Two approaches to Quantum Quenches

Vaios Ziogas

December 1, 2014

Introduction

- Entanglement Entropy Everywhere: In condensed matter physics characterizes phase transitions, used as diagnostic for universal properties of lattice models, important in quantum information and quantum computation, related to black hole entropy and can have deep implications in quantum gravity, ...
- Quantum quenches involve sudden change of parameters of Hamiltonian. That's a difficult problem in general, related to thermalization and relaxation of a quantum state.

Overview

Density Matrices

Quantum mechanics can be reformulated in terms of density matrices: Given a normalized state $|\psi\rangle$ we can form the *density matrix* $\rho = |\psi\rangle\langle\psi|$ with the following properties:

- $\rho = \rho^\dagger$
- $\rho \geq 0$ (ie all its eigenvalues are positive)
- $\mathrm{tr}\rho = 1$
- $\rho = \rho^2$

We can introduce some classical uncertainty by considering a *mixed* density matrix, which is a statistical ensemble of such density matrices

$$\rho_{mix} = \sum p_i \rho_i,$$

with $\sum p_i = 1$, and then $\rho \neq \rho^2$.

Reduced Density Matrices

Assume that the Hilbert space $\mathcal{H}$ factorizes: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.
The *reduced density matrix* associated to system A is

$$\rho_A = \mathrm{tr}_B \rho,$$

for a given ρ acting on $\mathcal{H}$.

For any observable $\mathcal{O}_A$ of A,

$$\langle \mathcal{O}_A \rangle = \mathrm{tr}_A(\mathcal{O}_A \rho_A).$$

In general ρ_A will be mixed, ie $|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$.

Entanglement Entropy

The *von Neumann entropy* of a density matrix ρ is

$$S = -\mathrm{tr}(\rho \log \rho),$$

eg for a pure density matrix ρ_0 we find $S_0 = 0$, while for a thermal density matrix $\rho_{th} = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$ we find $S = \beta\langle H\rangle + \langle \log Z\rangle = S_{th},\ Z = \mathrm{tr}(e^{-\beta H})$.

The *entanglement entropy* of system A is the von Neumann entropy of its reduced density matrix:

$$S_A = -\mathrm{tr}(\rho_A \log \rho_A).$$

All these definitions carry over to quantum field theory, where we usually take the system A to be a d-dimensional region in a d-dimensional time slice.

Properties of Entanglement Entropy

Some interesting properties of entanglement entropy:

- $S_A = S_{A^c}$ if ρ is pure (but $S_{A\cup A^c} = 0$)
- $S_{A\cup B} + S_{B\cup C} \geq S_B + S_{A\cup B\cup C}$
- $I(A,B) \equiv S_A + S_B - S_{A\cup B} \geq 0$
- For QFTs, it obeys the area law:

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\epsilon^{d-1}} + c_{d-3}\left(\frac{l}{\epsilon}\right)^{d-3} + \ldots + \begin{cases} c_1\left(\frac{l}{\epsilon}\right) + c_0, & d \text{ even} \\ \tilde{c}_1 \log\left(\frac{l}{\epsilon}\right) + \tilde{c}_0, & d \text{ odd} \end{cases}$$

Replica Trick

It can be seen from the definition that:

$$S_A = \lim_{n\to 1^+} \frac{1}{1-n} \log \mathrm{tr}\rho_A^n = -\lim_{n\to 1^+} \frac{\partial}{\partial n} \mathrm{tr}\rho_A^n.$$

So, alternatively, we can find $\mathrm{tr}\rho_A^n$, analytically continue to real n, differentiate and set $n = 1$.

Specialize to thermal density matrices (ground state ρ_0 is $\beta \to \infty$ limit of ρ_{th}).

Replica Trick

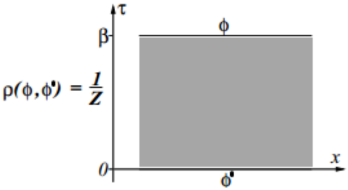
We can represent $\rho_{th}(\phi(x), \phi'(x))$ as a path integral in Euclidean space:

$$\begin{aligned}\rho_{th}(\phi(x), \phi'(x)) &= Z^{-1}\langle\phi(x)|e^{-\beta H}|\phi'(x)\rangle \\ &= Z^{-1}\int_{\varphi(\psi,0)=\phi'(x)}^{\varphi(\psi,\beta)=\phi(x)}[\mathrm{d}\varphi(\psi,\tau)]e^{-\int_0^\beta L_E \mathrm{d}\tau}\end{aligned}$$

