The Skyrme-Faddeev model

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Student Seminar

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The Skyrme-Faddeev model
(with a brief introduction to topological solitons)

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Outline

- What are topological solitons?
- Kinks
- σ-model lumps
- Skyrmions
- Hopfions

Topological solitons

- Topological solitons are stable, particle-like solutions to a field theory, where they differ topologically from the vacuum.
- The topological character is often captured by an integer (usually topological degree or generalised winding number) called the topological charge.
- Smooth deformations of the field does not change the topology and so solutions of non-trivial topological charge are stable.
- Energy density is smooth and concentrated in some finite region of space.

Topological solitons

- If there are static stable solitons in the theory - it must satisfy Derricks theorem.

Derrick's Theorem [*J. Math. Phys.* **5, 1252 (1964)**]

Consider a time independent field theory with a finite energy non-vacuum field configuration. Let $e(\mu)$ be the energy under spatial rescaling $\mathbf{x} \mapsto \mu\mathbf{x}$. Then if $e(\mu)$ has no stationary point, the theory has no static solutions with finite energy other than the vacuum.

- In many cases we can find a bound on the energy in terms of the topological charge (a Bogomolny bound).
- Examples are kinks, lumps, baby Skyrmions, Skyrmions, monopoles, instantons.

A basic example: The kink

- A one-dimensional theory of a real scalar field ϕ defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi),$$

for real non-negative function $U(\phi)$. Note for finite energy we need these to tend to a vacuum at spatial infinity.

- So long as there are multiple isolated vacua of the potential, solutions which go from one vacua to another are called kinks, and are topologically distinct from the vacuum solution.
- Examples of potential are the ϕ^4 kink, $U(\phi) = \lambda(m^2 - \phi^2)^2$ and the sine-Gordon kink $U(\phi) = 1 - \cos\phi$.

A basic example: The kink

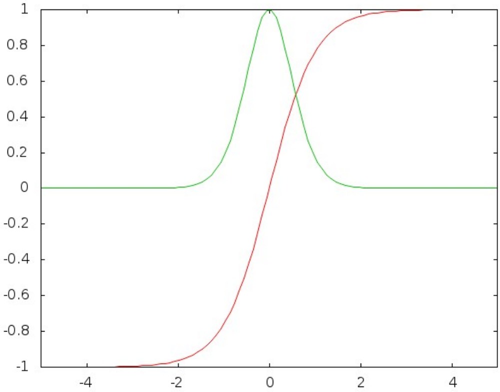
- Since the energy will be be comprised of two terms $E = E_2 + E_0$, and we are in one spatial dimension these will scale in opposite ways. Thus Derrick's theorem will not forbid static solutions.
- For the ϕ^4 kink we find associated topological charge

$$N = \frac{\phi_+ - \phi_-}{2m},$$

where $\phi_\pm = \lim_{x \to \pm\infty} \phi(x)$. Then $N \in \{-1, 0, 1\}$.

A basic example: The kink

- This case is analytically solvable, with a kink given by the field $\phi = m \tanh\left(\sqrt{2\lambda} m(x-a)\right)$, where a is the location of the kink.



The σ-model lump

- We now move to two dimensions!
- We upgrade ϕ from before to a three-component scalar field $\boldsymbol{\phi} \in S^2$, and have Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} - m^2 V(\phi)$$

- For the energy to be finite again the field must tend to a vacuum value. This compactifies the $\mathbb{R}^2$ to S^2, and means that we have a topological charge which is a winding number.
- Looking at Derrick's theorem though, in this extra spatial dimension we see that this does not have stable solutions since we have no non-trivial solution. So these σ-model lumps are not solitons and can suffer from scale instabilities.

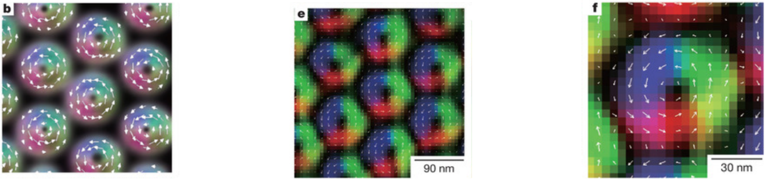
- To solve this instability we add another term to the Lagrangian.

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \cdot \partial_\mu \phi - \frac{\kappa}{4}(\partial_\mu \phi \times \partial_\nu \phi) \cdot (\partial^\mu \phi \times \partial^\nu \phi) - m^2 V(\phi)$$

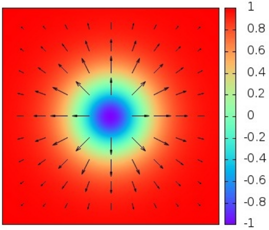
  where this term is the unique as the lowest order Lorentz invariant with field equations involving time derivatives of no more than second order. However becomes highly non-linear.

