Entanglement Entropy and Perturbed Black Holes

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Outline

- What is Entanglement Entropy?
  - Shannon Entropy,
  - Mixed States,
  - Von-Neumann Entropy, Entanglement Entropy and Mutual Information
  - Entanglement Entropy and QFT.
- Towards Holographic Entropy
  - ~~Conformal Field Theory and String Theory.~~
  - ~~AdS/CFT.~~
  - Holographic Entropy.
- The Thermofield Double State and the BTZ Black Hole.
  - Perturbations and 'Chaos'
  - Rotating Black Holes

Information

- How much information does a given message convey?
- How can we quantify the unpredictability or disorder of an information source?

Number of bits

How many bits are required to represent the results of:

- a coin toss?
- three coin tosses?
- the roll of an eight sided die?

Alternatively, how many 'yes-no' questions must you ask to find out the results?
This suggests an information content of

$$I = \log_2 n$$

where n is the number of possible outcomes.

Likely and unlikely outcomes

Every week your neighbour tells you whether he won the lottery jackpot. How surprised are you when he says that:

- he won the jackpot?
- he didn't win the jackpot?

Unlikely outcomes contain more information. We adjust our formula for the amount of information in a message to

$$I = \log_2 \frac{1}{p} = -\log_2 p,$$

where p is the probability of the outcome.

Shannon Entropy

Shannon entropy is the expected value of the information from an event, i.e.

$$S = -\sum_i p_i \log_2 p_i .$$

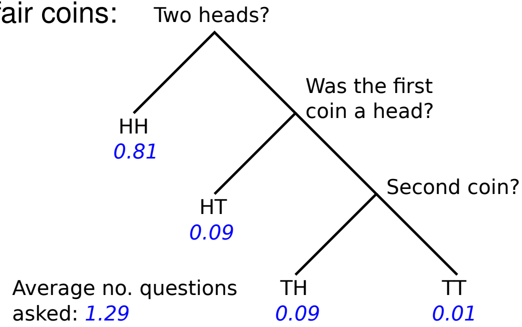
It measures the unpredictability of the event in question. We can think of this as the number of ‘yes-no’ one must ask, on average, to discuss the outcome of the event.

This is clear for the fair coin toss or eight sided dice. What about an unfair coin?

Multiple Unfair Coins

- If $P(\text{heads}) = 0.9$ then $S = 0.469$ bits. How can we ask 0.469 questions?
- Answer: Wait until a number of identical coins have been tossed before asking the questions.

For two unfair coins:



Multiple Unfair Coins

- Information content of the two unfair coin tosses is no more than 1.29 bits.
- We expect the information content of two coin tosses to be twice that of one coin toss.
- Hence the information content of one unfair coin toss is no more than 0.645 bits.
- As we consider more copies of the system, the number of questions required per coin approaches $S = 0.469$.

Density Matrices

For state $|\phi\rangle$, the density matrix ρ is simply

$$\rho_\phi = |\phi\rangle \langle\phi| \,.$$

We can use ρ_ϕ to find all physical quantities that could be determined from $|\phi\rangle$, e.g.

$$\langle O \rangle_\phi = \mathrm{tr}(\rho_\phi O) = \mathrm{tr}(O \rho_\phi)$$
$$\langle \phi|\psi\rangle = \mathrm{tr}(\rho_\phi \rho_\psi) = \mathrm{tr}(\rho_\psi \rho_\phi)$$

Mixed States

Mixed states are now defined as having density matrices of the form

$$\rho = \sum_{k=1}^{N} p_k \left|\psi_k\right\rangle \left\langle\psi_k\right|$$

where the $\left|\psi_k\right\rangle$ are some set of *pure* states and

$$\sum_{k=1}^{N} p_k = 1.$$

They describe a lack of complete information about the quantum state of the system.

Thermal States and Components of Entangled States

Mixed states are used to describe:

- Thermal states, e.g.

$$\rho = \sum_i e^{-\beta E_i} |\psi_i\rangle \langle\psi_i| = e^{-\beta\mathcal{H}},$$

- Components of entangled states. If

$$|\phi\rangle = \frac{1}{\sqrt{2}} \overset{\text{AB}}{|\uparrow\uparrow\rangle} + \frac{1}{\sqrt{2}} |\downarrow\downarrow\rangle$$

  and $\rho = |\phi\rangle\langle\phi|$ then

$$\rho_A = \mathrm{tr}_B\, \rho = \frac{1}{2} |\uparrow\rangle\langle\uparrow| + \frac{1}{2} |\downarrow\rangle\langle\downarrow| .$$

Von-Neumann Entropy

Given density matrix ρ, the Von-Neumann entropy is defined as

$$S(\rho) = -\operatorname{tr}(\rho \log \rho).$$

If we write

$$\rho = \sum_i \lambda_i \left|\phi_i\right\rangle \left\langle\phi_i\right| ,$$

where $|\phi_i\rangle$ are simply the eigenstates of ρ then this is

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i,$$

i.e. just the Shannon entropy that arises if we measure ρ as an observable.

