Many Moonshines: Monstrous, Mathieu and (M)Umbral

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May 2\textsuperscript{nd}, 2016
'The energy produced by the breaking down of the atom is a very poor kind of thing. Anyone who expects a source of power from the transformation of these atoms is talking *moonshine*.'

*Ernest Rutherford (1937), The Wordsworth Book of Humorous Quotations*
Outline

1. Introduction
2. Monstrous Moonshine
3. The Elliptic Genus of K3
4. $G_{24}$ and $M_{24}$
5. Mathieu Moonshine
6. Niemeier Lattices
7. Umbral Moonshine
Outline

1. Introduction

2. Monstrous Moonshine

3. The Elliptic Genus of K3

4. $G_{24}$ and $M_{24}$

5. Mathieu Moonshine

6. Niemeier Lattices

7. Umbral Moonshine
Here we present moonshine in the context of string theory, where the one-loop partition function has a world sheet with the topology of a torus.

\[ Z(\tau) = \text{Tr}_{\text{CFT}} \left[ q^{L_0 - \frac{c}{24}} \right] = e^{2\pi i \tau}. \]

The torus is described by a complex parameter \( \tau \) in the upper half plane. Two tori with moduli \( \tau, \tau' \) are conformally equivalent if their moduli are related by

\[ \tau' = a\tau + b \quad \text{and} \quad c\tau + d, \quad ad - bc = 1. \]

Therefore,

\[ Z(\gamma \tau) = Z(\tau), \quad \gamma \in \text{SL}_2(\mathbb{Z}). \]
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$$196884 = 196883 + 1$$

This function is known as the Hauptmodul for the 'genus 0' group $SL(2,\mathbb{Z})$; all modular functions for $SL(2,\mathbb{Z})$ are rational polynomials in $j(\tau)$. The numbers on the right are dimensions of irreducible representations of the Monster group $M$, the largest finite sporadic group of order $\approx 8 \times 10^{53}$. 
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\[ j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots \] (2)
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A Monster Module

The previous equalities suggest the existence of a graded Monster module $V$

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus \ldots$$

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Since the dimension of a representation \( \rho \) is given by the character of the identity, \( \chi_\rho(e) = Tr(\rho(e)) = \dim(\rho) \), we may write this as

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Written in this form it is natural to consider the \textit{McKay - Thompson series}

$$T_g(\tau) = \chi_{V_{-1}}(g)q^{-1} + \sum_{i=1}^{\infty} \chi_{V_i}(g)q^i$$  \hspace{1cm} (6)
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In physical terms, they constructed a 2d chiral CFT (Monster CFT) from bosonic strings on an orbifold of the Leech Lattice (even self-dual) torus.

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They showed that this module has automorphism group $\mathbb{M}$ and graded dimension $j(\tau) - 744$. That is, the CFT has $j(\tau) - 744$ as the partition function and has $\mathbb{M}$ symmetry.

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Borcherds\(^3\) showed that the graded characters of \(V^\natural, T_g(\tau)\) are the Hauptmoduls identified by Conway and Norton.

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- There are many different Calabi-Yau 3-folds, but the only Calabi-Yau two folds are complex tori and K3 surfaces.
- Alvarez-Gaumé and Freedman\(^4\) showed that a sigma model on a hyperkähler manifold has \(\mathcal{N} = 4\) symmetry (\(K3\) is hyperkähler).

The Partition Function and the Elliptic Genus

Mathieu Moonshine appears when considering the 2d conformal field theory on the worldsheet describing strings propagating on $K3$. 

