Toy models for Holographic Baryons

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Outline

- Introduce Topological Solitons
- Holographic QCD
- Baby Skyrmions and their variants

Topological Solitons

- For the purposes of this talk, a topological soliton can be defined as a stable, particle-like solution to a non-linear field theory.
- Stability is due to the presence of some topological charge, $B \in \mathbb{Z}$. This is usually a generalised winding number.
- Examples include sine-Gordon kinks, sigma-model lumps, monopoles, vortices, Skyrmions, Yang-Mills instantons...

Example: The Sine-Gordon Model

- Consider the Lagrangian in two spacetime dimensions:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - (1 - \cos\phi(x))$$

- Potential has an infinite number of vacua when $\phi = 2\pi n$, $n \in \mathbb{N}$.
- This leads to a topological charge

$$N = \frac{\phi_+ - \phi_-}{2\pi}$$

where $\phi(x) \to \phi_\pm$ as $x \to \pm\infty$.

Energy Bounds for Topological Solitons

- We can put a lower bound on the static energies of soliton solutions using a Bogomolny argument.
- The static energy in the sine-Gordon model is

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}\phi'^2 + (1 - \cos\phi) \right) dx$$

- By completing the square we can write

$$E = \int_{-\infty}^{\infty} \left\{ \left(\frac{1}{\sqrt{2}}\phi' \pm \sqrt{(1 - \cos\phi)} \right)^2 \mp \sqrt{2(1 - \cos\phi)}\phi' \right\} dx \geq 8|N|$$

- Saturation of the bound implies a first order equation of motion. In this example we can solve it (for $N = 1$) to find analytic solutions, but this is not always possible.

Scaling Arguments

- Derrick's Theorem provides a non-existence theorem for solitons.
- If a soliton is to be a stationary point of some energy functional, then its energy must be stationary with respect to spatial rescalings.
- As an example, we can write the sine-Gordon static energy as

$$E = \int_{-\infty}^{\infty} \phi'^2 \, dx + \int_{-\infty}^{\infty} (1 - \cos\phi) \, dx = E_2 + E_0$$

- Under a spatial rescaling $x \to \lambda x$ then

$$E \to E(\lambda) = \frac{1}{\lambda} E_2 + \lambda E_0$$

- Minimising this over λ gives a scale for the size of the soliton.

The $O(2) - \sigma$ Model

- Consider a three-component unit vector ϕ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi + \frac{\lambda}{2}(1 - \phi \cdot \phi)$$

- For a finite static energy we require $\phi \to \phi^\infty$ as $|\boldsymbol{x}| \to \infty$. W.L.O.G we can take $\phi^\infty = (0, 0, 1)$.
- In two spatial dimensions, this compactifies $\mathbb{R}^2$ to S^2, and we can write $\phi : S^2 \to S^2$.
- The map has an associated winding number

$$B = -\frac{1}{4\pi} \int \phi \cdot (\partial_x \phi \times \partial_z \phi)\, dx\, dz,$$

which we identify with the topological charge.

- The static energy is invariant under spatial rescalings.

Yang-Mills Instantons

- In four spatial dimensions, consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

- The "solitons" of this system are called Yang-Mills instantons, and they are also scale-invariant.
- YM instantons are highly symmetric and there is a wealth of mathematics concerning them!

Skyrmions

- Skyrmions are solitons of the Skyrme model, in three spatial dimensions
- It is a modified σ-model with target space $SU(2) = S^3$

$$\mathcal{L} = \frac{1}{2} Tr(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{16} Tr([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger])$$

- Obtained as a low-energy effective field theory for QCD in the large colour limit
- The topological charge is identified with baryon number
- Skyrmions possess interesting symmetries, although their binding energies are too large compared with experimental data

Holographic QCD: The Sakai-Sugimoto Model

- The Sakai-Sugimoto is the leading example of an AdS/QCD theory
- Formulated in a (4+1) dimensional spacetime with a warped metric of the form

$$ds^2 = H(z)dx_\mu dx^\mu + \frac{1}{H(z)}dz^2$$

where $H(z) = \left(1 + z^2\right)^p$

- The Sakai-Sugimoto takes $p = \frac{2}{3}$ and we say the spacetime is AdS-like

Holographic QCD: The Sakai-Sugimoto Model

- The Lagrangian, in suitable units, is:

$$\mathcal{L} = -\frac{1}{2} Tr(F_{\Gamma\Delta} F^{\Gamma\Delta}) + \frac{9\pi}{\lambda} \omega_5(A_\Gamma)$$

  where A_Γ and $F_{\Gamma\Delta}$ are some $U(2)$ gauge field and field strength respectively. λ is the 't Hooft coupling.

- The second term is a Chern-Simons term, which takes the form of a coupling between the gauge field and a topological term
- Topological solitons in the bulk correspond to Skyrmions on the boundary (coupled to a tower of vector mesons)
- This model is analytically and numerically very complicated

A Lower-Dimensional Analogue

- Studying a low-dimensional analogue of the SS model makes computations and numerical simulations more feasible
- We can use these toy models to study bulk solitons and test predictions made in the full SS model
- A natural analogue for the Yang-Mills term is an $O(2) - \sigma$ term, since both are scale invariant
- We still have a choice to make regarding the analogue of the Chern-Simons term