Questions → Measurements

- The Von-Neumann entropy is a measure of the impurity of the state.
- A pure state has zero Von-Neumann entropy.
- A maximally mixed state

$$\rho = \sum_i \frac{1}{n} |\phi_i\rangle \langle\phi_i|$$

  has entropy $S(\rho) = \log n$, where n is the dimensionality of the Hilbert space.
- Can be thought of as the expected number of binary measurements required in order to produce a pure state.

Entanglement Entropy

- Take Hilbert space $\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$, where A and B are two complementary subsystems.
- Consider (for now) pure state $|\phi\rangle$ with density matrix $|\phi\rangle\langle\phi|$.
- An observer with no access to subsystem B considers the system to be in reduced state $\rho_A = \mathrm{tr}_B\, \rho$.
- The *entanglement entropy* for subsystem A is then the Von-Neumann entropy of the reduced density matrix,

$$S_A = -\,\mathrm{tr}_A(\rho_A \log \rho_A).$$

- Can be though of as the number of measurements of A required to remove the entanglement, i.e. it is a measure of the 'amount' of entanglement.

Mutual Information

If the overall state is not pure, we can attempt to subtract the overall impurity from the entanglement entropy. We get

$$I(A;B) = S_A + S_B - S_{A\cup B}.$$

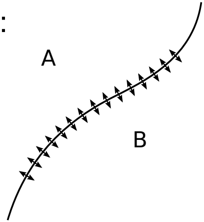
This

- is non negative,
- is symmetric,
- allows us to consider entanglement of non-complementary regions,
- but does not really solve the problem of separating entanglement from the overall impurity.

Entanglement Entropy in Quantum Field Theories

Problems with entanglement entropy in QFT:

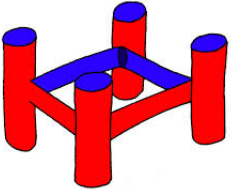
- It is infinite!
- It is a pain to calculate!



The Replica Trick

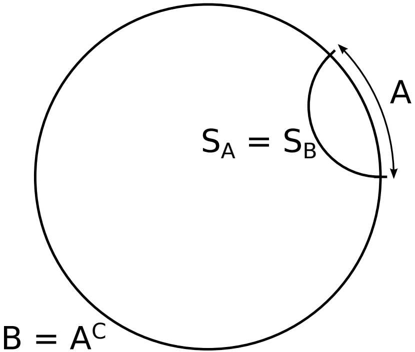
- $\operatorname{tr}\rho$: a Euclidean path integral, from time $t = 0$ to $t = \beta$, with identical conditions at the start and end — a path integral over a cylinder.
- $\rho_A = \operatorname{tr}_B \rho$: identical conditions at the start and end only for those points not in A — a cylinder with a cut.
- ρ_A^n: take n of these cylinders and attach the edge of the cut at $t = \beta$ of the cylinder i to $t = 0$ of cylinder $i + 1$.
- $\operatorname{tr}\rho_A^n$: attach cylinder n back to cylinder 1.
- Analytically continue to general n.
- Use

$$
\begin{aligned}
-\operatorname{tr}(\rho_A \log \rho_A) &= \lim_{n\to 1} \frac{\log(\operatorname{tr}\rho_A^n)}{1-n} \\
&= -\lim_{n\to 1} \frac{\partial}{\partial n} \operatorname{tr}\rho_A^n .
\end{aligned}
$$

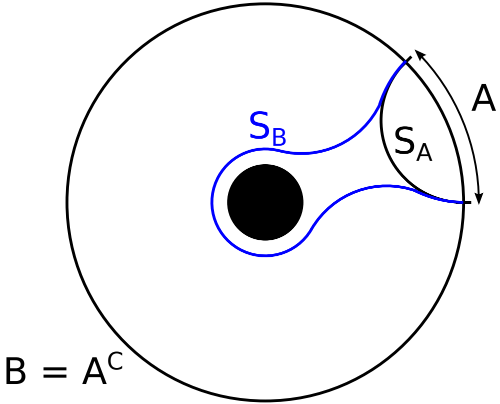


The Ryu-Takayanagi Conjecture

Entanglement entropy of A on the boundary theory is the surface area of the minimal surface of co-dimension 1 in the bulk time slice, homologous to, and with the same boundary as A, divided by $4G_N$.