The partition function is given by:

$$Z(\tau, z; \bar{\tau}, \bar{z}) = \text{Tr} \left[ H q^L_0 - c/24 \bar{y}^2 J_3^0 \bar{y}^2 q \right] = e^{2\pi i \tau}, y = e^{2\pi iz}.$$ (7)

Although the partition function is an important quantity containing the information about all states, it depends on where we are in the (80-dimensional) moduli space of $K3$. Too complicated to calculate at generic points in $M_{K3}$. For some purposes it is convenient to consider a related quantity known as the Elliptic Genus. This is moduli space independent:

$$\varepsilon_M(\tau, z) := Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z} = 0)$$ (8)
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We begin by considering the partition function:

**Definition**

The partition function for an $\mathcal{N} = (4, 4)$ theory is given by

$$Z(\tau, z; \bar{\tau}, \bar{z}) = \text{Tr}_H q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} y^{2J_0^3} \bar{y}^{2\bar{J}_0^3} q = e^{2\pi i \tau}, y = e^{2\pi iz}. \quad (7)$$
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\varepsilon_{K3}(\tau, z) := \text{Tr}_{\mathcal{H}^R} \left( (-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0} - \frac{\bar{c}}{24} y^2 J_0^3 \right)
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The elliptic genus of $K3$ can be shown\(^6\) to be a weak Jacobi form of weight 0 and index 1.

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This is independent of \( \tilde{q} \), becomes the Witten Index on the right\(^5\). The elliptic genus of K3 can be shown\(^6\) to be a weak Jacobi form of weight 0 and index 1. This space is one dimensional and so we have

\[
\varepsilon_{K3} = 8 \left[ \frac{\theta_2(\tau, z)^2}{\theta_2(\tau)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau)^2} \right]
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\(^6\)Wendland, “Snapshots of Conformal Field Theory”.

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$\varepsilon$ is a topological invariant and can be related to other invariants,

$$\varepsilon_{K3}(\tau, z = 0) = \chi(K3) = 24$$  \hspace{1cm} (11)

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$^6$Wendland, “Snapshots of Conformal Field Theory”.
The Elliptic Genus in $\mathcal{N} = 4$ Characters

Alvarez-Gaumé and Freedman\textsuperscript{7} showed that a sigma model on a hyperkähler manifold has $\mathcal{N} = 4$ symmetry.

\begin{equation}
\varepsilon_{K^3}(\tau, z) = 24 \text{ch} \sum_{l=0}^{\infty}(\tau, z) + \sum_{\tau} q^{1/8} \text{ch} \sum_{l=1}^{\infty}(\tau, z) (12)
\end{equation}

\begin{equation}
\sum(\tau) = q^{-1/8}(-2 + \sum_{n=1}^{\infty} A_n q^n) (13)
\end{equation}

\begin{equation}
\sum(\tau) = q^{-1/8}(-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + \ldots) (14)
\end{equation}

\textsuperscript{7}Alvarez-Gaume and Freedman, “Geometrical structure and ultraviolet finiteness in the supersymmetric $\sigma$-model”.

The Elliptic Genus in $\mathcal{N} = 4$ Characters

Alvarez-Gaumé and Freedman\(^7\) showed that a sigma model on a hyperkähler manifold has $\mathcal{N} = 4$ symmetry.

In terms of $\mathcal{N} = 4$ characters we can expand the elliptic genus as

$$\varepsilon_{K3}(\tau, z) = 24\text{ch}_{l=0}(\tau, z) + \Sigma(\tau)q^{\frac{1}{8}}\hat{\text{ch}}_{l=1/2}(\tau, z)$$  \hspace{1cm} (12)

where

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\Sigma(\tau) = q^{-\frac{1}{8}} (-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + \ldots)
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A sporadic group

Theorem

The Classification of Finite Simple Groups.
This theorem states that all finite simple groups fall into one of the following families:

1. Cyclic groups of order $n$ for $n$ prime.
2. Alternating groups of degree at least 5.
3. Simple Lie type groups.
4. The 26 sporadic simple groups.

M$_{24}$ is one of the sporadic finite simple groups. It is a subgroup of the Monster group $M$, as shown below.
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This theorem states that all finite simple groups fall into one of the following families:

1. Cyclic groups of order n for n prime.
2. Alternating groups of degree at least 5.
3. Simple Lie type groups.
4. The 26 sporadic simple groups.

