Galilean Fluids Through Null Reduction

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Banerjee, Dutta, Jain, Roychowdhury, "Entropy current for non-relativistic fluid," [arXiv:1405.5687]
Banerjee, Dutta, Jain, "Equilibrium partition function for nonrelativistic fluids," [arXiv:1505.05677]
Banerjee, Dutta, Jain, "Null Fluids – A New Viewpoint of Galilean Fluids," [arXiv:1509.04718]
Jain, "Galilean Anomalies and Their Effect on Hydrodynamics," [arXiv:1509.05777]

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CPT Student Seminars

Introduction

- System of interest: 4 dimensional ideal non-relativistic fluid.
- The respective relativistic system is well known.
- One can take a 'non-relativistic' limit ($v \ll c$) to get the non-relativistic counterpart. [Kaminski et al.'14]
- In [Rangamani et al.'08] an alternative approach – *null reduction* was suggested to get 'a' Galilean fluid starting from a 5 dimensional relativistic fluid.
- The goal of this talk is to revisit and address some issues with this construction – we want to find a relativistic system whose null reduction will give the most generic Galilean fluid.
- If time permits, we will also discuss how null reduction can be used to study anomalies in Galilean theories.

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Outline

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Null Reduction

Relativistic Hydrodynamics

Null Hydrodynamics

Anomalous Charged Null Backgrounds

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Null Reduction

- Fact: 4 dimensional Galilean algebra is a sub-algebra of the 5 dimensional Poincaré algebra, defined as the subset which leaves a null momenta invariant.

- Consider generators of a 5 dimensional Poincaré algebra,

$$\text{Translations:} \quad P_M, \qquad \text{Rotations and Boosts:} \quad M_{MN}, \tag{1}$$

where $M, N \ldots = 0, 1, 2, 3, 4$. They have commutation relations,

$$\begin{aligned} [P_M, P_N] &= 0, \\ [M_{MN}, P_R] &= \eta_{MR} P_N - \eta_{NR} P_M, \\ [M_{MN}, M_{RS}] &= \eta_{MR} M_{NS} - \eta_{MS} M_{NR} - \eta_{NR} M_{MS} + \eta_{NS} M_{MR}. \end{aligned} \tag{2}$$

- In null coordinates $x^M = \{x^\pm, x^i\}$ such that $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^{d+1}\right)$ and $i, j \ldots = 1, 2, 3$,

$$P_-, \quad P_+, \quad P_i, \quad M_{-+}, \quad M_{-i}, \quad M_{+i}, \quad M_{ij}. \tag{3}$$

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Null Reduction

- We want to pickup a subset of these generators which commute with null momenta P_-,

$$P_-, \quad P_+, \quad P_i, \quad B_i \equiv M_{i-}, \quad M_{ij}. \tag{4}$$

- This subset forms a 4 dimensional Galilean algebra with P_- as a Casimir,

$$\begin{aligned}
[P_-, P_+] = [P_-, P_i] = [P_-, B_i] = [P_-, M_{ij}] = 0, \\
[P_+, P_i] = [P_+, M_{ij}] = [P_i, P_j] = [B_i, B_j] = 0, \\
[B_i, P_+] = P_i, \qquad [B_i, P_j] = \eta_{ij} P_-, \\
[M_{ij}, P_k] = \eta_{ik} P_j - \eta_{jk} P_i, \qquad [M_{ij}, B_k] = \eta_{ik} B_j - \eta_{jk} B_i, \\
[M_{ij}, M_{kl}] = \eta_{ik} M_{jl} - \eta_{il} M_{jk} - \eta_{jk} M_{il} + \eta_{jl} M_{ik}.
\end{aligned} \tag{5}$$

- From here the generators of the Galilean algebra can be identified as,

$$\text{Continuity:} \quad P_-, \qquad \text{Time Translation:} \quad P_+, \qquad \text{Space Translations:} \quad P_i,$$
$$\text{Galilean Boosts:} \quad B_i, \qquad \text{Rotations:} \quad M_{ij}. \tag{6}$$

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Null Backgrounds

- Consider a 5 dimensional relativistic spacetime $\mathcal{M}$, equipped with a metric $\mathrm{d}s^2 = \mathrm{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N$. We describe a physical theory on $\mathcal{M}$ by a set of dynamical fields $\Phi^{(i)}$.