The Homology Constraint



The Covariant Generalization

- The covariant (or HRT) generalization suggests that the entanglement entropy is given by the surface area of an extremal, co-dimension 2 surface in the bulk.
- If there are multiple such surfaces, the one of smallest surface area is chosen.

The Thermofield Double State

Defined on two identical subsystems, L and R, the thermofield double state is given be

$$|\psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R .$$

Taking the trace over subsystem R, we get

$$\rho_L = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} |n\rangle \langle n| .$$

Entanglement Structure

- The thermofield double state has an “atypical local structure of entanglement”.
- Given $A \subset L$ and $B \subset R$, A and B may be highly entangled even if they are small subsystems of L and R.
- These correlations do not exist at earlier or later times.

The Holographic Dual

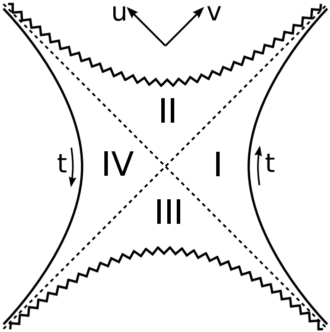
- Consider the thermofield double state on a pair of 1 + 1 dimensional CFTs.
- Holographic dual is the BTZ black hole in 2 + 1 dimensions:

$$ds^2 = -\frac{r^2 - R^2}{l^2}dt^2 + \frac{l^2}{r^2 - R^2}dr^2 + r^2 d\phi^2.$$

Kruskal Coordinates

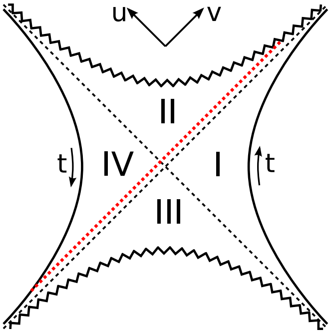
It is usually more convenient to use Kruskal coordinates:

$$ds^2 = \frac{-4l^2 du dv + R^2(1-uv)^2 d\phi^2}{(1+uv)^2}.$$

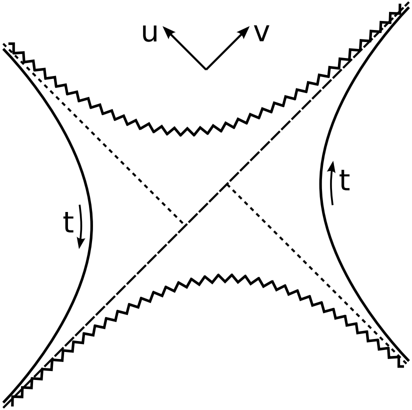


Adding a Perturbation

- We wish to add a small perturbation at an early time. We might expect to ruin the careful aim, destroying the special entanglement structure at $t = 0$.
- But what happens in the bulk?



Perturbed BTZ



$$ds^2 = \frac{-4l^2 du dv + R^2 \left(1 - u(v + \alpha\theta(u))\right)^2 d\phi^2}{(1 + u(v + \alpha\theta(u)))}, \qquad \alpha = \frac{E}{4M} e^{Rt_w/l^2}.$$

Extremal Surfaces in BTZ

- **Advantage of working in 2 + 1 dimensions:** Extremal surfaces are simply geodesics.
- **Advantage of working in (rotating) BTZ:** BTZ is just AdS in disguise. We can work in the AdS embedding coordinates and use the formula for geodesic length:

$$\cosh \frac{d}{l} = T_1 T_1' + T_2 T_2' - X_1 X_1' - X_2 X_2'.$$

Geodesics Lengths

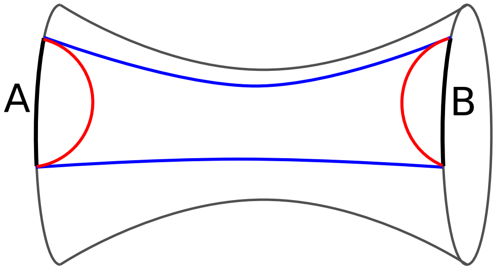
- Between two points on the same boundary, angular separation ϕ:

$$\frac{d}{l} = 2\log\frac{2r}{R} + 2\log\sinh\frac{R\phi}{2l}.$$

- Between matching points on opposite boundaries, $\phi = 0$:

$$\frac{d}{l} = 2\log\frac{2r}{R} + 2\log\left(1+\frac{\alpha}{2}\right).$$

Which Geodesics Do We Want?



