# AdS/CFT calculations of meson decay rates

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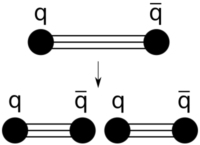
# Introduction to Meson Decay: QCD Picture

- Meson decay may be seem as $q\bar{q}$ pair production from a colour field flux tube



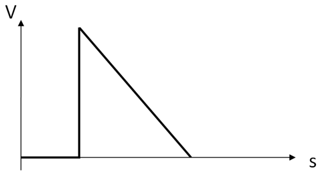

- May use Schwinger formula [1] to calculate decay rate $\Gamma$ for volume $V$ and energy density $\epsilon$:

$$\begin{aligned}\Gamma &= 2\mathrm{Im}\,\epsilon \\ &= -\frac{2}{V}\,\mathrm{Im}\,\ln\int \mathrm{d}X\,\mathrm{e}^{S}\end{aligned}$$

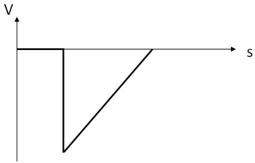
[1] Julian Schwinger. On gauge and vacuum polarizaion. *Physical Review*, 82(5):664, June 1951.

# Introduction to Meson Decay: Instanton Method

- After the new $q\bar{q}$ pair is produced it must gain sufficient energy from the field to come on shell. This may be seen as a tunneling process:




- By Wick rotating the time coordinate we flip the potential:




- The calculation may now be done semi-classically.

# Instaton Method: Point Particle Pair Production

- Consider Minkowski point particle action:

$$S_M = \int \mathrm{d}\tau \, \left( \frac{1}{2} \frac{\dot{X}^2}{e} - \frac{1}{2} e m^2 + A_\mu \dot{X}^\mu \right)$$

- Wick rotate to Euclidean spacetime:

$$\tau \to -i\tau$$
$$X^0 \to -iX^0$$
$$A^0 \to -iA^0$$

- Set periodic boundary condition:

$$X^\mu(\tau + 1) = X^\mu(\tau)$$

- Find Euclidean action:

$$S_E = -\int_0^1 \mathrm{d}\tau \, \left( \frac{\dot{X}^2}{4T} + m^2 T - iA_\mu \dot{X}^\mu \right)$$

# Instaton Method: Point Particle Pair Production

- Recall $A_\mu = -\frac{1}{2} F_{\mu\nu} X^\nu$
- Find Euler-Lagrange equation for $T$ and eliminate it from action:

$$T = \frac{\sqrt{\dot{X}^2}}{2m} \qquad \Rightarrow \qquad S_E = -\int_0^1 \mathrm{d}\tau \left( m\sqrt{\dot{X}^2} + \frac{i}{2} F_{\mu\nu} X^\nu \dot{X}^\mu \right)$$

- Find Euler-Lagrange equation for $X_\mu$:

$$iF_{\nu\mu}\dot{X}^\nu = \frac{2\ddot{X}_\mu}{\sqrt{\dot{X}^2}} - \frac{2\dot{X}_\mu}{\left(\dot{X}^2\right)^{\frac{3}{2}}} \dot{X}^\nu \ddot{X}_\nu + iF_{\mu\nu}\dot{X}^\nu$$

# Instaton Method: Point Particle Pair Production

- Choose constant field:

$$F_{01} = -F_{10} = -iE$$

- Problem admits the solution:

$$X_\mu = R \begin{pmatrix} \cos(2\pi n\tau) \\ sin(2\pi n\tau) \\ 0 \\ 0 \end{pmatrix}$$

- Subsitute $X_\mu$ into action and extremise value of $R$
- Action evaluates to expected result [2]:

$$S_E = -\frac{\pi m^2}{E} n$$

[2] Gordon W Semenoff and Konstantin Zarembo. Holographic schwinger effect, September 2011. arXiv:1109.2920 [hep-th].

