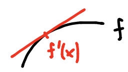
APPROXIMATION THEORY

Key idea: Replace a complicated function by a simple function.

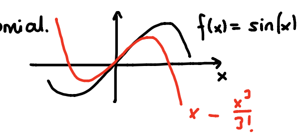
e.g. 1). $\pi \longrightarrow 3.14$.

2). function $f(x) \longrightarrow$ tangent $f'(x)$



3). function $f(x) \longrightarrow$ Taylor polynomial.

$\sum_{k=0}^{n} \frac{f^{(n)}(0)}{n!} x^n$



4). All of science/engineering measurements $\longrightarrow$ finite precision.

Why are simple functions better?

- Easier to store (discrete representation – e.g. mp3 or jpeg compression).
- Easier to evaluate.
- Easier to compute with (e.g. differentiate, find roots, etc.).

We aim to represent a complicated function $f(x)$ by a simpler function $\phi(x; a_0, \ldots, a_n)$ where $a_0, \ldots, a_n$ are parameters chosen to give the best approximation of f.

Common ways to do this are:

1). Interpolation – choose the a_i so that

$\phi(x_i; a_0, \ldots, a_n) = f(x_i)$ on a prescribed set of points $\{x_i \mid i = 0, \ldots, n\}$. ← seen in 2H Num. An. (will study more)

2). Least-squares approximation – choose the a_i to minimise

$\|f - \phi\|_2 := \left(\int_a^b (f(x) - \phi(x; a_0, \ldots, a_n))^2 dx \right)^{1/2}$. ← seen in 2H Num. An. (not in this course)

3). Minimax approximation – choose the a_i to minimise

$\|f - \phi\|_\infty := \max_{x \in [a,b]} |f(x) - \phi(x; a_0, \ldots, a_n)|$. ← will study.

COURSE OUTLINE

0). Revision of polynomial interpolation.

1). Piecewise polynomial interpolation (splines).

2). Minimax approximation.

3). Trigonometric interpolation. (discrete Fourier transform).

0) Revision of Polynomial Interpolation. (2H Numerical Analysis).

Main points:-

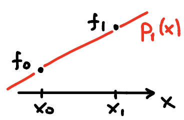
① Given $f \in C[a,b]$ and $n+1$ distinct points $a \le x_0 < x_1 < \ldots < x_n \le b$, there is a unique polynomial $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = f(x_i)$ for $i = 0, \ldots, n$.

polynomials of degree $\le n$.

② You have to be very careful where the points x_i are placed.

The proof of existence is by construction with Lagrange polynomials.

e.g. $n=1$



$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1.$$

In general,

$$p_n(x) = \sum_{i=0}^{n} f_i \, \ell_i(x) \quad \text{where} \quad \ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \quad \text{are the Lagrange polynomials.}$$

You should have seen the error formula:

Thm 0.1 — Let $f \in C^{n+1}[a,b]$ and $a \le x_0 < x_1 < \ldots < x_n \le b$. Let $p_n \in \mathcal{P}_n$ be such that $p_n(x_i) = f(x_i)$ for $i = 0, \ldots, n$. Then for every $x \in [a,b]$ there is a $\xi \in [a,b]$, depending on x, such that

$$f(x) - p_n(x) = \frac{w_{n+1}(x) \, f^{(n+1)}(\xi)}{(n+1)!}$$

where

$$w_{n+1}(x) = \prod_{j=0}^{n} (x - x_j) \quad \text{is the error polynomial.}$$

- Disadvantage: can only apply this formula if f is smooth enough.
- Can use this to put an upper bound:

$$\|f - p_n\|_\infty \le \frac{\|w_{n+1}\|_\infty \, \|f^{(n+1)}\|_\infty}{(n+1)!}$$

The ∞-norm means $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.

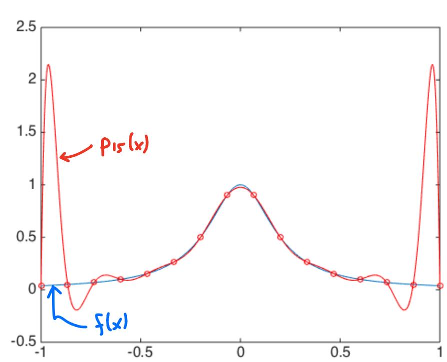
Suppose we take equally-spaced points $x_j = a + \left(\frac{b-a}{n}\right) j$ for $j = 0, \ldots, n$. As we increase the resolution n, we would like $\|f - p_n\|_\infty \to 0$ as $n \to \infty$.

Problem:- The sequence $\{p_n\}$ may not converge to f because $\|w_{n+1}\|_\infty \|f^{(n+1)}\|_\infty$ may tend to ∞ faster than $(n+1)!$.

A famous example is due to Runge (1901):

Example: $f(x) = \dfrac{1}{1 + 25x^2}$ $\quad x \in [-1, 1]$.

This function is well-behaved with all derivatives continuous and bounded, yet p_n diverges for equally-spaced points. →



Remark:- Actually it does converge near $x = 0$. These kind of interpolants are used in higher-order (e.g. 6^{th}) finite-difference methods for solving PDEs.