M\textsubscript{24} is one of the sporadic finite simple groups. It is a subgroup of the Monster group M, as shown below.

[Diagram of group relationships]
The Golay Code

Linear codes are linear subspaces of vector spaces over finite fields.
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There is a unique $[24, 12, 8]$ code up to equivalency, $G_{24}$. This code is known as the Extended Binary Golay Code.
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Lexicographic Code

$$c_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$
$$c_1 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$
$$c_2 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$
$$c_3 = (0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1)$$
...
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The Mathematical Game of Mogul

H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H H

H H T H H T H T H T H H H H H H H H H H H H H H H H H H H H T H T H T H
The Golay Code

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There are numerous ways to define $G_{24}$.

The Golay code was used to transmit photos back from the Voyager spacecraft.
We can define $M_{24}$ in many different ways, however one that suits us is the following.

**Definition**

\[
M_{24} := \text{Aut}(G_{24})
\]  

*That is, $M_{24} = \{ \tau \in S_{24} \mid \tau(c) \in G_{24} \quad \forall c \in G_{24} \}$*
We can define $M_{24}$ in many different ways, however one that suits us is the following.

**Definition**

\[ M_{24} := \text{Aut}(G_{24}) \]  

That is, $M_{24} = \{ \tau \in S_{24} | \tau(c) \in G_{24} \ \forall c \in G_{24} \}$

$M_{24}$ has order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244823040$
Outline

1. Introduction
2. Monstrous Moonshine
3. The Elliptic Genus of K3
4. $G_{24}$ and $M_{24}$
5. Mathieu Moonshine
6. Niemeier Lattices
7. Umbral Moonshine
### $M_{24}$ Representation Theory

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Mathieu Moonshine

We showed that the elliptic genus of $K3$ can be written as

$$
\varepsilon_{K3}(\tau, z) = 24 \text{ch}_{l=0}^\mathcal{R}(\tau, z) + \hat{\text{ch}}_{l=1/2}^\mathcal{R}(\tau, z)
\cdot (-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + \ldots)
$$

(16)
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The twining elliptic genera for $g \in M_{24}$ have been studied (eg.\textsuperscript{8})

$$
\phi_g(\tau, z) = \text{ch}_{H_0}(g)\text{ch}_{1/4,0}^R + \sum_{n=0}^{\infty} \text{ch}_{H_n}(g)\text{ch}_{n+1/4,1/2}^R(\tau, z)
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(17)

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$$
\phi_g(\tau, z) = \text{ch}_{H_{00}}(g)\text{ch}_R^{1/4,0} + \sum_{n=0}^{\infty} \text{ch}_{H_n}(g)\text{ch}_R^{n+1/4,1/2}(\tau, z)
$$

and Gannon\textsuperscript{9} proved that all* $H_n$ are indeed representations of $M_{24}$.

\textsuperscript{8} Gaberdiel, Hohenegger, and Volpato, “Mathieu twining characters for K3”.

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Lattices - First Definitions

Here we will be interested only in real lattices, that is subgroups of $\mathbb{R}^n$.

**Definition**

- A *lattice* of dimension $n$ is a free $\mathbb{Z}$-module, $L_n$ with a symmetric bilinear form $\langle \cdot, \cdot \rangle$.
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**Definition**

- A *lattice* of dimension $n$ is a free $\mathbb{Z}$-module, $L_n$ with a symmetric bilinear form $\langle \cdot, \cdot \rangle$.
- A lattice is *integral* if $\langle \lambda, \mu \rangle \in \mathbb{Z}$, $\forall \lambda, \mu \in L_n$.
- A lattice is *even* if $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ $\forall \lambda \in L_n$.
- For an even lattice $L_n$, we define the set of roots to be those elements of norm 2.
- We call a matrix, $M$ whose rows are a basis for $L_n$ a generator matrix for $L_n$. Then elements $\lambda \in L_n$ may be written as $\lambda = \xi M$, $\xi \in \mathbb{Z} \geq n$.
- The Gram matrix, $A$ is given by $M \cdot M^\top$.
- The determinant of the Gram matrix is known as the determinant of $L_n$, $\text{det}(L_n) = \text{det}(A)$. 
Lattices - First Definitions

Here we will be interested only in real lattices, that is subgroups of $\mathbb{R}^n$.