# AdS/CFT Correspondence

- 't Hooft suggested that that for large $N_C$ QCD is equivalent to theory of free strings [3]
- Seems to violate Weinberg-Witten theorem–which forbids the existance of gravitons in QCD [4]
- Maldacena suggested that a QFT in $D$ dimensions corresponds to string theory in $D+1$ dimensions. [5]
- Good correspondence between $\mathcal{N}=4$ super Yang-Mills theory and a string theory in the $AdS_5 \times S^5$

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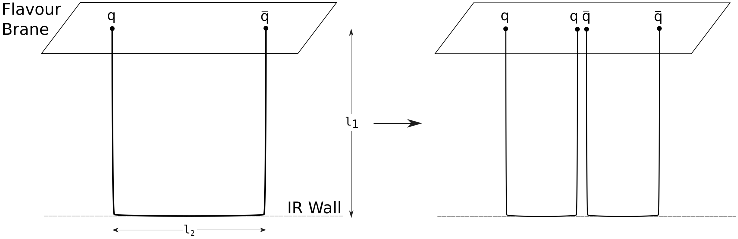
[3] Gerard 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 73:461, 1974.

[4] Steven Weinberg & Edward Witten. Limits on massless particles. *Nuclear Physics B*, 96(1-2):59, 1980.

[5] Juan Maldacena. The large-N limit of superconformal field theories and supergravity.*International Journal of Theoretical Physics*, 38(4):1113, 1999.

# Meson Decay in the Holographic Picture

- We will eventually want to work in AdS spacetime with a string of the following profile

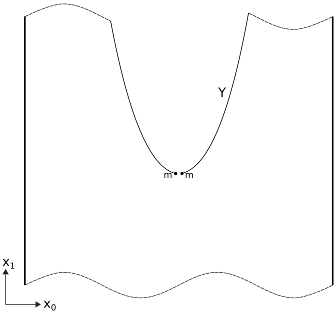



$l_1 \propto$ constituent quark mass $\qquad l_2 \propto$ colour flux tube energy

- Will build up to this using simpler examples
- Will work semi-classically using istanton method with a Wick rotated time coordinate

# Instaton Method: Setting up the Problems

- Consider string with massive endpoints in Eulidean spacetime:




- The Lagrangian for the string with massive endpoints is:

$$
\begin{aligned}
S_E &= \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma\, \mathcal{L}_{bulk} + \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \mathcal{L}_{end} \\
&= -\gamma \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma \sqrt{-\left(\dot{X}\cdot X'\right)^2 + \dot{X}^2 X'^2} \\
&\quad - m \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\sqrt{\dot{X}^2(\tau,\sigma=0)} + \sqrt{\dot{X}^2(\tau,\sigma=\pi)}\right)
\end{aligned}
$$

# Instaton Method: Equations of Motion

- Find variation of action:

$$
\begin{aligned}
0 = \delta S = &- \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma \left( \frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}_{bulk}}{\partial \dot{X}_\mu} \right) + \frac{\partial}{\partial \sigma} \left( \frac{\partial \mathcal{L}_{bulk}}{\partial X'_\mu} \right) \right) \delta X(\tau, \sigma) \\
&- \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left( \frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}_{end,\,\sigma=0}}{\partial \dot{X}_\mu} \right) + \frac{\partial \mathcal{L}_{bulk}}{\partial X'{}_\mu} \right) \Bigg|_{\sigma=0} \delta X_\mu(\tau, \sigma = 0) \\
&- \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left( \frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}_{end,\,\sigma=\pi}}{\partial \dot{X}_\mu} \right) - \frac{\partial \mathcal{L}_{bulk}}{\partial X'{}_\mu} \right) \Bigg|_{\sigma=\pi} \delta X_\mu(\tau, \sigma = \pi)
\end{aligned}
$$

- Find bulk equation of motion ($0 < \sigma < \pi$):

$$
\frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}_{bulk}}{\partial \dot{X}_\mu} \right) + \frac{\partial}{\partial \sigma} \left( \frac{\partial \mathcal{L}_{bulk}}{\partial X'_\mu} \right) = 0
$$

# Instaton Method: Equations of Motion

- Set parametrisation such that $\dot{X} \cdot X' = 0$ and $\dot{X}^2 = X'^2$ to get:

$$\frac{\partial \mathcal{L}_{bulk}}{\partial \dot{X}^\mu} = (-\gamma)\frac{X'^2 \dot{X}_\mu}{\sqrt{X'^2 \dot{X}^2}} = (-\gamma)\sqrt{\frac{X'^2}{\dot{X}^2}}\dot{X}_\mu = (-\gamma)\dot{X}_\mu$$

$$\frac{\partial \mathcal{L}_{bulk}}{\partial X'^\mu} = (-\gamma)\frac{\dot{X}^2 X'_\mu}{\sqrt{X'^2 \dot{X}^2}} = (-\gamma)\sqrt{\frac{\dot{X}^2}{X'^2}}X'_\mu = (-\gamma)X'_\mu$$