The normalization factor is

$$Z = \mathrm{tr}\rho_{th} = \int[\mathrm{d}\varphi(x)]\rho_{th}(\varphi(x), \varphi(x)) = \int[\mathrm{d}\varphi(\psi,\tau)]e^{-\int_0^\beta L_E \mathrm{d}\tau}.$$

(1+1)d picture:



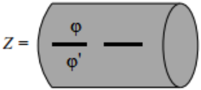
Replica Trick

Then, the reduced density matrix associated to a region A will be given by:

$$\begin{aligned}\rho_A(\phi(x), \phi'(x)) &= Z^{-1}\mathrm{tr}_{A^c}\langle\phi(x)|e^{-\beta H}|\phi'(x)\rangle \\ &= Z^{-1}\int_{\varphi(\psi,0)=\phi'(x)}^{\varphi(\psi,\beta)=\phi(x)}[\mathrm{d}\varphi(\psi,\tau)]e^{-\int_0^\beta L_E \mathrm{d}\tau},\end{aligned}$$

where now $x \in A$.

So now we integrate over



Replica Trick

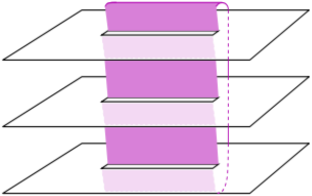
Finally, to obtain $\mathrm{tr}\rho_A^n$, we take n copies of our path integral and impose the identifications:

$$\begin{aligned}\phi_i(x,\tau=\beta^-) &= \phi_{i+1}(x,\tau=0^+),\ i=1,\ldots,n-1\\ \phi_n(x,\tau=\beta^-) &= \phi_1(x,\tau=0^+).\end{aligned}$$

So, we find $\mathrm{tr}\rho_A^n = \frac{Z_n(A)}{Z^n}$, where

$$Z_n(A) = \int_{\mathcal{C}}[\prod \mathrm{d}\varphi_i]e^{-\int_0^\beta \sum L_E[\varphi_i]},$$

and we denote by $\mathcal{C}$ the above conditions. In the (1+1)d case, they create an n-sheeted Riemann surface.



Twist operators

Now we restrict to (1+1)d CFTs and a single interval $[u, v]$. Formally, the twist operators σ and $\tilde{\sigma}$ are defined by requiring

$$\int_{\mathbb{C}^n} [\prod \mathrm{d}\varphi_i] \sigma(u, 0)\tilde{\sigma}(v, 0) \ldots e^{-\int_0^\beta \sum L_E[\varphi_i]} = \int_{\mathcal{C}} [\prod \mathrm{d}\varphi_i] \ldots e^{-\int_0^\beta \sum L_E[\varphi_i]},$$

ie they encode the boundary conditions.

The twist operators are local and fixed uniquely if we require them to be primary. Then

$$\langle \mathcal{O}(x, y, \text{sheet i}) \ldots \rangle_{\mathcal{L}, \mathcal{R}_n} = \frac{\langle \sigma(u, 0)\tilde{\sigma}(v, 0)\mathcal{O}_i(x, y) \ldots \rangle_{\mathcal{L}^n, \mathbb{C}}}{\langle \sigma(u, 0)\tilde{\sigma}(v, 0) \rangle_{\mathcal{L}^n, \mathbb{C}}},$$

Twist operators

Taking the limit $T \to 0$, the sheets get uncompactified. Then the conformal transformation $z \to w = \sqrt[n]{\frac{z-u}{z-v}}$ maps $\mathcal{R}_n$ to $\mathbb{C}$. The transformation of the stress energy tensor T gives

$$\langle T(w) \rangle_{\mathcal{R}_n} = \frac{c(n^2-1)}{24n^2} \frac{(u-v)^2}{(w-u)^2(w-v)^2}$$

By using the OPE of the twist operators with T, this gives the scaling dimensions of the twist operators:

$$d_n = \frac{c}{12}(n - n^{-1}).$$

Entanglement Entropy

So we find

$$\mathrm{tr}\rho_A^n \sim \left(\frac{u-v}{a}\right)^{-\frac{c}{6}(n-n^{-1})},$$

where a is the UV cut-off, arising from the normalization of the two-point function.
Finally

$$S_A = \frac{c}{3}\log\frac{l}{a} + c',$$

where $l = |u-v|$ and c' is a non-universal constant.