- If $\mathcal{M}$ admits a covariantly constant null Killing vector V^M,

$$\nabla_M V^N = 0, \qquad V^M V_M = 0, \qquad \pounds_V \mathrm{G}_{MN} = 0, \tag{7}$$

  it can be interpreted as a 4 dimensional Galilean manifold. We call such a $\mathcal{M}$ to be a *null background*.

- Similarly if V leaves dynamical fields $\Phi^{(i)}$ invariant,

$$\pounds_V \Phi^{(i)} = V^M \nabla_M \Phi^{(i)} = 0, \tag{8}$$

  the physical theory defined on $\mathcal{M}$ can be realized as a Galilean theory.

Null Backgrounds

- To get the 'conventional' representation of Galilean theories, we need to choose a *time field* T^M, and pick up a basis $x^M = \{x^-, t, x^i\}$ such that $V = \partial_-$ and $T = \partial_t$. t plays the role of Galilean time. Metric can be decomposed as,

$$\mathrm{d}s^2 = 2\left(n_t \mathrm{d}t + n_i \mathrm{d}x^i\right)\left(-\mathrm{d}x^- + B_t \mathrm{d}t + B_i \mathrm{d}x^i\right) + g_{ij}\mathrm{d}x^i \mathrm{d}x^j. \qquad (9)$$

- Conditions of null background will then dictate that $n_t, n_i, B_t, B_i, g_{ij}$ are just functions of Galilean time t and spatial coordinates x^i. Further the 'time metric is closed',

$$\partial_t n_i - \partial_i n_t = 0, \qquad \partial_i n_j - \partial_j n_i = 0. \qquad (10)$$

- Finally for a theory on $\mathcal{M}$ to have a Galilean interpretation we would need to demand,

$$\nabla_- \Phi^{(i)} = \partial_- \Phi^{(i)} = 0. \qquad (11)$$

Reduction of Ward Identities

- For simplicity lets consider spacetime to be flat.
- Consider a physical theory on $\mathcal{M}$ with an energy-momentum T^{MN}, which follows the Ward identity associated with diffeomorphism symmetry,

$$\partial_M T^{MN} = 0. \tag{12}$$

- On performing null reduction, these Ward identities will become,

$$\partial_t T^{t-} + \partial_i T^{i-} = 0, \qquad \partial_t T^{tt} + \partial_i T^{it} = 0, \qquad \partial_t T^{tj} + \partial_i T^{ij} = 0. \tag{13}$$

- They can be realized as energy, mass and momentum conservation equations respectively if we identify,

$$\text{Mass Density: } \rho = T^{tt}, \qquad \text{Mass Current: } \rho^i = T^{ti},$$
$$\text{Energy Density: } \epsilon = T^{-t}, \qquad \text{Energy Current: } \epsilon^i = T^{-i},$$
$$\text{Stress Tensor: } p^{ij} = T^{ij}. \tag{14}$$

- Respective conservation laws will look like,

$$\partial_t \epsilon + \partial_i \epsilon^i = 0, \qquad \partial_t \rho + \partial_i \rho^i = 0, \qquad \partial_t \rho^j + \partial_i p^{ij} = 0. \tag{15}$$

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Relativistic Ideal Hydrodynamics

- A fluid is characterized by a set of conserved currents (e.g. T^{MN}), with dynamics given by Ward identities imposed as equations of motion.
- A fluid configuration is described by a set of *fluid variables* which can be exactly solved for using the equations of motion,

$$\text{Temperature: } \vartheta_{rel}, \qquad \text{Velocity: } u^M_{rel} \quad \text{where} \quad u^M_{rel} u^N_{rel} \eta_{MN} = -1.$$

- In terms of fluid variables, the most generic expression of T^{MN}, known as *constitutive relations* of the fluid, at ideal order is given by,

$$T^{MN} = E_{rel}(\vartheta_{rel}) u^M_{rel} u^N_{rel} + P_{rel}(\vartheta_{rel}) \left(\eta^{MN} + u^M_{rel} u^N_{rel} \right), \tag{16}$$

  where E_{rel} is the energy density and P_{rel} is the pressure density.
- The fluid is required to follow the second law of thermodynamics, i.e. there must exist an entropy current J^M_s such that $\partial_M J^M_s \geq 0$.
- $J^M_s = S_{rel}(\vartheta_{rel}) u^M_{rel}$ does the job given the fluid satisfies,

$$\begin{aligned} \text{First Law of Thermodynamics:} \quad & \mathrm{d}E_{rel} = \vartheta_{rel} \mathrm{d}S_{rel}, \\ \text{Gibbs-Duhem Relation:} \quad & E_{rel} + P_{rel} = \vartheta_{rel} S_{rel}, \end{aligned} \tag{17}$$

  where S_{rel} is the entropy density.