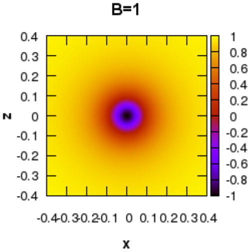
Stabilization via the Baby Skyrme Term

- One option is to represent the Chern-Simons term by a baby-Skyrme term

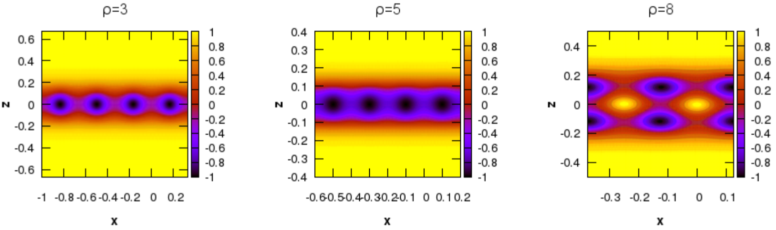
$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \cdot \partial_\nu \phi + \frac{\kappa^2}{4} g^{\mu\nu} g^{\alpha\beta} (\partial_\mu \phi \times \partial_\alpha \phi) \cdot (\partial_\nu \phi \times \partial_\beta \phi)$$

- For small κ (the analogue of large 't Hooft coupling) the solitons of this model are small compared to the curvature of spacetime
- We can approximate these solitons by flat-space σ-model lumps
- The curvature of spacetime and the baby-Skyrme interaction pick out a preferred size

- Here are some numerical static solutions to the baby-Skyrme model in this curved space, with parameter value $\kappa = 0.01$



- We can also use this model to study dense QCD (its low-dimensional analogue)
- At some critical density it has been shown that the baryon chains undergo a phase transition to produce “baryonic popcorn”. This is more energetically favourable than the “dyon salt”.



Stabilization via a Vector Meson Term

- Another (arguably better) candidate for our low-dimensional model is given by

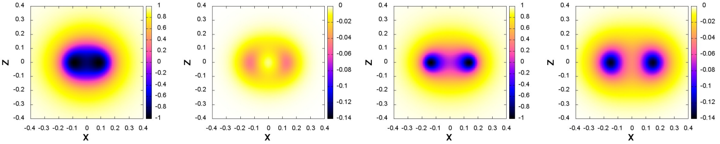
$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)(\partial_\alpha\omega_\beta - \partial_\beta\omega_\alpha) + \frac{1}{2}g^{\mu\nu}M^2\omega_\mu\omega_\nu + g\omega_\mu B^\mu$$

- $B^\mu = -\frac{1}{8\pi\sqrt{H}}\varepsilon^{\mu\alpha\beta}\phi\cdot(\partial_\alpha\phi\times\partial_\beta\phi)$ is the conserved topological current of the system

- It can be shown that the Vector Meson model tends to the Baby Skyrme model in the limit

$$g, M \to \infty\,,\ \frac{g}{M} \propto \kappa\,\text{constant}$$

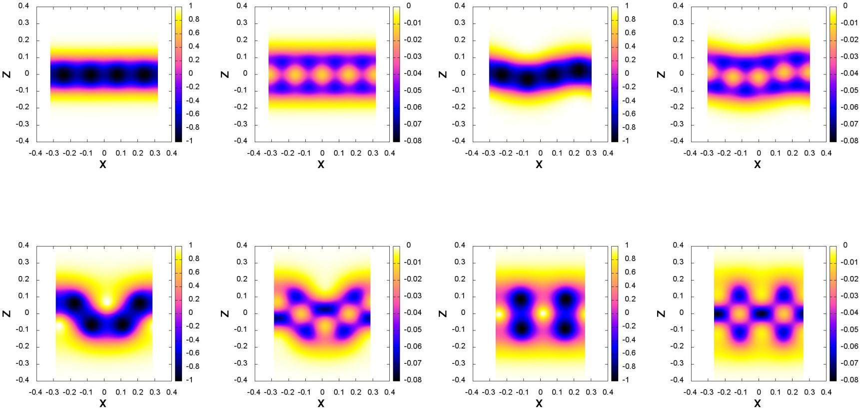
- For large g and M, it can be numerically confirmed that solitons in both model are similar.
- As we shrink g and M, keeping their ratio fixed, different qualitative behaviour can be observed.



Finite Density Chains

- In the full Sakai-Sugimoto model it is predicted that finite density baryon chains form a “zig-zag” pattern.
- This requires that the optimal separation between two solitons is greater than the size of a single soliton
- Can we find a parameter regime in the Vector Meson model to reproduce this behaviour?

Finite Density Chains

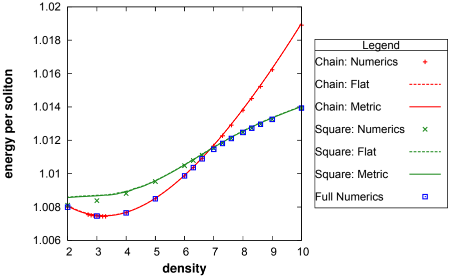


Finite Density Chains

- We find a two-stage popcorn transition in the VM model
- Differs from previous study of the BS model in which the popcorn transition was first-order
- In fact, going back to BS model reveals a two-stage transition, but may have been overlooked due to small energy differences

Approximations

- SS solitons are often approximated by flat-space self-dual Yang-Mills instantons
- Some SS model studies also approximate the warped metric by expanding it to leading order
- Our toy model for holographic baryons can be used to test these approximations in 2-d



Summary

- Topological solitons have been introduced and many examples have been mentioned
- Introduced the Sakai-Sugimoto model and looked at two lower-dimensional analogues
- Finite density chains of solitons have been investigated in both models
- In both models there seems to be a popcorn transition over some range of densities
- Common approximations used in studies of holographic QCD have been shown to be good approximations in our toy model
- Numerical results for parameter ranges in which both models differ have not yet been obtained

Thank you for listening.