

2.6). Remez/Exchange Algorithm

Unfortunately the Equioscillation Thm 2.3 doesn't tell us what the minimax polynomial is.

Suppose we have $n=1$, so $p_1^*(x) = a_0 + a_1x$, and suppose we have an alternating set $\{x_0, x_1, x_2\}$ for f, p_1^* , with $|E| = \|f - p_1^*\|_\infty$. Then the coefficients a_0, a_1 satisfy

$$\begin{aligned} f(x_0) - (a_0 + a_1x_0) &= E \\ f(x_1) - (a_0 + a_1x_1) &= -E \\ f(x_2) - (a_0 + a_1x_2) &= E. \end{aligned} \quad \leftarrow \text{cf. our initial examples.}$$

i.e.

$$\begin{aligned} a_0 + a_1x_0 + E &= f(x_0) \\ a_0 + a_1x_1 - E &= f(x_1) \\ a_0 + a_1x_2 + E &= f(x_2). \end{aligned}$$

Since $x_i, f(x_i)$ are known, this linear system would give a_0, a_1, E .

\leftarrow If the x_i are distinct, there will be a unique solution.

Unfortunately, we don't usually know an alternating set to start with. The idea of the Remez/Exchange algorithm is iterative: the size of A.set would need to be.

Step 1). Solve the linear system over a specified reference set of $n+2$ ordered points $\{x_i\}$.

Step 2). Update the reference set by an exchange procedure, and return to Step 1.

To update the reference set, we look for a set $\{y_i\}$, $i=0, \dots, n+1$, where

- 1). The existing error $f - p_n$ alternates sign on the y_i .
 \leftarrow the estimate from Step 1.
- 2). $|f(y_i) - p(y_i)| \geq |E|$ at every point y_i .
 \leftarrow found in Step 1.
- 3). $|f(y_i) - p(y_i)| > |E|$ for at least one y_i .

Example: $n=1$, $f(x) = e^x$ on $[0, 1]$. \leftarrow previously we found $p_1^*(x) = 0.894 + 1.718x$.

To illustrate the algorithm, start with $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$.

$$\text{Step 1: } \begin{cases} a_0 + E = e^0 = 1 \\ a_0 + \frac{1}{2}a_1 - E = e^{1/2} \\ a_0 + a_1 + E = e \end{cases} \Rightarrow \begin{cases} a_0 = 0.8948 \\ a_1 = 1.7183 \\ E = 0.1052 \end{cases} \rightarrow \text{already quite good!}$$

Step 2: Update the reference set.

A good way to do this is to find the point of maximum $|f - p_1|$ and swap this into the set.

Here,

$$\begin{aligned} f(x) - p_1(x) &= e^x - 0.8948 - 1.7183x \\ \Rightarrow \frac{\partial}{\partial x}(f(x) - p_1(x)) &= e^x - 1.7183 \quad \text{turning point at } x = \ln(1.7183) \approx 0.5413. \end{aligned}$$

We have

$$\begin{aligned} f(0.5413) - p_1(0.5413) &= -0.1067 \quad \leftarrow \text{close to optimum.} \\ f(1) - p_1(1) &= 0.1052. \end{aligned}$$

So we swap $x_1 = \frac{1}{2}$ for $x_1 = 0.5413$.

Step 1 :- Now solve

$$\begin{cases} a_0 + E = 1 \\ a_0 + 0.5413a_1 - E = e^{0.5413} \\ a_0 + a_1 + E = e \end{cases} \Rightarrow \begin{cases} a_0 = 0.8941 \\ a_1 = 1.7183 \\ E = 0.1059 \end{cases} \leftarrow \text{converging}$$

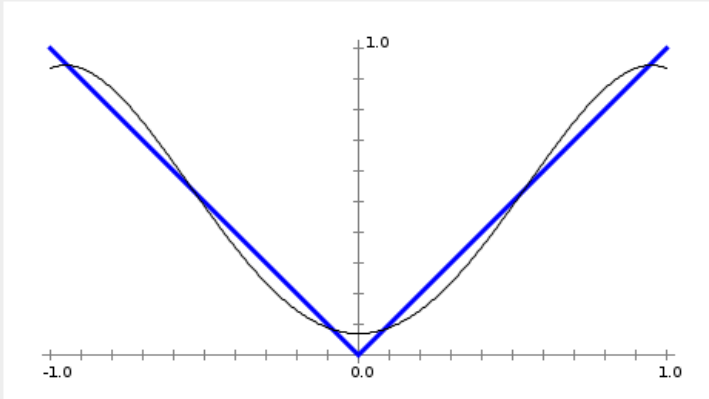
(note: E has increased — good because we want the alt. set with maximum E!)

In general, the Remez algorithm is found to converge if f is sufficiently smooth and the initial reference set is sufficiently close.

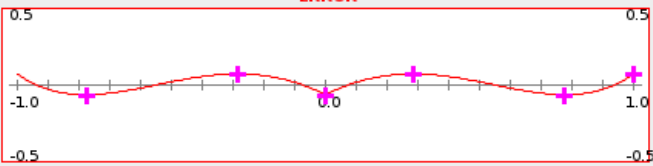
A common way to update the reference set is to find all of the local maxima/minima of $f - p_n$, and take a subset of these that alternate in sign.

e.g. [Mayans \(2006\)](#) — online article in *J. Online Math. & Appns.* → see app in §9.

Best Polynomial Approximation



ERROR



Nodes selected for iteration 5

COEFFICIENTS			
0.06762	0	1.93033	0
-1.06557			

Sketch a Function:

Clear

Sketch

Or Select a Function:

$y = |x|$

$y = \exp(x)/3$

$y = (1 + \sin(\pi x))/2$

Remes Iteration:

Degree (1 to 7)

$n = 1$

$n = 2$

$n = 3$

$n = 4$

$n = 5$

$n = 6$

$n = 7$