- S_A is the length of the red geodesic to the left.
- $S_{A \cup B}$ may be the combined length of the red geodesics, or the blue ones, whichever is smaller.
- Remember, mutual information is

$$I(A;B) = S_A + S_B - S_{A \cup B}.$$

Mutual Information and Local Entanglement

Hence

$$I(A;B) = \max\left(\frac{l}{G_N}\left(\log\sinh\frac{R\phi}{2l} - \log\left(1+\frac{\alpha}{2}\right)\right), 0\right).$$

- Without the perturbation, there is local entanglement provided ϕ is large enough.
- Remember that α depends exponentially on the time since the perturbation was introduced.
- Any local entanglement structure is destroyed by an arbitrarily small perturbation, provided it occurs far enough in the past.

Rotating BTZ

$$ds^2 = -(N^{\perp})^2(r)dt^2 + f^{-2}(r)dr^2 + r^2\left[N^{\phi}(r)dt + d\phi\right]^2$$

where

$$\begin{aligned} N^{\perp}(r) &= f(r)N(r) \\ N(r) &= N(\infty) \\ N^{\phi} &= -\frac{J}{2r^2}N(\infty) + N^{\phi}(\infty) \end{aligned}$$

and

$$f^2(r) = -M + \left(\frac{r}{l}\right)^2 + \frac{J^2}{4r^2}.$$

The rotating BTZ solution will have two horizons, at $r = r_+$ and $r = r_-$.

Kruskal Coordinates

We can convert to Kruskal-like coordinates in the usual way, though these are only valid for $r > r_-$:

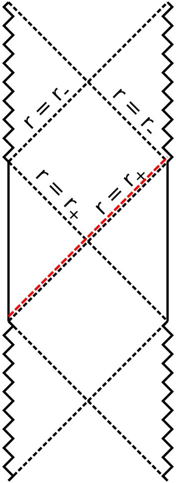
$$ds^2 = \frac{-4l^2 dUdV \mp 4lr_-(UdV - VdU)d\phi + \left[(1-UV)^2 r_+^2 + 4UVr_-^2\right] d\phi^2}{(1+UV)^2}.$$

It is possible to find an alternative set of Kruskal coordinates for $r < r_+$.

Adding the Perturbation

- We can confirm that curves of constant ϕ and U (or V) are geodesics.
- The perturbation now adds both angular momentum and energy.
- Glue two copies of the spacetime as before and take the same limit.
- We get a step change in the V coordinate across the shock, of size

$$\alpha = \frac{E}{4M}\frac{r_+^2(r_+^2+r_-^2)}{(r_+^2-r_-^2)^2}\exp\left(\frac{r_+^2-r_-^2}{l^2 r_+}t_w\right) - \frac{L}{2J}\frac{r_+^2 r_-^2}{(r_+^2-r_-^2)^2}\exp\left(\frac{r_+^2-r_-^2}{l^2 r_+}t_w\right).$$



Geodesic Lengths

- Between two points on the same boundary:

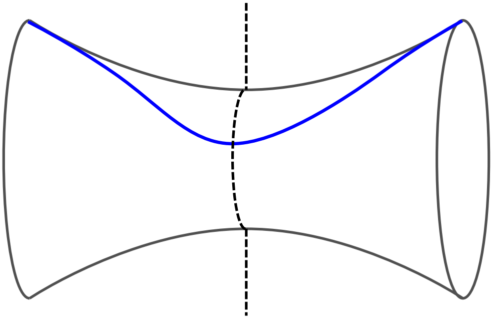
$$\frac{d}{l} = 2\log\frac{2r}{\sqrt{r_+^2 - r_-^2}} + \log\sinh\frac{(r_+ + r_-)\,|\phi|}{2l} + \log\sinh\frac{(r_+ - r_-)\,|\phi|}{2l}.$$

- Between matching points on opposite boundaries:

$$\frac{d}{l} = 2\log\frac{2r}{\sqrt{r_+^2 - r_-^2}} + p(\alpha),$$

where $p(\alpha)$ is positive and increases with α, seemingly without limit.

An Angular Diversion



Mutual Information and Local Entanglement

We find that

$$I(A;B) = \max\left(\frac{l}{2G_N}\left(\log\sinh\frac{(r_+ + r_-)\phi}{2l} + \log\sinh\frac{(r_+ - r_-)\phi}{2l} - p(\alpha)\right), 0\right).$$

- Similar qualitative features as before.
- Any local entanglement can be destroyed if the (small) perturbation occurs early enough.
- But for black holes close to extremality, there may be no local entanglement to begin.

Conclusions

- The thermofield double state has unusual local entanglement structure.
- This entanglement structure can be destroyed by the addition of a small perturbation at early time.
- In the dual, this is seen as the back reaction of the blue-shifted perturbation.
- Geodesics across the perturbation increase in length, reducing mutual information.