Many Moonshines: Monstrous, Mathieu and (M)Umbral

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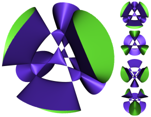

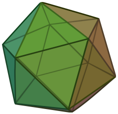

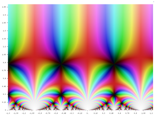

‘The energy produced by the breaking down of the atom is a very poor kind of thing. Anyone who expects a source of power from the transformation of these atoms is talking **moonshine**’

Ernest Rutherford (1937), The Wordsworth Book of Humorous Quotations

Outline

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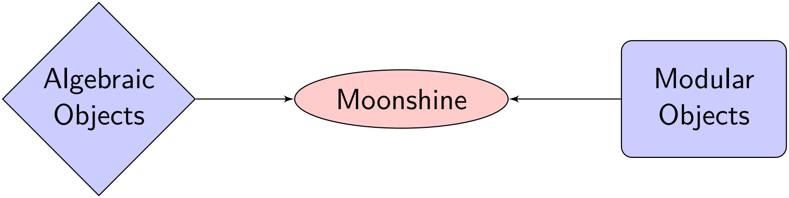

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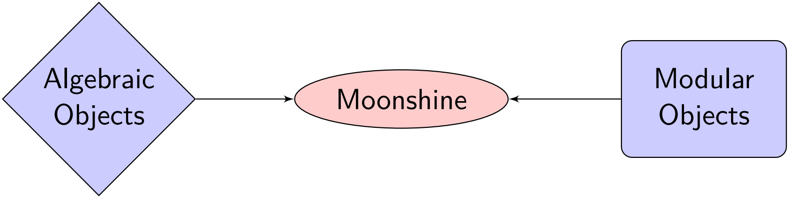

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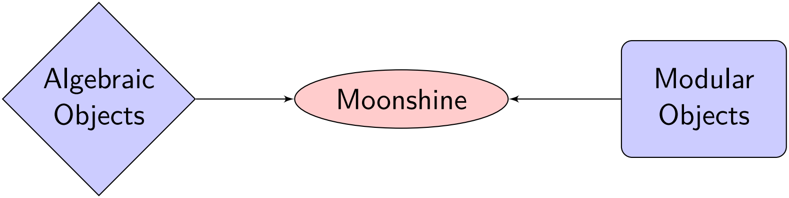

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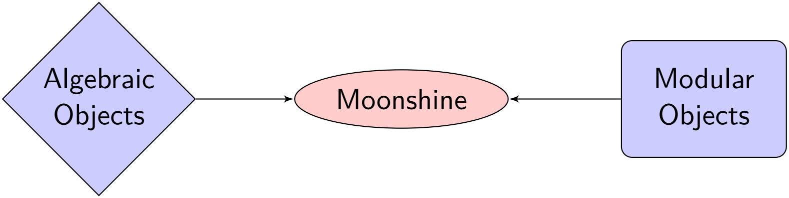

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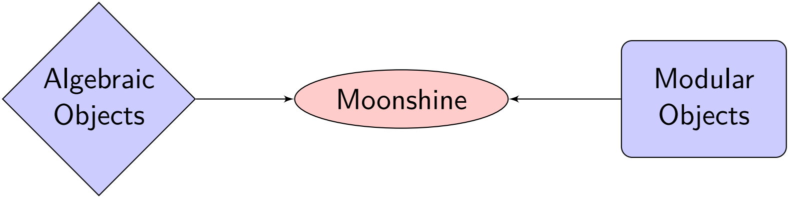

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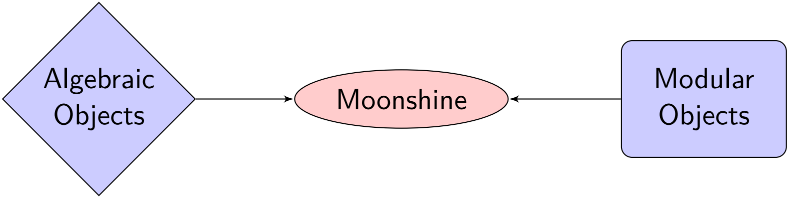

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Therefore,

$$Z(\gamma\tau) = Z(\tau), \qquad \gamma \in SL_2(\mathbb{Z})$$

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The numbers of the left hand sides are coefficients of the j-invariant, a modular function (form of weight 0, where we allow meromorphicity) for $SL(2, \mathbb{Z})$ with $q \equiv e^{2\pi i \tau}$ expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots \tag{2}$$

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The numbers on the right are dimensions of irreducible representations of the Monster group $\mathbb{M}$, the largest finite sporadic group of order $\approx 8 \times 10^{53}$.