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Null Reduction of Relativistic Hydrodynamics

- Using our dictionary of relativistic and Galilean Ward identities, we can read out Galilean constitutive relations,

$$\rho = T^{tt} = R, \qquad \rho^i = T^{ti} = Rv^i, \qquad p^{ij} = T^{ij} = Rv^i v^j + P\eta^{ij},$$
$$\epsilon = T^{-t} = \frac{1}{2}R\vec{v}^2 + E, \qquad \epsilon^i = T^{-i} \left(\frac{1}{2}R\vec{v}^2 + E + P\right) v^i,$$
$$s = J_s^t = S, \qquad s^i = J_s^i = Sv^i, \tag{18}$$

  where we have identified,

$$R = (E_{rel} + P_{rel})(u_{rel}^t)^2, \qquad v^i = \frac{u_{rel}^i}{u_{rel}^t},$$
$$E = \frac{1}{2}(E_{rel} - P_{rel}), \qquad P = P_{rel}, \qquad S = S_{rel} u_{rel}^t. \tag{19}$$

  Here we have used the velocity normalization: $2u_{rel}^t u_{rel}^- = 1 + \vec{u}_{rel}^2$.
- We have recovered the standard constitutive relations of a Galilean fluid.
- Note that these identifications also imply,

$$R = 2(u_{rel}^t)^2(E + P). \tag{20}$$

Null Reduced Thermodynamics

- We finally proceed to reduce the thermodynamics. Using the mapping of thermodynamic function, it trivially follows that,

$$\mathrm{d}E = -\frac{1}{2(u^t_{rel})^2}\mathrm{d}R + \frac{\vartheta_{rel}}{u^t_{rel}}\mathrm{d}S, \qquad E + P = -\frac{1}{2(u^t)^2_{rel}}R + \frac{\vartheta_{rel}}{u^t_{rel}}S. \tag{21}$$

- From here we can read out Galilean temperature $\vartheta = \frac{\vartheta_{rel}}{u^t_{rel}}$ and Galilean mass chemical potential $\mu_{\mathrm{M}} = -\frac{1}{2(u^t_{rel})^2}$.
- Con: Note that the thermodynamics is restricted,

$$E + P + R\mu_{\mathrm{M}} = 0, \tag{22}$$

  due to identification $R = 2(u^t_{rel})^2(E + P)$.
- Con: Thermodynamic functions E, P, S are effectively arbitrary functions of only one variable (as their relativistic parents are arbitrary functions of one variable), however for a Galilean fluid we expect them to be arbitrary functions of two variables $\vartheta, \mu_{\mathrm{M}}$.
- Conclusion: Null reduction of a relativistic fluid *does not* give the most generic Galilean fluid.

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Null Hydrodynamics

- Idea: Instead of starting with the relativistic fluid, we should construct a theory of fluids on null backgrounds from scratch – *null fluids*.

- We choose *fluid variables* to be,

$$\text{Temperature: } \vartheta, \qquad \text{Mass Chemical Potential: } \mu_{\mathrm{M}},$$
$$\text{Null Velocity: } u^M \quad \text{where} \quad u^M u_M = 0, \quad u^M V_M = -1. \tag{23}$$

- Constitutive relations of a null fluid, at ideal order, are given as,

$$\begin{aligned} T^{MN} = {} & R(\vartheta, \mu_{\mathrm{M}}) u^M u^N + 2E(\vartheta, \mu_{\mathrm{M}}) u^M V^N \\ & + P(\vartheta, \mu_{\mathrm{M}}) \left(\eta^{MN} + V^M u^N + u^M V^N\right) + \#(\vartheta, \mu_{\mathrm{M}}) V^M V^N. \end{aligned} \tag{24}$$

- Requiring null fluid to follow the second law of thermodynamics we find $J_s^M = S(\vartheta, \mu_{\mathrm{M}}) u^M$, provided the system follows,

$$\mathrm{d}E = \mu_{\mathrm{M}} \mathrm{d}R + \vartheta \mathrm{d}S, \qquad E + P = \mu_{\mathrm{M}} R + \vartheta S. \tag{25}$$

Note that the thermodynamics already looks Galilean.