- This gives

$$\ddot{X}_\mu + X''_\mu = 0$$

# Instaton Method: Boundary Condtions

- Find boundary equations of motions:

$$\left(\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}_{end,\,\sigma=0}}{\partial\dot{X}_\mu}\right)+\frac{\partial\mathcal{L}_{bulk}}{\partial X'_\mu}\right)\Bigg|_{\sigma=0}\delta X_\mu(\tau,\sigma=0)=0$$
$$\left(\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}_{end,\,\sigma=\pi}}{\partial\dot{X}_\mu}\right)-\frac{\partial\mathcal{L}_{bulk}}{\partial X'_\mu}\right)\Bigg|_{\sigma=\pi}\delta X_\mu(\tau,\sigma=\pi)=0$$

- Before the split, we may choose both endpoints to have Dirichlet boundary condition:

$$\delta X_\mu=\frac{\partial X_\mu}{\partial\tau}\Big|_{\sigma=0}=0\qquad \delta X_\mu=\frac{\partial X_\mu}{\partial\tau}\Big|_{\sigma=\pi}=0$$

- After the split, the new endpoints must satisfy the moving endpoint conditions:

$$\frac{m^2}{\gamma}\ddot{X}_\mu=X'_\mu\qquad\text{and}\qquad \dot{X}^2=\frac{1}{m^2}$$

# Instaton Method: General Solution

- Before the split, we may choose

$$X_\mu^{in} = \frac{1}{m}\begin{pmatrix}\tau\\ \sigma\\ 0\\ 0\end{pmatrix}$$

- After the split we retain $X_0^{out} = \frac{1}{m}\tau$ but wish to add time evolution to $X_1$ such that

$$X_1^{out} = \frac{1}{m}\left[\alpha + \beta\tau + i\sum_{n\in\mathbb{Z}}\left(\alpha_n e^{\omega_n(-\tau+i\sigma)} + \tilde{\alpha}_n e^{\omega_n(-\tau-i\sigma)}\right)\right]$$

# Instaton Method: Applying Boundary Conditions

- Boundary conditions give

$$\beta = 0$$
$$\alpha_n + \tilde{\alpha}_n = 0$$
$$\tan(\omega_n \sigma^*) = -\frac{\gamma}{m^2}\frac{1}{\omega_n}$$