A Monster Module

The previous equalities suggest the existence of a graded Monster module V

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus \dots \tag{3}$$

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Since the dimension of a representation ρ is given by the character of the identity, $\chi_\rho(e) = Tr(\rho(e)) = dim(\rho)$, we may write this as

$$j(\tau) - 744 = \chi_{V_{-1}}(e_{V_{-1}})q^{-1} + \chi_{V_1}(e_{V_1})q + \chi_{V_2}(e_{V_2})q^2 + \dots \tag{5}$$

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Written in this form it is natural to consider the *McKay - Thompson series*

$$T_g(\tau) = \chi_{V_{-1}}(g)q^{-1} + \sum_{i=1}^{\infty} \chi_{V_i}(g)q^i \tag{6}$$

Monstrous Moonshine

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[2] Igor B Frenkel, James Lepowsky, and Arne Meurman. "A natural representation of the Fischer-Griess Monster with the modular function J as character". In: *Proceedings of the National Academy of Sciences* 81.10 (1984), pp. 3256–3260.

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Borcherds[3] showed that the graded characters of $V^\natural$, $T_g(\tau)$ are the Hauptmoduls identified by Conway and Norton.

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[2] Frenkel, Lepowsky, and Meurman, "A natural representation of the Fischer-Griess Monster with the modular function J as character".

[3] Richard E Borcherds. "Monstrous moonshine and monstrous Lie superalgebras". In: *Inventiones mathematicae* 109.1 (1992), pp. 405–444.

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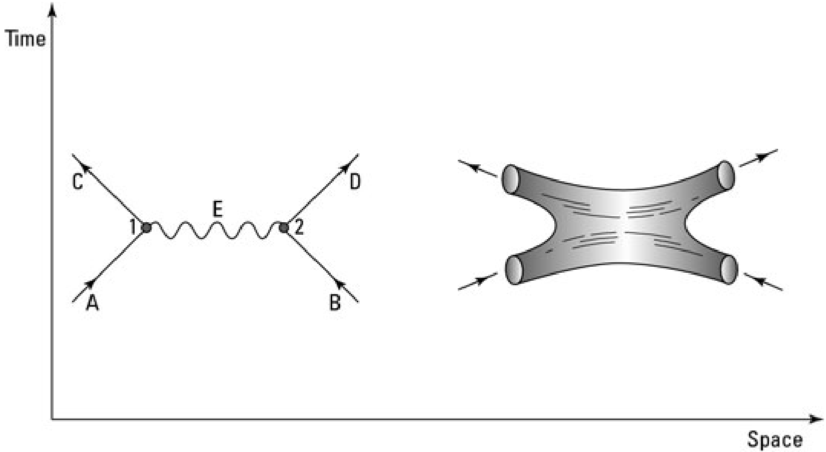
String Theory Reminder

String Theory gives us a quantum theory of gravity and is therefore a promising 'Theory of Everything'.



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- There are many different Calabi-Yau 3-folds, but the only Calabi-Yau two folds are complex tori and K3 surfaces.
- Alvarez-Gaumé and Freedman[4] showed that a sigma model on a hyperkähler manifold has $\mathcal{N} = 4$ symmetry ($K3$ is hyperkähler).

[4] Luis Alvarez-Gaume and Daniel Z Freedman. "Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model". In: *Communications in Mathematical Physics* 80.3 (1981), pp. 443–451.

The Partition Function and the Elliptic Genus

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$$Z(\tau, z; \bar{\tau}, \bar{z}) = \mathrm{Tr}_{\mathbf{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L_0} - \frac{c}{24}} y^{2J_0^3} \bar{y}^{2\bar{J}_0^3} \qquad q = e^{2\pi i \tau}, y = e^{2\pi i z}. \quad (7)$$

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For some purposes it is convenient to consider a related quantity known as the Elliptic Genus. This is moduli space independent.

$$\varepsilon_{\mathcal{M}}(\tau, z) := Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z} = 0) \quad (8)$$

The Elliptic Genus Of K3

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$$\varepsilon_{K3}(\tau, z) := Tr_{\mathcal{H}^R}\left((-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L_0} - \frac{\bar{c}}{24}} y^{2J_0^3}\right) \tag{9}$$