- Has applications as an approximation in condensed matter theories

[Yu, Onose et al. Nature 465, 901 (2010)]



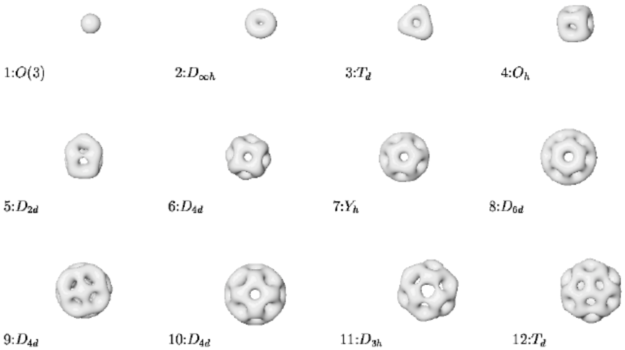
- Numerical solutions are well known, with solutions looking like localised lumps of energy.
- With the standard analogue of the pion mass term, $V(\phi) = 1 - \phi_3$, we find charge one solution.



The Skyrme model

- Can be easily extended into three dimensions by letting $\phi \in S^3$ be a four-component unit vector. Again is a winding number as another one-point compactification occurs.
- This is a theory of pions, with the solitons (called Skyrmions) of the theory representing baryons. It can be regarded as a low-energy effective theory of QCD in the large-colour limit.

- Solutions to this are well-known, where platonic symmetries are used to generate solutions.



[Battye, Sutcliffe; *Rev. Math. Phys.* **14**, 29 (2002)]

The Skyrme–Faddeev model

- Three-dimensional theory with links to QCD and condensed matter physics. [Faddeev; Princeton preprint IAS-75-QS70 (1975)]
- It is defined by the static energy functional

$$E = \frac{1}{32\pi^2\sqrt{2}} \int \partial_i\phi \cdot \partial_i\phi + \frac{1}{2}(\partial_i\phi \times \partial_j\phi) \cdot (\partial_i\phi \times \partial_j\phi)\, d^3x,$$

where ϕ is a three-component unit vector.

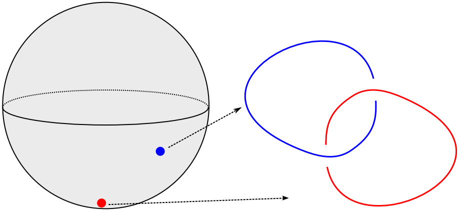
- Finite energy considerations lead to $\phi(\infty) = (0, 0, 1)$, so now $\phi : S^3 \to S^2$. Center of soliton taken to be antipodal point.
- Derrick's scaling theorem allows static solutions with a non-zero size.

- We have topological charge, the Hopf charge, given by

$$Q = \frac{1}{4\pi^2} \int_{S^3} F \wedge A,$$

where $F = dA$ is the pull-back of the area two-form on the target S^2.

- Alternatively, we can interpret this more geometrically in terms of the linking number of the preimages of two distinct points.



- Initial conditions can be generated via rational maps.

[Sutcliffe; *Proc. R. Soc. Lond.* **A463**, 3001 (2007)]

- We map $(x_1, x_2, x_3) \in \mathbb{R}^3$ to the unit three-sphere $S^3 \subset \mathbb{C}^2$ via the map

$$(Z_1, Z_0) = \left((x_1 + ix_2)\frac{\sin f}{r}, \cos f + i\frac{\sin f}{r}x_3 \right),$$

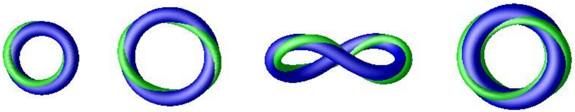
where $r^2 = x_1^2 + x_2^2 + x_3^2$ and $f(r)$ is monotonically decreasing function satisfying $f(0) = \pi$, $f(\infty) = 0$.

- The Riemann sphere coordinate, W, of the field are given by rational map

$$W = \frac{\phi_1 + i\phi_2}{1 + \phi_3} = \frac{p(Z_1, Z_0)}{q(Z_1, Z_0)}.$$

The Skyrme–Faddeev model

- We see that $W = Z_1^n / Z_0^m$ generates axially symmetric fields, denoted $\mathcal{A}_{n,m}$, with topological charge $Q = mn$. This rational map generates the static energy configurations for charge one to four.