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Unimodular Lattices

Definition

We define the dual of a lattice, $L^*_n = \{ \mu \in \mathbb{R}^n \mid \mu \cdot L_n \subseteq \mathbb{Z} \}$. 

If $M$ is the generating matrix for $L$ then $(M^{-1})^\top$ is a generating matrix for $L^*$ and $A^{-1}$ is the Gram matrix.

Clearly in an integral lattice we have $L_n \subseteq L^*_n$, if we have $L_n = L^*_n$ we say $L_n$ is self-dual or unimodular.

For an integral lattice with generating matrix $M$ we have, for $\mu \in L^*$,

$\mu = \xi (M^{-1})^\top = \xi (M^{-1})^\top M = \det(A^{-1}) \xi \text{adj}(A) M = \det(L) \xi \text{adj}(L) M$ 

hence we also have $L^*_n \subseteq \det(L) L_n$.

When $\det(L) = 1$ then $L$ is unimodular.
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The Leech lattice is the unique even, unimodular (Type II) lattice of dimension 24 without roots, vectors of norm 2. The only 16 dimensional Type II lattices are $E_8 \times E_8$ and the weight lattice of $\text{Spin}(32)/\mathbb{Z}_2$. Related to gauge groups for Heterotic string theory.
The Niemeier Lattices

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The Niemeier lattices can be constructed\(^\text{10}\) by *gluing* the root lattices using *glue vectors* whose components are given by elements of \(L^*/L\). We consider \((A^2_{24})^+\) and \((A^1_{12})^+\).

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The Niemeier Lattices

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\((A_{24}^+)^+\) is constructed using the Extended binary Golay code, \(G_{24}\) as glue. The minimum weight of \(G_{24}\) ensures this doesn’t add roots and maintains evenness.

\((A_{12}^+)^+\) is constructed using the Extended ternary Golay code, \(G_{12}\). \(G_{12}\) is a \([12, 6, 6]_3\) code. Therefore using this as glue doesn’t add roots and maintains evenness.

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Outline

1. Introduction
2. Monstrous Moonshine
3. The Elliptic Genus of K3
4. $G_{24}$ and $M_{24}$
5. Mathieu Moonshine
6. Niemeier Lattices
7. Umbral Moonshine
The Umbral Forms

Cheng, Duncan and Harvey\textsuperscript{11} described a particular way to associate to each Niemeier lattice a (vector-valued) mock modular form.

\begin{itemize}
  \item Begin with a weight 0, index \(m-1\) (holo.) weak Jacobi form \(\phi(\tau, z)\).
  \item Form a weight 1, index \(m\) Jacobi form as \(\psi(\tau, z) = \mu_{1, 0}(\tau, z) \phi(\tau, z)\) (18)
  \item Define the Polar part of \(\psi\) as \(\psi_P(\tau, z) = \chi_{m, 0}(19)\) where \(\chi = \phi(\tau, 0)\) is a weight 0 modular form and hence is constant.
\end{itemize}

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This construction, in the simplest cases of $A_{1}^{24}$ and $A_{2}^{12}$ may be summarised as follows:

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This construction, in the simplest cases of $A_{12}^{24}$ and $A_{22}^{12}$ may be summarised as follows:

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• Form a weight 1, index $m$ Jacobi form as

$$\psi(\tau, z) = \mu_{1,0}(\tau, z)\phi(\tau, z)$$  \hspace{1cm} (18)

where $\mu_{1,0}$ is a meromorphic Jacobi form of weight 1, index 1.

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Cheng, Duncan and Harvey\textsuperscript{11} described a particular way to associate to each Niemeier lattice a (vector-valued) mock modular form.

