3). TRIGONOMETRIC INTERPOLATION

Many real-life functions (e.g. sounds) are periodic, meaning

$$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}.$$ ← by suitably rescaling x we can convert any other period to 2π.

Such functions are well-approximated by trigonometric polynomials

$$p_n(x) = \sum_{k=0}^{n-1} \left(a_k \cos(kx) - b_k \sin(kx)\right),$$

called degree n if a_n or b_n are the highest non-zero coefficients.

Using Euler's identity $e^{ikx} = \cos(kx) + i\sin(kx)$, it is nicer to re-write $p_n(x) = \mathrm{Re}\{q_n(x)\}$ for the complex polynomial

$$q_n(x) = \sum_{k=0}^{n-1} (a_k + ib_k)\left(\cos(kx) + i\sin(kx)\right)$$

$$= \sum_{k=0}^{n-1} (a_k + ib_k)e^{ikx} = \sum_{k=0}^{n-1} c_k e^{ikx}.$$ ← $= (\cos x + i\sin x)^k$ ⟶ explains why it is a "polynomial"

3.1). THE DISCRETE FOURIER TRANSFORM

Consider the problem of interpolating f at n equally-spaced points $x_j = \frac{2\pi j}{n}$ for $j = 0, \dots, n-1$.

i.e. we want

$$q_n(x_j) = f_j \quad \text{for } j = 0, \dots, n-1.$$

This gives

$$\sum_{k=0}^{n-1} c_k e^{ikx_j} = f_j \iff \sum_{k=0}^{n-1} c_k e^{i\frac{2\pi k}{n}j} = f_j \iff \sum_{k=0}^{n-1} c_k \omega^{jk} = f_j \quad \text{where } \omega = e^{i\frac{2\pi}{n}}.$$

This is a system of n linear equations

$$\underbrace{\begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & & \omega^{n-1} \\ \omega^0 & \omega^2 & \omega^4 & & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}}_{F_n} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

Inverting would give the interpolation coefficients $\vec{c} = F_n^{-1}\vec{f}$.

The matrix F_n is called the Fourier matrix. ← Note: books differ in whether they include the $\frac{1}{\sqrt{n}}$, or swap F_n and F_n^{-1}.

In general, for any n-vector $\vec{x}$, $F_n^{-1}\vec{x}$ is called the discrete Fourier transform (DFT) of $\vec{x}$, and $F_n\vec{x}$ is the inverse DFT of $\vec{x}$.

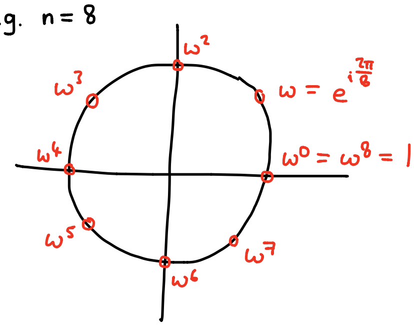
the DFT matrix.

The trigonometric interpolation coefficients $\vec{c}$ are given by the DFT of the data F.

Notice that $\omega = e^{i\frac{2\pi}{n}}$ is a primitive n^{th} root of unity, so F_n contains roots of unity.

$\omega^n = 1$ and n is smallest integer of $k = 1, \dots, n$ for which $\omega^k = 1$.

e.g. $n = 8$



it is "almost" unitary, $F_n^{-1} = F_n^*$ up to $\frac{1}{n}$.

Thm 3.1 — The Fourier matrix is symmetric and satisfies $F_n^{-1} = \frac{1}{n}\overline{F_n}$.

complex conjugate (same as F_n^* since symmetric).

Proof :-

We want to show that $\frac{1}{n}\overline{F}_n F_n = I_n$, i.e.

$$\frac{1}{n}\begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & & \omega^{-(n-1)} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \ddots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}\begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{1} & \omega^{2} & & \omega^{(n-1)} \\ \omega^0 & \omega^{2} & \omega^{4} & \ddots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Let

$$\vec{v}^{(k)} = (1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k})$$

denote the k^{th} row of F_n.

Diagonal entries:

$$\frac{1}{n}(\overline{F}_n F_n)_{kk} = \frac{1}{n}\overline{\vec{v}^{(k)}} \cdot \vec{v}^{(k)} = \frac{1}{n}\sum_{\ell=0}^{n-1}\overline{\omega^{k\ell}}\,\omega^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1}\omega^{-k\ell}\omega^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1} 1 = 1.$$

Off-diagonal entries:

$$\frac{1}{n}(\overline{F}_n F_n)_{jk} = \frac{1}{n}\overline{\vec{v}^{(j)}}\,\vec{v}^{(k)} = \frac{1}{n}\sum_{\ell=0}^{n-1}\overline{\omega^{j\ell}}\,\omega^{k\ell}$$

$$= \frac{1}{n}\sum_{\ell=0}^{n-1}\omega^{(k-j)\ell}$$

$$= \frac{1}{n}\sum_{\ell=0}^{n-1}(\omega^{k-j})^\ell$$

geometric series.

$$= \frac{1}{n}\frac{(\omega^{k-j})^n - 1}{\omega^{k-j} - 1}$$

$$= \frac{1}{n}\frac{\overbrace{e^{i(k-j)2\pi}}^{=1} - 1}{\underbrace{e^{i(k-j)\frac{2\pi}{n}}}_{\neq 1} - 1} = 0.$$

∎

So there is a simple expression for the DFT matrix F_n^{-1}.