2.4). ALTERNATING SETS

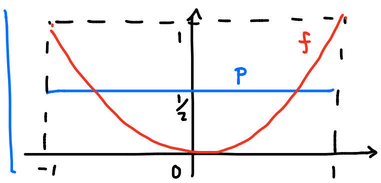
We have seen that the maximum error in the minimax polynomial p_n^* seems to occur $n+2$ times, with alternating sign of $f - p^*$.

Def^n:- Let $f \in C[a,b]$ and p be a polynomial approximation. An alternating set / alternant of f, p of length n is a sequence of points $x_0, \dots, x_{n-1}$ such that

1). $a \le x_0 < x_1 < \dots < x_{n-1} \le b$.

2). $f(x_i) - p(x_i) = (-1)^i E$ for $i = 0, \dots, n-1$ where either $E = \|f-p\|_\infty$ or $E = -\|f-p\|_\infty$.

Example:- Let $f(x) = x^2$ on $[-1, 1]$ and $p(x) = \frac{1}{2}$.



Here we have $\|f-p\|_\infty = \frac{1}{2}$.
The points $\{-1, 0, 1\}$ form an alternating set of length 3, with
$E = \|f-p\|_\infty = \frac{1}{2}$.

↳ will see that this implies that $p = \frac{1}{2}$ is the minimax degree-1 polynomial.

Before we can prove our general result (that maximum error in minimax polynomial always alternates), we need some preliminary results.

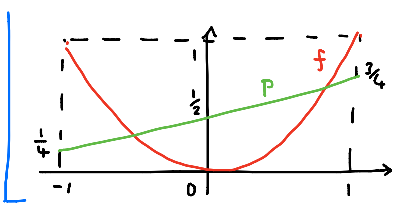
Def^n:- Let $f \in C[a,b]$ and p be a polynomial. A non-uniform alternating set of length n is a sequence of points $x_0, \dots, x_{n-1}$ such that

1). $a \le x_0 < x_1 < \dots < x_{n-1} \le b$.

2). $f(x_i) - p(x_i) = (-1)^i e_i$ for $i = 0, \dots, n-1$,

where the e_i's are either all positive or all negative.

Example:- $f(x) = x^2$ on $[-1, 1]$ and $p(x) = \frac{x}{4} + \frac{1}{2}$



Now $\{-1, 0, 1\}$ is a non-uniform alternating set of length 3, with
$e_0 = \frac{3}{4}$, $e_1 = \frac{1}{2}$, $e_2 = \frac{1}{4}$.

The following result lets us bound $\|f-p\|_\infty$ if we can find a suitable non-uniform alternating set.

Thm 2.2 (de la Vallée Poussin) —— Let $f \in C[a,b]$, $q_n \in P_n$ and p_n^* the minimax polynomial of degree n for f on $[a,b]$. If f, q_n have a non-uniform alternating set of length $n+2$, then

$$\|f - p_n^*\|_\infty \ge \min_{i \in [0, n+1]} |e_i|.$$

critical

$|e_i| = |f(x_i) - q_n(x_i)|$ where x_i are the n-u.a.set

- We see that this holds in the above example, where
$\min |e_i| = \frac{1}{4}$ and $\|f - p_1^*\|_\infty = \frac{1}{2}$.

De la Vallée Poussin is most famous for proving the Prime Number Thm.

Proof:- (by contradiction).

Assume that $\|f - p_n^*\|_\infty < \min_{i \in [0,n+1]} |e_i|$. ⊛

Let $X = \{x_0, \dots, x_{n+1}\}$ be the non-uniform alternating set for f, q_n.

Define

$$r_n(x) = p_n^*(x) - q_n(x) \in \mathcal{P}_n.$$

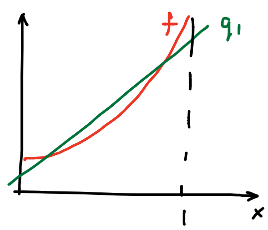
$$= \left(f(x) - q_n(x)\right) - \left(f(x) - p_n^*(x)\right)$$

so $r_n(x_i)$ has the same sign as $f(x_i) - q_n(x_i)$ by ⊛.

But then r_n changes sign $n+1$ times, which is a contradiction. ▨

If we don't know p_n^* we can use Thm 2.2 to estimate the error in p^* by choosing a suitable q_n.

Example:- Let $f(x) = e^x$ on $[0,1]$ and $q_1(x) = 0.9 + 1.6x$.



$f - q_1$ changes sign twice in $[0,1]$ and $\{0, 0.5, 1\}$ is a non-uniform alternating set, with

$$f(0) - q_1(0) = 0.1$$
$$f(0.5) - q_1(0.5) = -0.05$$
$$f(1) - q_1(1) = 0.22$$

Hence by Thm 2.2 we have

$$\|f - p_1^*\|_\infty \geq 0.05 \quad \text{in } [0,1].$$

We can also get an upper bound using the fact that $\|f - p_1^*\|_\infty \leq \|f - q_1\|_\infty$. We need to find $\|f - q_1\|_\infty$:

$$f - q_1 = e^x - 0.9 - 1.6x \quad \Rightarrow \quad f' - q_1' = e^x - 1.6.$$

Turning point is at $x = \ln(1.6)$.

We have

$$f(0) - q_1(0) = 0.1, \quad f(\ln(1.6)) - q_1(\ln(1.6)) = -0.05, \quad f(1) - q_1(1) = 0.22,$$

so $\|f - q_1\|_\infty = 0.22$.

Hence

$$0.05 \leq \|f - p_1^*\|_\infty \leq 0.22. \quad ⊛$$

We can check this using our result from last lecture,

$$p_1^*(x) = 0.894 + 1.718x.$$

The maximum error was at $x=0$, so $\|f - p_1^*\|_\infty = |1 - 0.894| = 0.106$ — consistent with ⊛.