Example:- Find the DFT of the vector $\vec{x} = (1, 0, -1, 0)^T$.

Let $\omega = e^{i\frac{2\pi}{4}}$ be a primitive 4^{th} root of unity.

Note that $\omega = e^{i\pi/2} = i$.

The DFT gives

$$F_4^{-1}\vec{x} = \frac{1}{4}\begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{bmatrix} \overset{\vec{x}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

Example:- Find a trigonometric polynomial which interpolates the data $(0,1)$, $(\frac{2\pi}{4}, 0)$, $(\frac{4\pi}{4}, -1)$, $(\frac{6\pi}{4}, 0)$.

The complex interpolating polynomial is

$$q_4(x) = \sum_{k=0}^{3} c_k e^{ikx}$$

and the coefficients are given by the DFT

$$\vec{c} = F_4^{-1}\vec{f} = F_4^{-1}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix} \quad \text{from previous example.}$$

Hence

$$q_4(x) = \tfrac{1}{2}\left(e^{ix} + e^{3ix}\right)$$
$$= \tfrac{1}{2}\left(\cos(x) + \cos(3x)\right) + \tfrac{1}{2}i\left(\sin(x) + \sin(3x)\right).$$

Since the data $\vec{f}$ are real, a real trigonometric polynomial which interpolates the data is

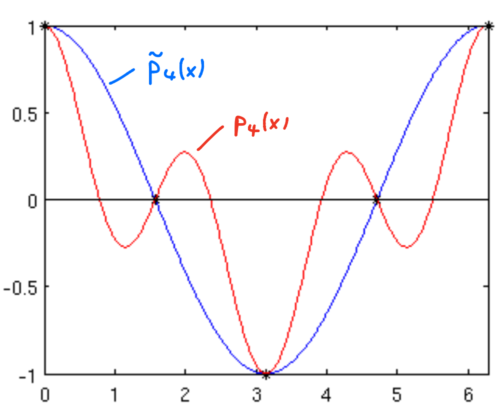
$$p_4(x) = \mathrm{Re}\{q_4(x)\} = \tfrac{1}{2}\left(\cos(x) + \cos(3x)\right).$$

3.2). ALIASING

Trigonometric interpolation as defined so far is not unique.

In the previous example, notice that $\cos(3x_j) = \cos(x_j)$ at all four nodes. So we could also interpolate the data with

$$\tilde{p}_4(x) = \cos(x).$$



This is called aliasing. If there are n nodes, then $\cos(k'x_j) = \cos(kx_j)$ for all j

$$\Leftrightarrow \cos\left(\tfrac{2\pi j}{n}k'\right) = \cos\left(\tfrac{2\pi j}{n}k\right) \quad \text{for all } j$$

$$\Leftrightarrow \frac{k'}{n} = \pm\frac{k}{n} + \mathbb{Z}.$$

because $\cos(x) = \cos(-x)$

e.g. $n=4$: We saw that $\cos(3x_j) = \cos(x_j)$

These are aliases for $n=4$ $(k'=3,\ k=1)$ since

$$\frac{3}{4} = -\frac{1}{4} + 1.$$

Other aliases would be e.g. $\cos(5x)$, $\cos(7x)$.

In fact, half of the expansion

$$p_4(x) = \sum_{k=0}^{n-1}\left(a_k\cos(kx) - b_k\sin(kx)\right)$$

is always redundant. For at x_j we have

$$\cos\left((n-k)x_j\right) = \cos\left(2\pi j - \frac{2\pi k j}{n}\right)$$
$$= \cos(2\pi j)\cos\left(\frac{2\pi k j}{n}\right) - \sin(2\pi j)\sin\left(\frac{2\pi k j}{n}\right)$$
$$= \cos\left(\frac{2\pi k j}{n}\right) = \cos(kx_j)$$

and similarly

$$\sin\left((n-k)x_j\right) = \sin\left(2\pi j - \frac{2\pi k j}{n}\right)$$
$$= \sin(2\pi j)\cos\left(\frac{2\pi k j}{n}\right) - \cos(2\pi j)\sin\left(\frac{2\pi k j}{n}\right)$$
$$= -\sin(kx_j).$$

Therefore for n even we can always interpolate the data with the shorter series

$$\tilde{p}_n(x) = a_0 + \sum_{k=1}^{n/2-1}\left((a_k + a_{n-k})\cos(kx) - (b_k - b_{n-k})\sin(kx)\right) + a_{n/2}\cos\left(\tfrac{n}{2}x\right).$$

(and similar for n odd except that there is no "middle" term $k = n/2$).

Example:- In the example above, $a_0 + ib_0 = 0$, $a_1 + ib_1 = \frac{1}{2}$, $a_2 + ib_2 = 0$, $a_3 + ib_3 = \frac{1}{2}$.

so
$$\tilde{p}_4(x) = \left(\tfrac{1}{2}+\tfrac{1}{2}\right)\cos(x) = \cos(x).$$

This is the idea behind an important result: not explicitly written down by Nyquist (1920s) but shown by Claude Shannon (1949), the founder of information theory

Thm 3.2 (Nyquist-Shannon Sampling Thm) — A band-limited function $f(x)$ (ie. with no Fourier components of frequency $k > K$) is completely determined by its values $f(x_j)$ for $x_j = \frac{2\pi j}{n}$, $j = 0,\ldots,n-1$, providing that $n > 2K$.

Conversely, for a given n, the highest-frequency Fourier component that can be perfectly reconstructed is $k < n/2$. This is called the Nyquist frequency.

Example:- If we had sampled $f(x) = \cos(x)$ with $n > 6$ points, we could be sure that the signal had no $\cos(3x)$ component.

Remark:- If the signal is "sparse" then it may be possible to recover it from fewer samples than required by Thm 3.2 — called compressed sensing.

Some physical examples of aliasing:

1). Moiré patterns in poorly pixelized images (see "Aliasing" in Wikipedia).
2). The "wagon-wheel effect" in video (temporal aliasing).

e.g. if camera shutter clicks 24 times per second and spokes pass vertical 20 times per second, it looks as though wheel is rotating −4 times per second, i.e. backwards!

$$\sin(20x_j) = \sin(-4x_j) \quad \text{for } x_j = \frac{2\pi}{24}j, \quad j = 0,\ldots,23. \quad \left[\frac{20}{24} = \frac{-4}{24} + 1\right]$$