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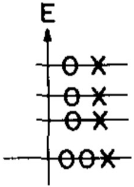

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$$\varepsilon_{K3} = 8\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau)^2}\right] \tag{10}$$

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ε is a topological invariant and can be related to other invariants,

$$\varepsilon_{K3}(\tau,z=0) = \chi(K3) = 24 \tag{11}$$

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The Elliptic Genus in $\mathcal{N} = 4$ Characters

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In terms of $\mathcal{N} = 4$ characters we can expand the elliptic genus as

$$\varepsilon_{K3}(\tau, z) = 24\mathrm{ch}^{\tilde{R}}_{l=0}(\tau, z) + \Sigma(\tau) q^{\frac{1}{8}} \hat{\mathrm{ch}}^{\tilde{R}}_{l=1/2}(\tau, z) \tag{12}$$

where

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Σ can be expanded as

$$\Sigma(\tau) = q^{-\frac{1}{8}}(-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + \dots) \tag{14}$$

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A sporadic group

Theorem

The Classification of Finite Simple Groups.

This theorem states that all finite simple groups fall into one of the following families:

1. Cyclic groups of order n for n prime.
2. Alternating groups of degree at least 5.
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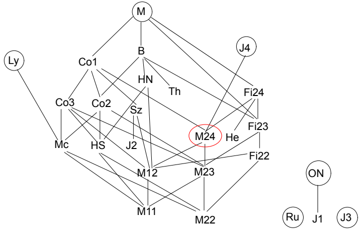
M_{24} is one of the sporadic finite simple groups. It is a subgroup of the Monster group M, as shown below.



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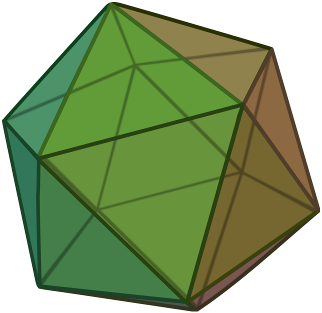

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Lexicographic Code

$$\begin{aligned}
c_0 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\\
c_1 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1)\\
c_2 &= (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1)\\
c_3 &= (0,0,0,0,0,0,0,0,0,0,1,1,0,0,1,1,0,0,1,1,0,0,1,1)\\
&\vdots
\end{aligned}$$

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The Mathematical Game of Mogul

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↓

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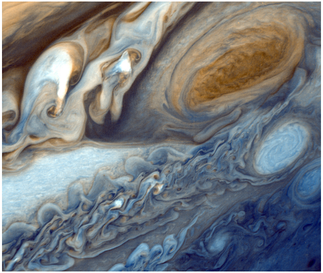
The Golay code was used to transmit photos back from the Voyager spacecraft.



M_{24}

We can define M_{24} in many different ways, however one that suits us is the following.

Definition

$$M_{24} := Aut(\mathcal{G}_{24}) \tag{15}$$

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M_{24} has order $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244823040$

Outline

M_{24} Representation Theory

K_g	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ρ_1	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
ρ_2	45	-3	5	0	3	-3	1	1	0	0	-1	α_+^-	α_-^-	-1	0	1	0	1	α_-^+	α_+^+	0	0	α_+^-	α_-^-	-1	-1
$\overline{\rho_2}$	45	-3	5	0	3	-3	1	1	0	0	-1	α_-^-	α_+^-	-1	0	1	0	1	α_+^+	α_-^+	0	0	α_-^-	α_+^-	-1	-1
ρ_3	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	β_+^-	β_-^-	0	0	1	1
$\overline{\rho_3}$	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	β_-^-	β_+^-	0	0	1	1
ρ_4	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
ρ_5	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
ρ_6	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
ρ_7	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	γ_+^-	γ_-^-
$\overline{\rho_7}$	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	γ_-^-	γ_+^-
ρ_8	990	-18	-10	0	3	6	2	-2	0	0	-1	α_+^-	α_-^-	0	0	0	0	1	α_+^-	α_-^-	0	0	α_+^-	α_-^-	1	1
$\overline{\rho_8}$	990	-18	-10	0	3	6	2	-2	0	0	-1	α_-^-	α_+^-	0	0	0	0	1	α_-^-	α_+^-	0	0	α_-^-	α_+^-	1	1
ρ_9	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
ρ_{10}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\alpha_+^-$	$2\alpha_-^-$	-1	0	1	0	-1	0	0	0	0	α_-^+	α_+^+	0	0
$\overline{\rho_{10}}$	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\alpha_-^-$	$2\alpha_+^-$	-1	0	1	0	-1	0	0	0	0	α_+^+	α_-^+	0	0
ρ_{11}	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
ρ_{12}	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
ρ_{13}	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
ρ_{14}	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
ρ_{15}	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
ρ_{16}	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
ρ_{17}	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
ρ_{18}	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
ρ_{19}	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
ρ_{20}	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