Null Reduction of Null Hydrodynamics

- A straight away reduction of the constitutive relations will yield,

$$\rho = T^{tt} = R, \qquad \rho^i = T^{ti} = Ru^i, \qquad p^{ij} = T^{ij} = Ru^i u^j + P\eta^{ij},$$
$$\epsilon = T^{-t} = \frac{1}{2}R\vec{u}^2 + E, \qquad \epsilon^i = T^{-i} = \left(\frac{1}{2}R\vec{u}^2 + E + P\right) u^i,$$
$$s = J^t_s = S, \qquad s^i = J^i_s = Su^i. \tag{26}$$

  Here we have used the velocity normalization: $u^t = 1$, $u^- = \frac{1}{2}\vec{u}^2$.

- Pro: There is no need of an identification; the constitutive relations are already in their Galilean form.

- Pro: Thermodynamics is unrestricted, and is same as the null fluid,

$$\mathrm{d}E = \mu_{\mathrm{M}}\mathrm{d}R + \vartheta\mathrm{d}S, \qquad E + P = \mu_{\mathrm{M}}R + \vartheta S. \tag{27}$$

- Pro: Thermodynamic variables are a function of two variables $\vartheta, \mu_{\mathrm{M}}$.

- Conclusion: Null reduction of a null fluid gives the most generic Galilean fluid, and the map between quantities in both the theories is trivial (at least at ideal order).

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Null Fluids as an Embedding of Galilean Fluids

- Claim: Null fluid is an embedding of the Galilean fluid into a spacetime of one higher dimension. It is merely a 'nicer' covariant boost-invariant language for the same thing.
- There is a better known 'covariant formulation' of Galilean hydrodynamics in terms of Newton-Cartan geometries [Jensen '14], which however is not explicitly boost invariant.
- Attempts were made by [Geracie et.al.'15] to make this formulation boost invariant by embedding the fluid into a spacetime of one higher dimension – the *extended space representation*.
- It can be showed that the extended space representation is just the bottom-up approach to the null fluid [Jain '15].

Further Developments on Null Backgrounds/Null Fluids

- The nice trivial mapping between constitutive relations of a null fluid and a Galilean fluid also works for dissipative fluids.
- The entire null background story can be extended to include a slowly varying curved torsional spacetime.
- The null background construction can also be extended to include a U(1) gauge field, and with a corresponding anomalous U(1) current.
- Null backgrounds can also describe a Galilean system with a non-abelian and spin anomaly (equivalent of relativistic gravitational anomaly).

Outline

Charged Null Backgrounds

- Consider a relativistic theory coupled to a 5 dim spacetime $\mathcal{M}$, equipped with a metric G_{MN}, a U(1) gauge field A_M (with associated field strength F_{MN}).
- We define $\mathcal{M}$ as a *null background* by introduction of a covariantly constant compatible null Killing vector V^M,

$$\nabla_M V^N = 0, \quad \underline{A_M V^M = 0}, \quad V_M V^M = 0, \quad \pounds_V \mathrm{G}_{MN} = \pounds_V A_M = 0. \tag{28}$$

- Coming back to flat space, Ward identities corresponding to diffeomorphisms and gauge invariance are given as,

$$\partial_M T^{MN} = F^{NR} J_R, \qquad \partial_M J^M = 0, \tag{29}$$

where T^{MN} is the energy-momentum tensor and J^M is the charge current.