Entanglement Entropy

By mapping $\mathbb{C}$ to a cylinder, we can immediately obtain the results for a thermal state with temperature $T = \beta^{-1}$

$$S_A = \frac{c}{3} \log(\frac{\beta}{\pi a} \sinh \frac{\pi l}{\beta}) + c'.$$

In the large T limit $l \gg \beta$ we get an extensive behaviour $S_A = \frac{\pi c}{3\beta} l$.
Similarly, for a CFT on a circle of circumference L,

$$S_A = \frac{c}{3} \log(\frac{L}{\pi a} \sin \frac{\pi l}{L}) + c'.$$

However, we cannot use this technique for finite T and L or multiple intervals.

Global Quantum Quenches

Prepare the system in ground state $|\psi_0\rangle$ of a Hamiltonian H_0. At $t = 0$, a sudden change of parameters leads to a new Hamiltonian H. The density matrix of the full system will undergo unitary evolution

$$\rho(t) = e^{-itH}|\psi_0\rangle\langle\psi_0|e^{itH}$$

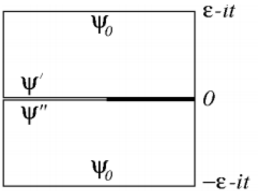
and it can be written in a path integral representation as well:

$$\rho(t)(\psi'', \psi) = Z^{-1}\langle\psi''|e^{-itH-\epsilon H}|\psi_0\rangle\langle\psi_0|e^{itH-\epsilon H}|\psi'\rangle,$$

where ϵ is a dumping factor to ensure absolute convergence and $Z = \langle\psi_0|e^{-2\epsilon H}|\psi_0\rangle$.

Global Quantum Quenches

Now, to find the reduced density matrix associated to an interval A, we need to "sew" ψ' and ψ'' in A^c and integrate over them. We thus obtain the following geometry:



and we have continued to complex time.

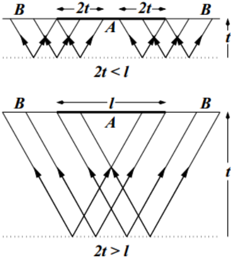
At this point we assume that H is conformally invariant and H_0 has a mass gap m_0, but is close to H under the RG flow. Then $|\psi_0\rangle$ will flow to a conformal boundary state $|\psi_0^*\rangle$ and $\epsilon \sim m_0^{-1}$.

Global Quantum Quenches

Then, we can find

$$S_A = \frac{c}{3}\log m_0 + \begin{cases} \frac{m_0 \pi c}{6} t, & t < l/2 \\ \frac{m_0 \pi c}{12} l, & t \geq l/2 \end{cases},$$

assuming $l, t \gg m_0^{-1}$. We observe a period of linear growth and saturation at an effective temperature $T_{eff} = m_0/4$.
This is well described by a quasi-particle picture (really justified only for weak coupling), valid also for more intervals:



AdS/CFT

The AdS/CFT correspondence suggests that certain quantum gravity theories in (D+2) dimensions are equivalent to certain field theories in (D+1) dimensions. More precisely, the partition functions describing those theories should be equal:

$$Z_{QG}[\phi \to \phi_0] = Z_{FT}[\phi_0],$$

and so all physical observables of one theory should match the physical observables of the other.

The most studied instance of this correspondence involves asymptotically AdS spacetimes, which are dual to CFTs living on their conformal boundary. If we restrict to large-c, strongly coupled CFTs, classical GR dominates in the LHS, and we can use the saddle-point approximation to evaluate it.

AdS

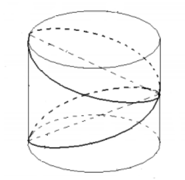
Pure AdS_{d+2} space is a solution of Einstein equations with negative cosmological constant, described globally by the metric

$$ds^2 = -(r^2+1)dt^2 + (r^2+1)^{-1}dr^2 + r^2 d\Omega_d^2,$$

where $t \in (-\infty, \infty)$, $r \in [0, \infty)$ and $d\Omega_d^2$ is the metric on the unit sphere S^d (we have set the AdS radius to 1). In Poincare coordinates the metric becomes

$$ds^2 = -r^2 dt^2 + r^{-2}dr^2 + r^2 d\vec{x}^2,$$

where now $\vec{x}$ are flat coordinates, but only a part of the spacetime is covered.



Pure AdS is dual to the vacuum state of a CFT.

BTZ Black Hole

In 3 dimensions, there is also the BTZ black hole solution, described by the metric

$$ds^2 = -(r^2 - r_+^2)dt^2 + (r^2 - r_+^2)^{-1}dr^2 + r^2 d\phi^2,$$

where now r_+^2 is the location of the event horizon. In Poincare coordinates, the angular coordinate ϕ gets uncompactified $\phi \to \vec{x}$.
Note that this is asymptotically AdS_3.