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We showed that the elliptic genus of $K3$ can be written as

$$\begin{aligned}\varepsilon_{K3}(\tau, z) = {} & 24\mathrm{ch}^{\tilde{R}}_{l=0}(\tau, z) + \hat{\mathrm{ch}}^{\tilde{R}}_{l=1/2}(\tau, z) \\ & \cdot \left(-2 + \mathbf{90}q + \mathbf{462}q^2 + \mathbf{1540}q^3 + \mathbf{4554}q^4 + \ldots\right)\end{aligned} \tag{16}$$

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The *twining elliptic genera* for $g \in \mathrm{M}_{24}$ have been studied (eg.[8])

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[8] Matthias R Gaberdiel, Stefan Hohenegger, and Roberto Volpato. "Mathieu twining characters for K3". In: *Journal of High Energy Physics* 2010.9 (2010), pp. 1–20.

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and Gannon[9] proved that all* H_n are indeed representations of M_{24}.

[8] Gaberdiel, Hohenegger, and Volpato, "Mathieu twining characters for K3".

[9] Terry Gannon. "Much ado about Mathieu". In: *arXiv preprint arXiv:1211.5531* (2012).

Outline

Lattices - First Definitions

Here we will be interested only in real lattices, that is subgroups of $\mathbb{R}^n$.

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- The *Gram matrix*, A is given by $M \cdot M^\top$.
- The determinant of the Gram matrix is known as the *determinant* of L_n, $\det(L_n) = \det(A)$.

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\mu &= \xi (M^{-1})^\top = \xi (M^{-1})^\top M^{-1} M \\
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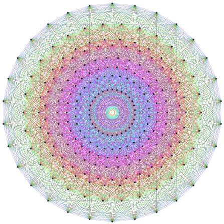

hence we also have $L_n^* \subseteq \det(L)^{-1} L_n$.
When $\det(L) = 1$ then L is unimodular.

The Niemeier Lattices

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The only 16 dimensional Type II lattices are $E_8 \times E_8$ and the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$. Related to gauge groups for Heterotic string theory.

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It can be shown[10] that an even, unimodular lattice either has no roots (Leech) or its roots are given by the union of irreducible, simply-laced root systems of the same Coxeter number. Verified by Mass Formula.

[10] John Horton Conway and Neil James Alexander Sloane. *Sphere packings, lattices and groups*. Vol. 290. Springer Science and Business Media, 2013.

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The Niemeier lattices can be constructed [10] by *gluing* the root lattices using *glue vectors* whose components are given by elements of L^*/L. We consider $(A_1^{24})^+$ and $(A_2^{12})^+$.

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$(A_1^{24})^+$ is constructed using the Extended binary Golay code, $\mathcal{G}_{24}$ as glue. The minimum weight of $\mathcal{G}_{24}$ ensures this doesn't add roots and maintains evenness.

$(A_2^{12})^+$ is constructed using the Extended ternary Golay code, $\mathcal{G}_{12}$. $\mathcal{G}_{12}$ is a $[12, 6, 6]_3$ code. Therefore using this as glue doesn't add roots and maintains evenness.

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Outline

The Umbral Forms

Cheng, Duncan and Harvey[11] described a particular way to associate to each Niemeier lattice a (vector-valued) mock modular form.

[11] Miranda CN Cheng, John FR Duncan, and Jeffrey A Harvey. "Umbral moonshine". In: *arXiv preprint arXiv:1204.2779* (2012); Miranda CN Cheng, John FR Duncan, and Jeffrey A Harvey. "Umbral moonshine and the Niemeier lattices". In: *arXiv preprint arXiv:1307.5793* (2013).

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- Form a weight 1, index m Jacobi form as

$$\psi(\tau, z) = \mu_{1,0}(\tau, z)\phi(\tau, z) \tag{18}$$

where $\mu_{1,0}$ is a meromorphic Jacobi form of weight 1, index 1.