  where $\sigma = \sigma^* = a\pi$ is the new endpoint

- Taking semi-classical approximation $L \gg \sqrt{\alpha}$, we find

$$\cot \omega_n \sigma^* = 0$$
$$\omega_n = \frac{2n+1}{2a}$$

# Instaton Method: Applying Initial Conditions

- We rescale such that

$$\tilde{\sigma} = \frac{\sigma}{a} \qquad \tilde{\tau} = \frac{\tau}{a}$$

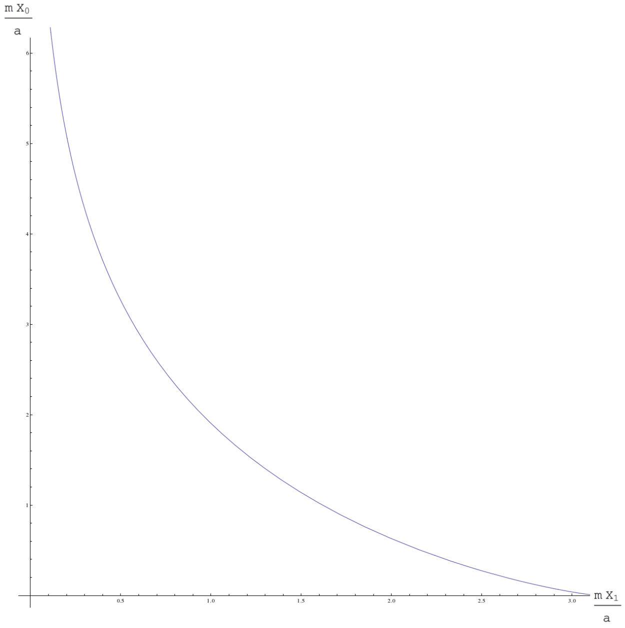
- We then apply the intial condition

$$X_1^{in}(\tilde{\tau} = 0, \tilde{\sigma}) = X_1^{out}(\tilde{\tau} = 0, \tilde{\sigma})$$
$$a\tilde{\sigma} = \alpha + i \sum_{n \in \mathbb{Z}} \alpha_n \left( e^{+i\frac{(2n+1)}{2}\tilde{\sigma}} - e^{-i\frac{(2n+1)}{2}\tilde{\sigma}} \right)$$

to find

$$X_\mu^{out} = \frac{1}{m} \begin{pmatrix} \tau \\ \frac{8\sigma^*}{\pi^2} \sum_{n \geqslant 0} \frac{(-1)^n}{(2n+1)^2} e^{-\frac{2n+1}{2}\frac{\tilde{\tau}}{\sqrt{a}}} \sin\left(\frac{2n+1}{2}\tilde{\sigma}\right) \\ 0 \\ 0 \end{pmatrix}$$

# Instaton Method: Motion of String



# Instaton Method: Evaluating Action

- We find the action evaluates to

$$S = -\frac{\gamma}{m^2}\int_0^{\tilde{\tau}_2} \mathrm{d}\tilde{\tau} \int_0^{\pi} \mathrm{d}\tilde{\sigma}\, \frac{4a^{\frac{3}{2}}}{\pi} \left| \sum_{n\geqslant 0} \frac{(-1)^n}{2n+1} e^{-\frac{2n+1}{2}\frac{\tilde{\tau}}{\sqrt{a}}} \cos\left(\frac{2n+1}{2}\tilde{\sigma}\right) \right|$$

$$-\sqrt{a}\int_0^{\tilde{\tau}_2} \mathrm{d}\tilde{\tau} \left(1 + \sqrt{1 + \frac{16}{\pi^2}\left(\sum_{n\geqslant 0} \frac{1}{2n+1} e^{-\frac{2n+1}{2}\frac{\tilde{\tau}}{\sqrt{a}}}\right)^2}\right)$$

# Instanton Method: Return to point particle system

- The problem may also be done in the static gauge:

$$X_0 = t \qquad X_1 = x \qquad\qquad A_0 = iEx \qquad A_1 = 0$$

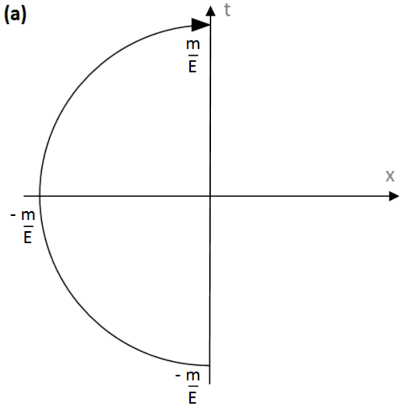
- The Lagrangian reduces to

$$L = m\sqrt{1 + \dot{x}^2} + Ex$$

- We obtain the same circular path

$$x = \pm\frac{1}{E}\sqrt{m^2 - E^2 t^2}$$

# Instanton Method: Return to point particle system



# Instanton Method: Applying conditions

- For the string, we used the condition

$$\frac{\mathrm{d}X_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \frac{1}{m^2}$$

  where $\tau$ is the proper time.
- For the Euclidean point particle we find

$$\begin{aligned}
\mathrm{d}\tau^2 &= m^2\left(\mathrm{d}t^2 + \mathrm{d}x^2\right) \\
\tau &= m\int\sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\mathrm{d}t \\
\tau &= m^2\int\frac{1}{\sqrt{m^2 - E^2t^2}}\mathrm{d}t
\end{aligned}$$

# Instanton Method: Applying conditions

- We therefore find

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\sqrt{m^2 - E^2 t^2}}{m^2}$$

- Therefore

$$\frac{\mathrm{d}X_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \left(\left(\frac{\mathrm{d}t}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

$$= \left(\frac{E^2 t^2}{m^2 - E^2 t^2} + 1\right)\frac{m^2 - E^2 t^2}{m^4} = \frac{1}{m^2}$$

# Instanton Method: Limits of integration

- We may wish to consider the potential of the system
- We find that

$$\frac{\partial V}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = E$$
$$V = Ex$$

- Not certain if this method would be useful for the string.