This construction, in the simplest cases of $A_{24}^1$ and $A_{12}^2$ may be summarised as follows:

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Umbral Forms Continued

- Now form the finite part of $\psi$ as

$$\psi^F(\tau, z) = \psi(\tau, z) - \psi^P(\tau, z)$$  \hspace{1cm} (20)

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$\psi^F$ is known as a mock Jacobi form of weight 1 and index $m$.

• It can be shown\(^{12}\) that such a form has a theta expansion given by

$$\psi^F(\tau, z) = \sum_{r=1}^{m-1} h_r(\tau) \hat{\theta}_{m,r}(\tau, z)$$  \hspace{1cm} (21)

the theta coefficients $h_r$ are the components of a vector-valued mock modular form of weight $\frac{1}{2}$.

The first few coefficients

The link to the Niemeier lattices $A_1^{24}$ and $A_2^{12}$, comes by taking $\phi$ to be an extremal Jacobi form of index the Coxeter number of the Niemeier root system.

We define $f_i(\tau, z) := \frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)}$ (22)

$\phi_1^{24} = 8(f_2^2 + f_2^3 + f_2^4), \quad (23)$

$\phi_1^{12} = 4(f_2^2 f_2^3 + f_2^3 f_2^4 + f_2^4 f_2^2). \quad (24)$

We find the Umbral Forms ($H_r$)

$H_2^1 = 2q^{-1}/8(-1 + 45q + 231q^2 + 770q^3 + \ldots) \quad (25)$

$H_3^1 = 2q^{-1}/12(-1 + 16q + 55q^2 + 144q^3 + \ldots) \quad (26)$

$H_3^2 = 2q^2/3(10 + 44q + 110q^2 + 280q^3 + \ldots) \quad (27)$
The first few coefficients

The link to the Niemeier lattices $A_{12}^{24}$ and $A_{22}^{12}$, comes by taking $\phi$ to be an extremal Jacobi form of index the Coxeter number of the Niemeier root system. A weak Jacobi form of weight 0, index $m$ is extremal if it admits a particular decomposition into $N = 4$ characters.
The first few coefficients

The link to the Niemeier lattices $A_1^{24}$ and $A_2^{12}$, comes by taking $\phi$ to be an extremal Jacobi form of index the Coxeter number of the Niemeier root system. A weak Jacobi form of weight 0, index $m$ is extremal if it admits a particular decomposition into $N = 4$ characters.

The spaces of extremal Jacobi forms of index 2, and index 3, are both known to be of dimension 1. We define

$$f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0)$$ (22)

$$\phi_1^2 = 8(f_2^2 + f_3^2 + f_4^2),$$ (23)

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For the Niemeier lattices the Automorphism group is given by the product $\bar{G}_X W X G_1$, where $\bar{G}_X$ is the group of permutations of the components of $X$ induced by automorphisms of $L^X$, and $G_1$ is given by $\hat{W}_X / W X$ where $\hat{W}_X$ is automorphisms of $L^X$ that stabilise the components of $X$. In the cases $A_{24}$ and $A_{12}$ we find $G_{A_{24}} = M_{24}$ (29) $G_{A_{12}} = 2^4 M_{12}$ (30).
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We define the Umbral group $G^X$ to be the corresponding quotient

$$G^X := \text{Aut}(L^X)/W^X \quad (28)$$
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For the Niemeier lattices the Automorphism group is given by the product $\tilde{G}^X W^X G_1$, where $\tilde{G}^X$ is the group of permutations of the components of $X$ induced by automorphisms of $L^X$, and $G_1$ is given by $\hat{W}^X/ W^X$ where $\hat{W}^X$ is automorphisms of $L^X$ that stabilise the components of $X$. 