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- We define the *Polar part* of ψ as

$$\psi^P(\tau, z) = \chi\mu_{m,0} \tag{19}$$

  where $\chi = \phi(\tau, 0)$ is a weight 0 modular form and hence is constant.

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Umbral Forms Continued

- Now form the *finite* part of ψ as

$$\psi^F(\tau, z) = \psi(\tau, z) - \psi^P(\tau, z) \tag{20}$$

ψ^F is known as a *mock Jacobi form* of weight 1 and index m.

Umbral Forms Continued

- Now form the *finite* part of ψ as

$$\psi^F(\tau, z) = \psi(\tau, z) - \psi^P(\tau, z) \tag{20}$$

ψ^F is known as a *mock Jacobi form* of weight 1 and index m.

- It can be shown[12] that such a form has a theta expansion given by

$$\psi^F(\tau, z) = \sum_{r=1}^{m-1} h_r(\tau)\hat{\theta}_{m,r}(\tau, z) \tag{21}$$

the theta coefficients h_r are the components of a vector-valued mock modular form of weight $\frac{1}{2}$.

[12] Atish Dabholkar, Sameer Murthy, and Don Zagier. "Quantum black holes, wall crossing, and mock modular forms". In: *arXiv preprint arXiv:1208.4074* (2012).

The first few coefficients

The link to the Niemeier lattices A_1^{24} and A_2^{12}, comes by taking ϕ to be an *extremal* Jacobi form of index the Coxeter number of the Niemeier root system.

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The spaces of extremal Jacobi forms of index 2, and index 3, are both known to be of dimension 1. We define

$$f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0) \tag{22}$$

$$\phi_1^2 = 8(f_2^2 + f_3^2 + f_4^2), \tag{23}$$

$$\phi_1^3 = 4(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2). \tag{24}$$

We find the Umbral Forms (H_r)

$$H_1^2 = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + \dots) \tag{25}$$

$$H_1^3 = 2q^{-1/12}(-1 + 16q + 55q^2 + 144q^3 + \dots) \tag{26}$$

$$H_2^3 = 2q^{2/3}(10 + 44q + 110q^2 + 280q^3 + \dots) \tag{27}$$

The Umbral Groups

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For the Niemeier lattices the Automorphism group is given by the product $\bar{G}^X W^X G_1$, where $\bar{G}^X$ is the group of permutations of the components of X induced by automorphisms of L^X, and G_1 is given by $\hat{W}^X/W^X$ where $\hat{W}^X$ is automorphisms of L^X that stabilise the components of X.

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In the cases A_1^{24} and A_2^{12} we find

$$G^{A_1^{24}} = M_{24} \tag{29}$$

$$G^{A_2^{12}} = 2.M_{12} \tag{30}$$

(M)Umbral Moonshine

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	11	11	−1	3	3	2	2	−1	−1	−1	3	1	1	−1	0	0	−1	−1	1	1	−1	−1	0	0	0	0
χ_3	$+$	11	11	−1	3	3	2	2	−1	−1	3	−1	1	1	−1	0	0	1	1	−1	−1	−1	−1	0	0	0	0
χ_4	$\circ$	16	16	4	0	0	−2	−2	1	1	0	0	1	1	1	0	0	0	0	0	0	−1	−1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	$\circ$	16	16	4	0	0	−2	−2	1	1	0	0	1	1	1	0	0	0	0	0	0	−1	−1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	$+$	45	45	5	−3	−3	0	0	3	3	1	1	0	0	−1	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	$+$	54	54	6	6	6	0	0	0	0	2	2	−1	−1	0	0	0	0	0	0	0	1	1	−1	−1	−1	−1
χ_8	$+$	55	55	−5	7	7	1	1	1	1	−1	−1	0	0	1	1	1	−1	−1	−1	−1	0	0	0	0	0	0
χ_9	$+$	55	55	−5	−1	−1	1	1	1	1	3	−1	0	0	1	−1	−1	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	$+$	55	55	−5	−1	−1	1	1	1	1	−1	3	0	0	1	−1	−1	1	1	−1	−1	0	0	0	0	0	0
χ_{11}	$+$	66	66	6	2	2	3	3	0	0	−2	−2	1	1	0	−1	−1	0	0	0	0	1	1	0	0	0	0
χ_{12}	$+$	99	99	−1	3	3	0	0	3	3	−1	−1	−1	−1	−1	0	0	1	1	1	1	−1	−1	0	0	0	0
χ_{13}	$+$	120	120	0	−8	−8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	−1	−1	−1	−1
χ_{14}	$+$	144	144	4	0	0	0	0	−3	−3	0	0	−1	−1	1	0	0	0	0	0	0	−1	−1	1	1	1	1
χ_{15}	$+$	176	176	−4	0	0	−4	−4	−1	−1	0	0	1	1	−1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	$\circ$	10	−10	0	−2	2	1	−1	−2	2	0	0	0	0	0	1	−1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	−1	1	−1	1
χ_{17}	$\circ$	10	−10	0	−2	2	1	−1	−2	2	0	0	0	0	0	1	−1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	−1	1	−1	1
χ_{18}	$+$	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	0	0	0	1	−1	1	−1
χ_{19}	$-$	32	−32	0	0	0	−4	4	2	−2	0	0	2	−2	0	0	0	0	0	0	0	0	0	−1	1	−1	1
χ_{20}	$\circ$	44	−44	0	4	−4	−1	1	2	−2	0	0	−1	1	0	1	−1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	$\circ$	44	−44	0	4	−4	−1	1	2	−2	0	0	−1	1	0	1	−1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	$\circ$	110	−110	0	−6	6	2	−2	2	−2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	$\circ$	110	−110	0	−6	6	2	−2	2	−2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	$+$	120	−120	0	8	−8	3	−3	0	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	−1	1	−1	1
χ_{25}	$\circ$	160	−160	0	0	0	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$
χ_{26}	$\circ$	160	−160	0	0	0	−2	2	−2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\overline{b_{11}}$	$\overline{b_{11}}$	$-b_{11}$	b_{11}

