2.6). Remez/Exchange Algorithm

Unfortunately the Equioscillation Thm 2.3 doesn't tell us what the minimax polynomial is.

Suppose we have $n=1$, so $p_1^*(x) = a_0 + a_1 x$, and suppose we have an alternating set $\{x_0, x_1, x_2\}$ for f, p_1^*, with $|E| = \|f - p_1^*\|_\infty$. Then the coefficients a_0, a_1 satisfy

$$f(x_0) - (a_0 + a_1 x_0) = E$$
$$f(x_1) - (a_0 + a_1 x_1) = -E$$
$$f(x_2) - (a_0 + a_1 x_2) = E.$$

← cf. our initial examples.

i.e.

$$a_0 + a_1 x_0 + E = f(x_0)$$
$$a_0 + a_1 x_1 - E = f(x_1)$$
$$a_0 + a_1 x_2 + E = f(x_2).$$

Since x_i, $f(x_i)$ are known, this linear system would give a_0, a_1, E.

If the x_i are distinct, there will be a unique solution.

Unfortunately, we don't usually know an alternating set to start with. The idea of the Remez/Exchange algorithm is iterative:

the size an A. set would need to be.

Step 1). Solve the linear system over a specified reference set of $n+2$ ordered points $\{x_i\}$.

Step 2). Update the reference set by an exchange procedure, and return to Step 1.

To update the reference set, we look for a set $\{y_i\}$, $i = 0, \dots, n+1$, where

1). The existing error $f - p_n$ alternates sign on the y_i.

the estimate from Step 1.

found in Step 1.

2). $|f(y_i) - p(y_i)| \geq |E|$ at every point y_i.

3). $|f(y_i) - p(y_i)| > |E|$ for at least one y_i.

Example:- $n=1$, $f(x) = e^x$ on $[0,1]$. ← previously we found $p_1^*(x) = 0.894 + 1.718x$.

To illustrate the algorithm, start with $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$.

Step 1:- $\begin{cases} a_0 + E = e^0 = 1 \\ a_0 + \frac{1}{2}a_1 - E = e^{1/2} \\ a_0 + a_1 + E = e \end{cases} \Rightarrow \begin{array}{l} a_0 = 0.8948 \\ a_1 = 1.7183 \\ E = 0.1052 \end{array}$ → already quite good!

Step 2:- Update the reference set.

A good way to do this is to find the point of maximum $|f - p_1|$ and swap this into the set.

Here,

$$f(x) - p_1(x) = e^x - 0.8948 - 1.7183x$$

$$\Rightarrow \frac{\partial}{\partial x}(f(x) - p_1(x)) = e^x - 1.7183$$ — turning point at $x = \ln(1.7183) \approx 0.5413$.

We have

$$f(0.5413) - p_1(0.5413) = -0.1067$$ ← close to optimum.

$$f(1) - p_1(1) = 0.1052.$$

So we swap $x_1 = \frac{1}{2}$ for $x_1 = 0.5413$.

Step 1 :- Now solve

$$\begin{cases} a_0 + \quad E = 1 \\ a_0 + 0.5413 a_1 - E = e^{0.5413} \\ a_0 + a_1 + E = e \end{cases} \Rightarrow \quad \begin{aligned} a_0 &= 0.8941 \\ a_1 &= 1.7183 \\ E &= 0.1059 \end{aligned}$$

← converging
(note: E has increased — good because we want the alt. set with maximum E!)

In general, the Remez algorithm is found to converge if f is sufficiently smooth and the initial reference set is sufficiently close.

A common way to update the reference set is to find all of the local maxima/minima of $f - p_n$, and take a subset of these that alternate in sign.

e.g. Mayans (2006) — online article in J. Online Math. & Appns. → see app in §9.