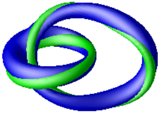


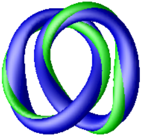
- Note that Q3 solution buckles.

- We also have the possibility of linked solutions. Fields linked once, denoted $\mathcal{L}^{1,1}_{n,n}$ (where subscript denotes constituent charges and superscript denotes linking number) are generated via

$$W = \frac{Z_1^{n+1}}{Z_1^2 - Z_0^2} = \frac{Z_1^n}{2(Z_1 - Z_0)} + \frac{Z_1^n}{2(Z_1 + Z_0)}.$$

- Solutions linked once gain charge two via linking. In general the charge is given by the sum of subscripts and superscripts. Energy minimum for charges five and six.

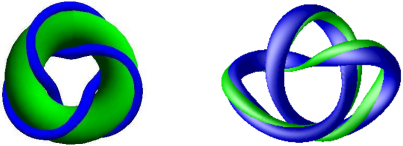




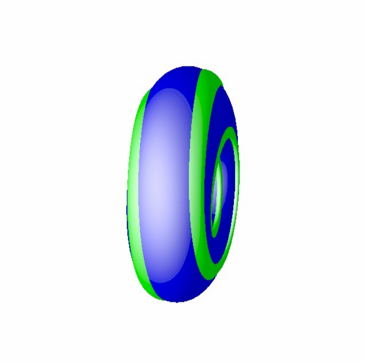
- Also can have solutions which are torus knots (i.e. the knot can be drawn on the surface of a torus) the first example of which is the trefoil knot
- An (a, b) torus knot is generated by rational map

$$W = \frac{Z_1^{\alpha} Z_0^{\beta}}{Z_1^a + Z_0^b},$$

where α positive integer and β non-negative which has charge $Q = \alpha b + \beta a$. First appears at charge seven.

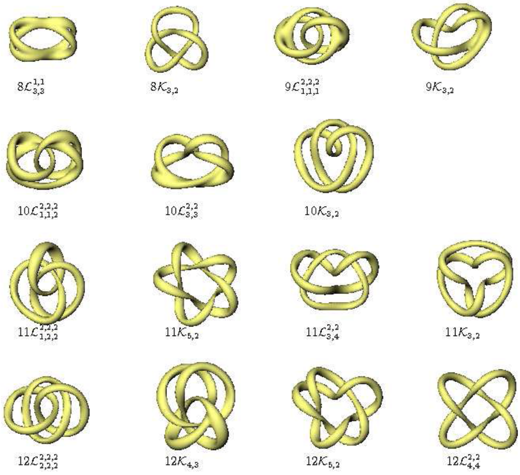


- We then take our initial ansatz and follow an energy minimisation algorithm to relax to a (quasi-) stable energy minimum, which give static solutions.
- For example, an initial field $7\mathcal{A}_{17}$ relaxes to $7\mathcal{K}_{32}$:



- Solutions have been found for charges up to charge sixteen.

[Sutcliffe; *Proc. R. Soc. Lond.* **A463**, 3001 (2007)]



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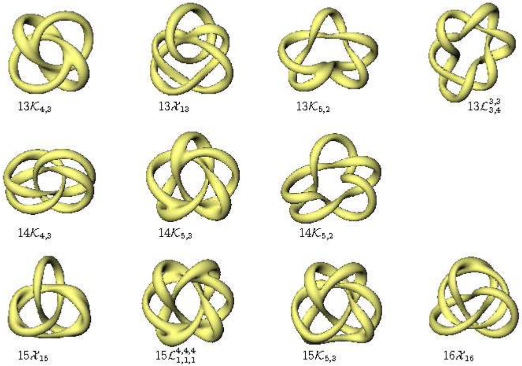
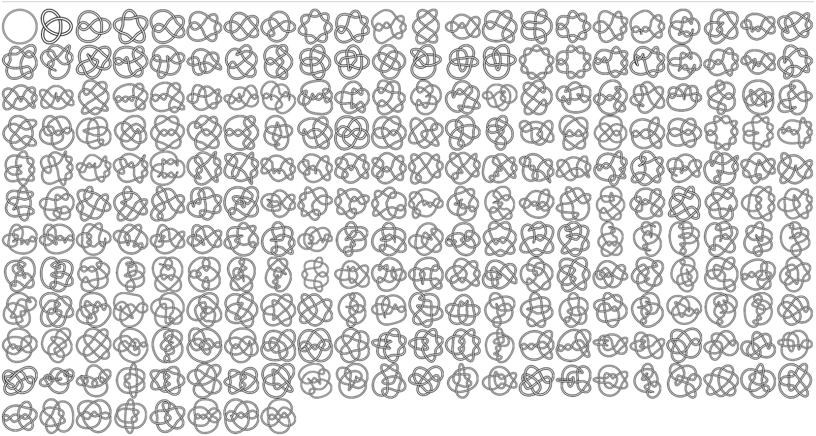