- Null reduction of these Ward identities will yield,

$$\begin{aligned}
\partial_t T^{tt} + \partial_i T^{it} &= -J^t \partial_t A^t - J^i \partial_i A^t, \\
\partial_t T^{t-} + \partial_i T^{i-} &= J^- \partial_t A^t - \left(\partial_t A_i + \partial_i A^-\right) J^i, \\
\partial_t T^{tj} + \partial_i T^{ij} &= -J^- \partial^j A^t - \left(\partial_t A^j + \partial^j A^-\right) J^t + \left(\partial^j A_i - \partial_i A^j\right) J^i, \\
\partial_t J^t + \partial_i J^i &= 0.
\end{aligned} \tag{30}$$

Reduction of Charged Ward Identities

- These can be identified as mass, energy, momentum and charge conservation equations of a Galilean theory if we identify,

$$\rho = T^{tt}, \quad \rho^i = T^{ti}, \quad \epsilon = T^{-t}, \quad \epsilon^i = T^{-i}, \quad p^{ij} = T^{ij},$$

$$\text{Charge Density: } q = J^t, \quad \text{Charge Current: } j^i = J^i,$$

$$\text{Electric Field: } e_i = -\partial_i A^- - \partial_t A_i, \quad \text{Dual Magnetic Field: } \beta_{ij} = \partial_i A_j - \partial_j A_i.$$

- Having done the identification, the corresponding Ward identities look like,

$$\begin{aligned}
\partial_t \rho + \partial_i \rho^i &= -q\partial_t A^t - j^i \partial_i A^t, \\
\partial_t \epsilon + \partial_i \epsilon^i &= J^- \partial_t A^t + j^i e_i, \\
\partial_t \rho^j + \partial_i p^i{}_j &= -J^- \partial_j A^t + q e_j + [\vec{j} \times \vec{b}]_j, \\
\partial_t q + \partial_i j^i &= 0,
\end{aligned} \tag{31}$$

where $b^i = \frac{1}{2}\epsilon^{ijk}\beta_{jk}$ is the magnetic field and $\varepsilon^{ijk} = -\epsilon^{-tijk}$.

- We see that A^t serves as a source of mass. For physically realizable theories we need to get rid of these mass sources, which is indeed done by our compatibility condition: $A_M V^M = -A^t = 0$.

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Reduction of $\mathrm{U}(1)$ Anomaly

- Finally we want to make some comments on obtaining anomalies via null reduction.
  - 5 dimensional relativistic theories do not have $\mathrm{U}(1)$ anomaly, so the 4 Galilean theory gained via null reduction is non-anomalous.
  - 4 dimensional relativistic theories do have a $\mathrm{U}(1)$ anomaly,

    $$\partial_M J^M = \frac{3}{4} C^{(4)} \epsilon^{MNRS} F_{MN} F_{RS}, \tag{32}$$

    but it vanishes upon null reduction,

    $$\partial_t q + \partial_i j^i = 3 C^{(4)} \left(-\varepsilon^{ij} \beta_{ij} \partial_t A^t + 2 \varepsilon^{ij} e_i \partial_j A^t \right) = 0, \tag{33}$$

    where $\varepsilon^{ij} = -\epsilon^{-tij}$. Hence 3 dimensional Galilean theories gained via null reduction are non-anomalous.

- Conclusion: Galilean theories gained via null reduction are non-anomalous (at least for $\mathrm{U}(1)$ anomalies).

Modified $\mathrm{U}(1)$ Anomaly on Null Backgrounds

- Do Galilean systems exhibit anomalies? Yes, [Bakas et al.'11] computed U(1) anomaly for a Galilean system (Lifshitz fermions), using path integral methods.

- Idea: We can introduce a 'modified U(1) anomaly' on null backgrounds which gives the correct Galilean anomalies upon null reduction.

- Consider a modified U(1) anomaly,

$$\partial_M J^M = \frac{3}{4} C^{(4)} \epsilon^{MNRST} \bar{V}_M F_{NR} F_{ST}, \tag{34}$$

  such that $\bar{V}_M V^M = -1$.

- It does not matter what $\bar{V}_M$ we choose; upon reduction it gives rise to,

$$\partial_t q + \partial_i j^i = -3 C^{(4)} \epsilon^{ijk} e_i \beta_{jk} = -6 C^{(4)} e_i b^i. \tag{35}$$

  This is the correct U(1) anomaly as found by [Bakas et al.'11].

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Conclusions

- There exists a well defined relativistic system – null fluid, whose null reduction gives the most generic Galilean fluid.
- Null backgrounds can be seen as a nicer representation of Galilean backgrounds, which is covariant, boost invariant, and is easier to handle.
- Null backgrounds can be used to translate various exotic relativistic phenomenon (like anomalies) to Galilean theories.

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