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In the cases $A^2_{14}$ and $A^1_{12}$ we find

$$G^{A^2_{14}} = M_{24}$$

(29)

$$G^{A^1_{12}} = 2M_{12}$$

(30)
Umbral Moonshine

\[ \{g\} \]

<table>
<thead>
<tr>
<th>[g^2]</th>
<th>FS</th>
<th>1A 2A 4A 2B 2C 3A 6A 3B 6B 4B 4C 5A 10A 12A 6C 6D 8A 8B 8C 8D 20A 20B 11A 22A 11B 22B</th>
</tr>
</thead>
<tbody>
<tr>
<td>3A</td>
<td>1A 2A 4A 2B 2C 3A 6A 3B 6B 4B 4C 1A 2A 12A 6C 6D 8B 8A 8D 8C 4A 4A 11A 22A 11B 22B</td>
<td></td>
</tr>
<tr>
<td>6A</td>
<td>1A 2A 4A 2B 2C 3A 6A 3B 6B 4B 4C 5A 10A 12A 6C 6D 8A 8B 8C 8D 20B 20A 1A 2A 1A 2A</td>
<td></td>
</tr>
</tbody>
</table>

\[ \{g^3\} \]

\[ \{g^5\} \]

\[ \{g^{11}\} \]

\[ \chi_1 \]

\[ \chi_2 \]

\[ \chi_3 \]

\[ \chi_4 \]

\[ \chi_5 \]

\[ \chi_6 \]

\[ \chi_7 \]

\[ \chi_8 \]

\[ \chi_9 \]

\[ \chi_{10} \]

\[ \chi_{11} \]

\[ \chi_{12} \]

\[ \chi_{13} \]

\[ \chi_{14} \]

\[ \chi_{15} \]

\[ \chi_{16} \]

\[ \chi_{17} \]

\[ \chi_{18} \]

\[ \chi_{19} \]

\[ \chi_{20} \]

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(M)Umbral Moonshine

We see that the coefficients of the forms $H_1^2$, $H_1^3$ and $H_2^3$ are given by dimensions of representations of the Umbral groups $M_{24}$ and $2.M_{12}$ respectively.

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13 Cheng, Duncan, and Harvey, “Umbral moonshine and the Niemeier lattices”.

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(M)Umbral Moonshine

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A similar procedure can be used to find vector-valued mock modular forms associated to each of the 23 Niemeier lattices, each of which is found to have coefficients encoding dimensions of representations of the 23 Umbral groups as defined above.

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Umbral Moonshine conjectures that there exists a graded module $K^X$ associated to each Niemeier lattice $L^X$ such that the characters associated to elements $g \in G^X$ give the umbral forms $H_g^{X13}$.

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Hidden Physics?

Recall:

- Monstrous Moonshine was hidden in the partition function of a particular CFT
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Umbral moonshine can also be seen in terms of the elliptic genus of $K3$: Recall that we split the elliptic genus into massless and massive characters of $\mathcal{N} = 4$. We can instead split the elliptic genus into a part corresponding to some surface singularities of the $K3$ and the remaining ‘Moonshine’ part which encodes the moonshine form\(^\text{14}\).

Hidden Physics

Kachru et al.\textsuperscript{15} consider 3d gravity theories by for instance compactifying the Type II string on $K3 \times T^3$. The moduli space of such theories can be thought of as the space of 32-dimensional even unimodular lattices of signature (8,24). In a neighbourhood of some particular points in this moduli space the theory has Umbral symmetry.

Questions?
Mathieu and Monstrous Moonshine

- Both moonshines involve the representation theory of finite simple groups and objects with particular modular transformations.

- Monstrous Moonshine involved modular functions (in fact Hauptmodul) but Mathieu Moonshine (and Umbral Moonshine) involves mock-modular forms.

- Monstrous moonshine can be explained in terms of a string propagating on an orbifold of the 'Leech Torus' $\mathbb{R}^{24}/\Lambda$ where the $j$-invariant describes the partition functions for the theory. In Mathieu Moonshine we don’t consider the full partition function but the elliptic genus which only counts half BPS states (right moving ground states).
Mathieu and Monstrous Moonshine

- Both moonshines involve the representation theory of finite simple groups and objects with particular modular transformations.
- In both cases we have been able to learn more about the representations involved by twisting the functions involved.

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