# Instanton Method: Limits of integration

- Alternate method is to consider when particle comes on shell, that is

$$p^\mu p_\mu = m^2$$

- Then

$$p_0 = \frac{\partial L}{\partial \dot{X}_0} = \frac{m\dot{X}_0^2}{\dot{X}_0^2 + \dot{X}_1^2} - iA_0 = \frac{m}{\sqrt{1+\dot{x}^2}} + Ex$$
$$p_1 = \frac{\partial L}{\partial \dot{X}_1} = \frac{m\dot{X}_1^2}{\dot{X}_0^2 + \dot{X}_1^2} - iA_1 = \frac{m\dot{x}}{\sqrt{1+\dot{x}^2}}$$

- Therefore

$$p^\mu p_\mu = E^2 t^2$$

which is equal to $m^2$ at $t = \frac{m}{E}$, as expected.

# Motivating Further Work

- Appropriate metric[6] is:

$$\mathrm{d}s^2 = Y_\mu Y^\mu = \frac{U^{\frac{3}{2}}}{{R_0}^{\frac{3}{2}}}\left(\mathrm{d}X^\mu \mathrm{d}X^\nu \eta_{\mu\nu} + f(U)\mathrm{d}\psi^2\right) + K(U)\left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}{\Omega_4}^2\right)$$

$$= \frac{U^{\frac{3}{2}}}{{R_0}^{\frac{3}{2}}}\left(\mathrm{d}X^\mu \mathrm{d}X^\nu \eta_{\mu\nu} + f(U)\mathrm{d}\psi^2\right) + K(U)\left(\mathrm{d}\lambda^2 + \lambda^2 \mathrm{d}{\Omega_2}^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2\right),$$

where

$$f(U) = 1 - \frac{{U_\Lambda}^3}{U^3}$$

$$U(\rho) = \left(\rho^{\frac{3}{2}} + \frac{{U_\Lambda}^3}{4\rho^{\frac{3}{2}}}\right)^{\frac{2}{3}}$$

$$K(U) = {R_0}^{\frac{3}{2}} U^{\frac{1}{2}} \rho^{-2},$$

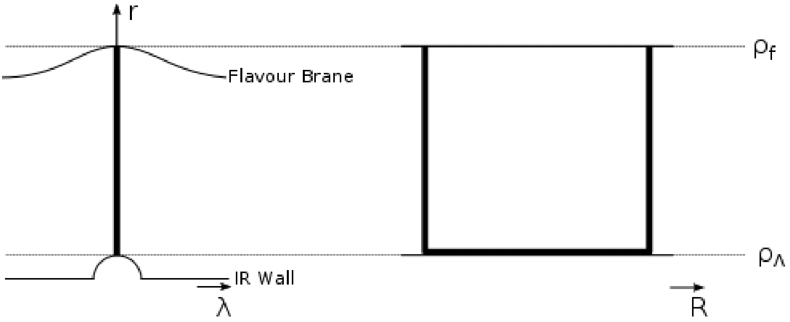
and, after Wick rotating the time coordinate,

$$\mathrm{d}X^\mu \mathrm{d}X^\nu \eta_{\mu\nu} = \left(\mathrm{d}X^0\right)^2 + \mathrm{d}R^2 + R^2 \mathrm{d}\theta^2 + \left(\mathrm{d}X^3\right)^2.$$

[6] Martin Kruczenski, Leopoldo A. Pando Zayas, Jacob Sonnenschein, and Varman Diana. Regge trajectories for mesons in the holographic dual of large-$N_c$ QCD. *Journal of High Energy Physics*, 2005(06):046, June 2005. arXiv:hep-th/0410035.

# Motivating Further Work

- Use string profile[7]:




[7]Kasper Peeters, Jacob Sonnenschein, and Marija Zamaklar. Holographic decays of large spin mesons. JHEP, 0602:009, 2005. arXiv:hep-th/0511044.