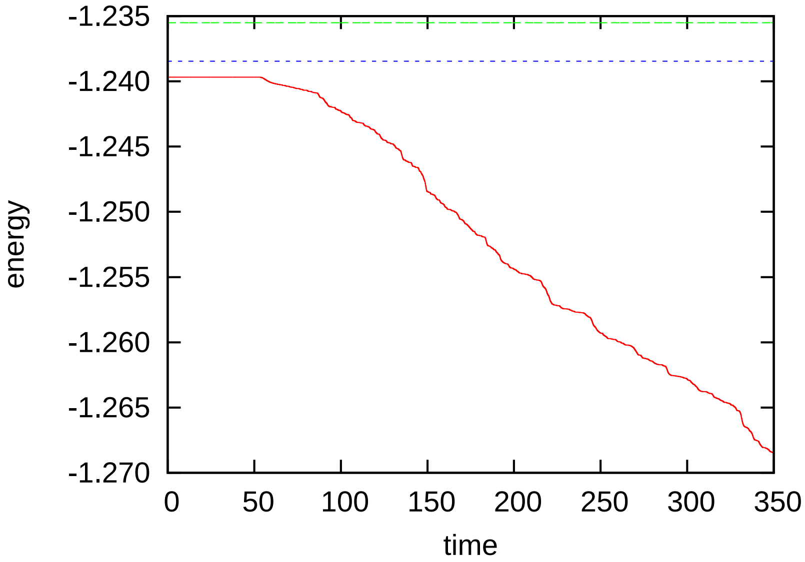


$\epsilon = 0.06$



$\epsilon = -0.06$

red/green: LH/RH approximation, pink/blue: LH/RH full simulation



red: $\delta = 0$, blue: $\delta = \frac{\pi}{4}$, green: $\delta = \frac{\pi}{2}$

Anomalous curvature condition:

$$\partial_t A_x - \partial_x A_t + [A_x, A_t] = X T_3^0$$

$$X \equiv -i\,\partial_x \left(\frac{\delta V}{\delta |\psi|^2} - 2\,\eta\,|\psi|^2 \right)$$

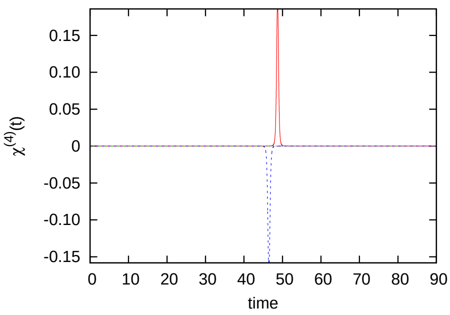
Anomalies:

$$\frac{dQ^{(n)}}{dt} = \beta_n(t)$$

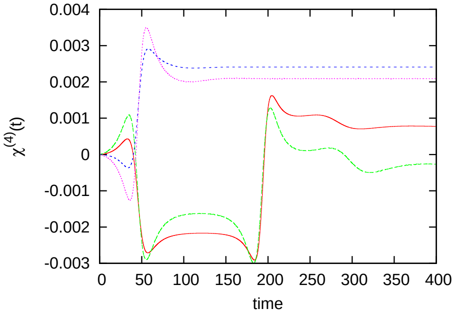
Time-integrated anomaly:

$$\chi^{(4)}(t) \equiv \int_{-\infty}^{t} dt'\,\beta_4$$

If $\quad P(\psi) = (-1)^n \psi^* \quad \Rightarrow \quad Q^{(n)}(t = +\infty) = Q^{(n)}(t = -\infty)$

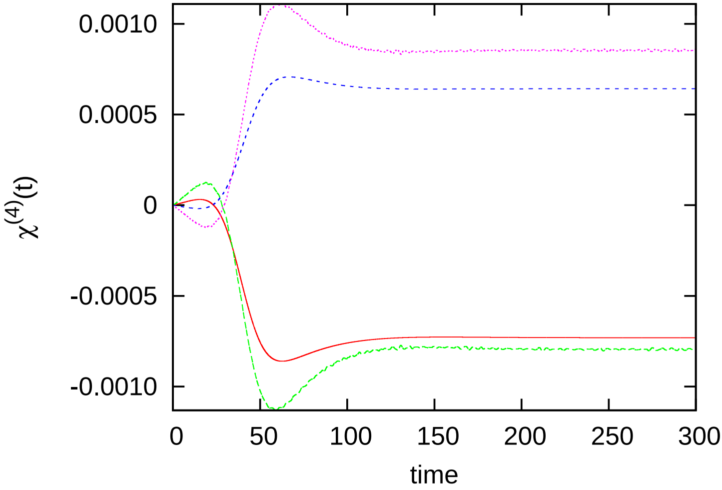


$\delta = 0$



$\delta = \frac{\pi}{4}$

red/blue: $\epsilon = +/-$ 0.06 approximation, green/pink: $\epsilon = +/-$ 0.06 full simulation