Non-torus knots

250 prime knots with minimal crossing number up to 10



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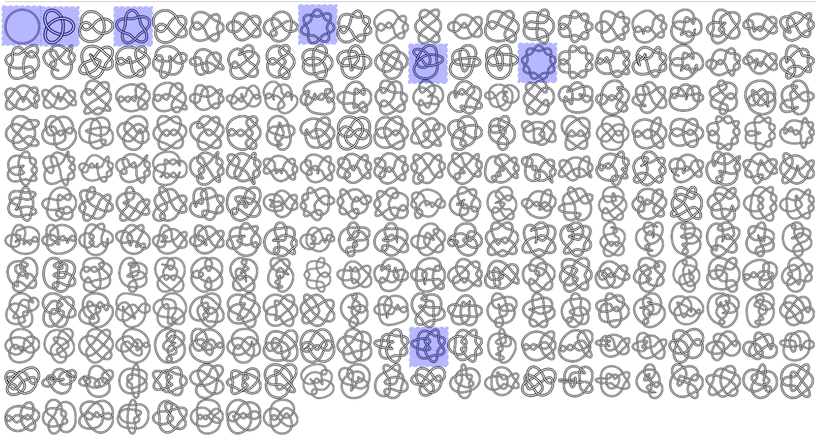
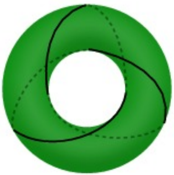


Only 7 of which are torus knots.

Cable Knots

- First step towards finding non-torus knots - cable knots are the obvious extension of torus knots.
- What is a cable knot? Take torus knot $\mathcal{K}_2$



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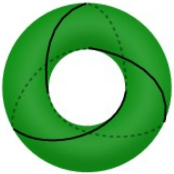


- We call this knot the $\mathcal{K}_2$ cable on $\mathcal{K}_1$.

Cable knot construction

- We know that a cable knot can be generated by the map

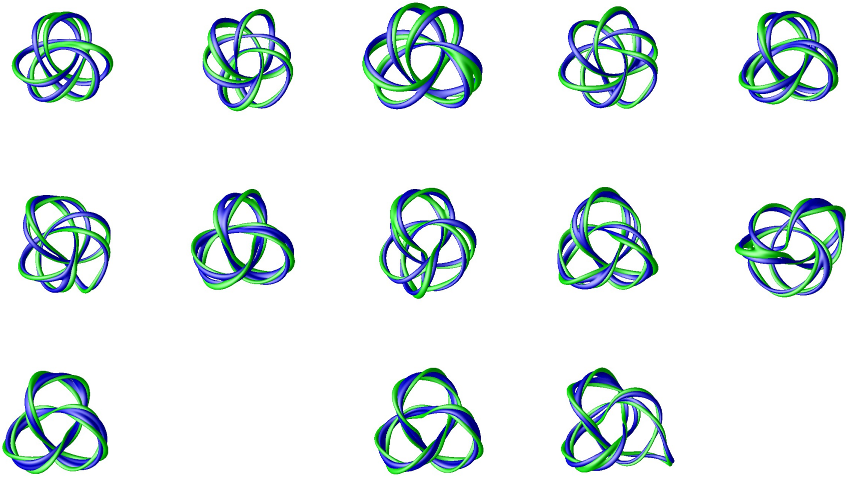
$$W = \frac{Z_1^{\alpha} Z_0^{\beta}(Z_1 - Z_0)^{\gamma}}{Z_0^4 - 2Z_1^3 Z_0^2 - 4\eta^2 Z_1^3 Z_0 + Z_1^6 - \eta^4 Z_1^3},$$

  for some $\eta \neq 0$, which describes a $\mathcal{K}_{32}$ cable on $\mathcal{K}_{32}$. Choice of positive integer α, non-negative integer β and $\gamma \in \{0, 1\}$.

- Lower charges relax to torus knots or links of torus knots, above charge 22 we find solutions of the right form.

Cable Knotted Hopfions

- Solutions with the form of cable knots and links for charges $22-35$ with the exception of charge 33.



Conclusion and outlook

- We have seen a range of topological solitons.
- We have seen the SF model, and the types of solution.
- We have seen the first known examples of non-torus knots.
- What happens for even higher charges? Iterated torus knots?
- What about the other non-torus knots? Non-prime knots?
- What is the behaviour of these knots under classical isospin?

Thank you for listening.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