(M)Umbral Moonshine

We see that the coefficients of the forms H_1^2, H_1^3 and H_2^3 are given by dimensions of representations of the Umbral groups M_{24} and $2.M_{12}$ respectively.

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A similar procedure can be used to find vector-valued mock modular forms associated to each of the 23 Niemeier lattices, each of which is found to have coefficients encoding dimensions of representations of the 23 Umbral groups as defined above.

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Umbral Moonshine conjectures that there exists a graded module K^X associated to each Niemeier lattice L^X such that the characters associated to elements $g \in G^X$ give the umbral forms H_g^X[13].

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Recall:

- Monstrous Moonshine was hidden in the partition function of a particular CFT

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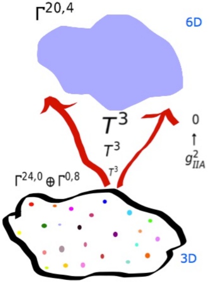
- The Elliptic Genus of $K3$, which revealed Mathieu Moonshine when written in terms of $\mathcal{N} = 4$ characters, described the right-moving ground states of the theory.

Umbral moonshine can also be seen in terms of the elliptic genus of K3: Recall that we split the elliptic genus into massless and massive characters of $\mathcal{N} = 4$. We can instead split the elliptic genus into a part corresponding to some surface singularities of the $K3$ and the remaining 'Moonshine' part which encodes the moonshine form[14].

[14] Miranda CN Cheng and Sarah Harrison. "Umbral Moonshine and K3 Surfaces". In: *arXiv preprint arXiv:1406.0619* (2014).

Hidden Physics

Kachru et al.[15] consider 3d gravity theories by for instance compactifying the Type II string on $K3xT^3$. The moduli space of such theories can be thought of as the space of 32-dimensional even unimodular lattices of signature (8,24). In a neighbourhood of some particular points in this moduli space the theory has Umbral symmetry.



[15] Shamit Kachru, Natalie M Paquette, and Roberto Volpato. "3D String Theory and Umbral Moonshine". In: *arXiv preprint arXiv:1603.07330* (2016).


UNCLE COOTER'S
GENUINE
MOONSHINE
XXX
Hooch, Fire-water, Booze, Snake Oil,
Adult Beverage, Medicine, Elixir,
Sauce, Home Brew, Hair-of-the-dog.
NOT FOR BABIES

Questions?

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- Monstrous Moonshine involved modular functions (in fact Hauptmodul) but Mathieu Moonshine (and Umbral Moonshine) involves mock-modular forms.
- Monstrous moonshine can be explained in terms of a string propagating on an orbifold of the 'Leech Torus' $\mathbb{R}^{24}/\Lambda$ where the j-invariant describes the partition functions for the theory. In Mathieu Moonshine we don't consider the full partition function but the elliptic genus which only counts half BPS states (right moving ground states).