$\delta = \frac{\pi}{2}$

red/blue: $\epsilon = +/-$ 0.06 approximation, green/pink: $\epsilon = +/-$ 0.06 full simulation

Sine-Gordon

(1+1) dim sine-Gordon eq:

$$L = \int dx \, \frac{1}{2} \left((\partial_t \psi)^2 - (\partial_x \psi)^2 \right) - V(\psi)$$

$$V_{SG} = \frac{1}{8} \sin^2(2\psi)$$

1-soliton solution:

$$\psi = \mathrm{ArcTan} \left(e^{\pm(x - x_0)} \right)$$

Modified sine-Gordon

Take a change of variable $\psi \to \phi$

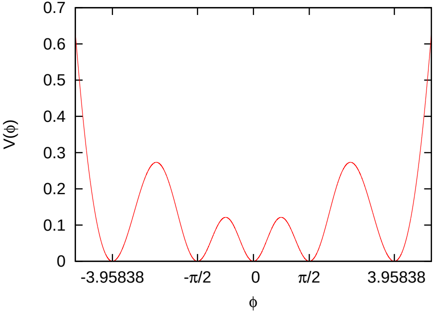
$$\psi(\phi) = \frac{c\phi}{\sqrt{1 + \epsilon\phi(\phi - 2\gamma)}}$$

where c is chosen to be

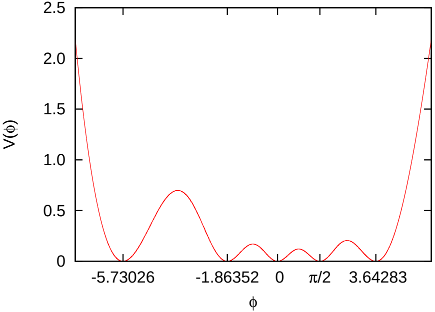
$$c = \sqrt{1 + \epsilon\pi\left(\frac{\pi}{4} - \gamma\right)}$$

then this solves the sine-Gordon equation with potential:

$$V(\phi) = \left(\frac{d\phi}{d\psi}\right)^2 V_{SG} = \frac{1}{8}\frac{\left(1 + \epsilon\phi\left(\phi - 2\gamma\right)\right)^3}{c^2\left(1 - \epsilon\gamma\phi\right)^2}\sin^2\left(2\psi(\phi)\right).$$



$\epsilon = 0.05, \gamma = 0$

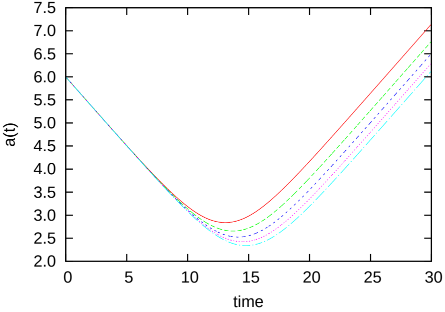


$\epsilon = 0.05, \gamma = 1$

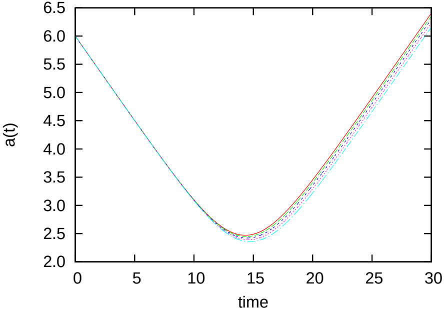
$$\psi = \psi_1 + \psi_2 = \mathrm{ArcTan}\left(2\,\mathrm{Sinh}(x)\,e^{-a(t)}\right)$$

then take a change of variable $\psi \to \phi$

$$\begin{aligned}
\phi &= \frac{\psi^2\epsilon\gamma + \sqrt{\psi^2 c^2 + \psi^4\epsilon\left(-1+\gamma^2\epsilon\right)}}{\psi^2\epsilon - c^2} \qquad \text{for} \quad x < 0 \\
\phi &= \frac{\psi^2\epsilon\gamma - \sqrt{\psi^2 c^2 + \psi^4\epsilon\left(-1+\gamma^2\epsilon\right)}}{\psi^2\epsilon - c^2} \qquad \text{for} \quad x > 0
\end{aligned}$$