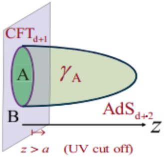
The BTZ black hole is dual to a thermal state in an (1+1)d CFT with temperature $T \sim r_+$.

Holographic Entanglement Entropy

Prescription to compute the entanglement entropy of a region A on the boundary of a static spacetime:

$$S_A = \mathrm{Min}_{\Sigma_A} \frac{\mathrm{Area}(\Sigma_A)}{4G_N}$$

where Σ_A are Euclidean surfaces anchored on ∂A and subject to a homology constraint, and G_N is the Newton constant of the bulk.



Here $z = 1/r \sim a$ is the UV cut-off, on which the theory is defined before taking $\epsilon \to 0$, and γ_A is the surface that minimizes the area of all Σ_A.

Checks & Extension

- Reproduces the divergence structure of entanglement entropy
- In particular, computation of length of geodesics in Poincare AdS_3 reproduces the vacuum result for a (1+1)d CFT $S_A = \frac{c}{3} \log \frac{l}{a}$, after the identification $c \equiv \frac{3}{2G_N}$.
- In global coordinates we recover the compact result and in BTZ the finite temperature result.
- Easy proof of strong subadditivity.
- ...

For non-static spacetimes, there is a covariant proposal that replaces "minimal" surfaces with "extremal" surfaces.

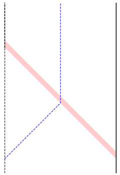
Vaidya-BTZ

The Vaidya-BTZ solution describes an incoming null shell in AdS that collapses and forms a BTZ black hole.

$$ds^2 = -(r^2 + 1 - \theta(v)(r_+^2 + 1))dv^2 + 2dvdr + r^2 d\phi,$$

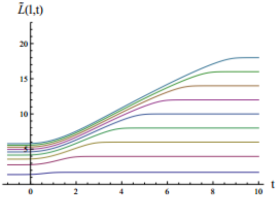
where $v = \arctan r + t - \frac{\pi}{2}$ are infalling null coordinates. For $v < 0$ have pure AdS_3 and for $v > 0$ have BTZ.

From the boundary point of view, looks like we start from vacuum, inject energy uniformly into our a system and end up with a thermal state$\longrightarrow$ "dual" to global quantum quench.

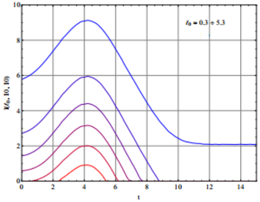


HEE & Mutual Information

Compute (regularized) length of spacelike geodesics in Poincare coordinates numerically. For a fixed interval:



Mutual information for two equal length intervals:



HEE & Mutual Information

Observations:

- Reproduction of linear growth and saturation of entanglement entropy.
- "Bump" in mutual information visible in many cases.

So we see that the qualitative behaviour of the CFT calculations is quite universal, observed also in these "holographic" quenches (note that before there was no mention of the strength of the coupling or the central charge).

However, there is an important difference: we started from a CFT and not a gapped theory, so we have initial long range correlations, accounting for the differences observed.

Conclusion and Future Directions

- We have seen the definition and basic properties of entanglement entropy, and the basic techniques for its calculation in (1+1)d CFTs.
- We discussed the quasi-particle interpretation of the evolution of entanglement entropy under a quantum quench.
- Finally, we gave a quick overview of the holographic approach to similar processes.

- Local Quenches
- Many other approaches to holographic quantum quenches: other geometries, E/M quenches, probe fields and branes on fixed background, AdS/BCFT,...
- We can also generalize to global Vaidya-BTZ; novel phenomena may arise!

Thank you

References

Figures taken from

P. Calabrese and J. Cardy (2009)
Entanglement entropy and conformal field theory
J. Phys. A: Math. Theor. 42(50).

C. A. B. Bayona, and N. R. F. Braga (2007)
Anti-de Sitter boundary in Poincare coordinates
Gen.Rel.Grav. 42, 1367–1379.

V. E. Hubeny, H. Maxfield (2013)
Holographic probes of collapsing black holes
arXiv:1312.6887.

T. Takayanagi (2012)
Entanglement Entropy from a Holographic Viewpoint
Class.Quant.Grav. 29 (2012) 153001.

J. Abajo-Arrastia, J. Aparicio, E. Lopez (2010)
Holographic Evolution of Entanglement Entropy
JHEP 1011:149 (2010).

References

A. Allais, E. Tonni (2011)
Holographic evolution of the mutual information
JHEP 1201:102 (2012).