$\epsilon = -0.2, -0.1, 0, 0.1, 0.2$ and $\gamma = 0$



$\gamma = -0.4, -0.2, 0, 0.2, 0.4$ and $\epsilon = 0.1$

Anomalous curvature condition:

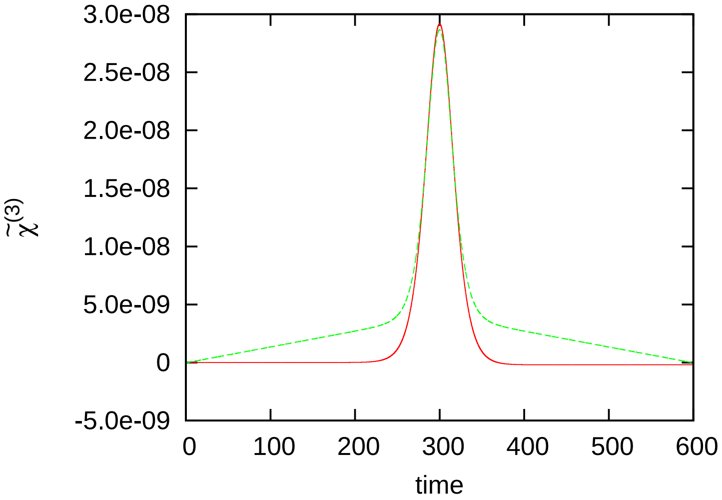
$$\partial_t A_x - \partial_x A_t + [A_x, A_t] = X T_3^0$$

$$X \equiv \frac{iw}{2} \partial_- \phi \left[\frac{d^2 V}{d\phi^2} + w^2 V - m \right]$$

(X vanishes for sine-Gordon for $m = 1, w = 4$)
Time-integrated anomaly:

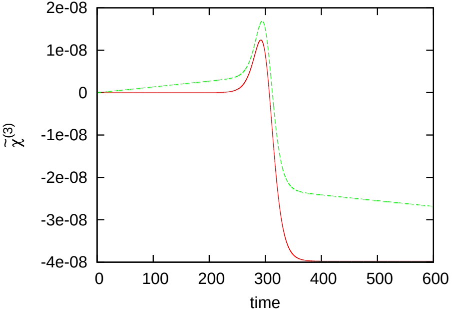
$$\chi^{(3)} = -\frac{1}{2} \int_{t_0}^{t} dt' \, \beta^{(3)}$$

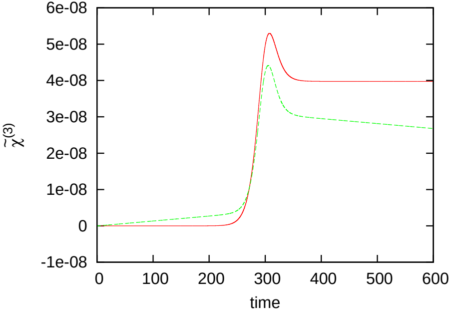
If $\quad P(\phi) = -\phi + const \quad$ and $\quad P(V) = V$



$\epsilon = 0.000001, \gamma = 0.00001$

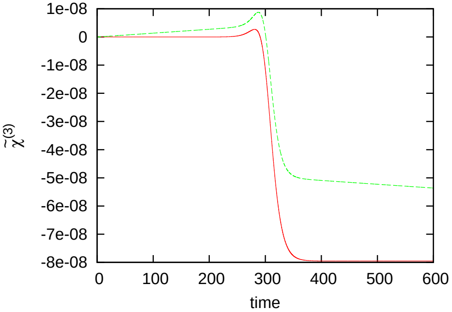
red: approximation, green: full simulation

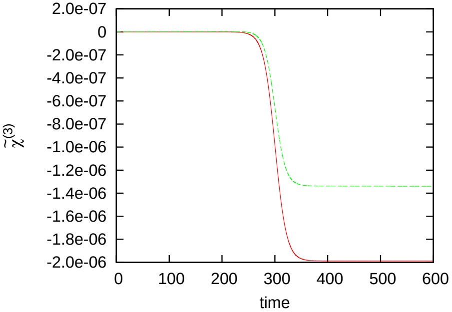




$\epsilon = 0.000001, \gamma = 0.002$ $\qquad$ $\epsilon = 0.000001, \gamma = -0.002$

red: approximation, green: full simulation





$\epsilon = 0.000001, \gamma = 0.004$

$\epsilon = 0.000001, \gamma = 0.1$

red: approximation, green: full simulation

Conclusion

- Collective coordinate approximation generally accurate in all considered systems
- In our modified NLS approximation was less reliable the more time the solitons spent in close proximity
- In our modified sine-Gordon the values for the time-integrated anomaly are better when the fields possess suitable parity symmetry