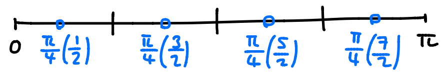
## 3.4). THE DISCRETE COSINE TRANSFORM

**Recall**:- When interpolating $n$ points $x_j = \frac{2\pi j}{n}$ in $[0, 2\pi]$ by a function $q_n(x) = \sum_{k=0}^{n-1} c_k e^{ikx}$ we got the DFT $\vec{c} = F_n^{-1}\vec{f}$, where the matrix $F_n^{-1}$ had complex entries.

We can get a matrix of real entries if we interpolate a function on $[0, \pi]$ (no longer needs to be periodic) at $n$ equally-spaced points

$$x_j = \frac{\pi}{n}\left(j + \tfrac{1}{2}\right), \quad j = 0, \dots, n-1$$

e.g. $n = 4$




with a function of the form

$$p_n(x) = \frac{1}{\sqrt{n}} a_0 + \sqrt{\frac{2}{n}} \sum_{k=1}^{n-1} a_k \cos(kx).$$

coefficients are for convenience (to make matrix orthogonal).

This is a real basis.

The interpolation conditions give

$$f_j = p_n(x_j) = \frac{1}{\sqrt{n}} a_0 + \sqrt{\frac{2}{n}} \sum_{k=1}^{n-1} a_k \cos\left(\frac{k\pi}{n}\left(j + \tfrac{1}{2}\right)\right) \quad \text{for } j = 0, \dots, n-1,$$

i.e.

$\vec{f} = C_n \vec{a}$ where

$$C_n = \sqrt{\frac{2}{n}} \begin{array}{c} \begin{array}{cccc} k=0 & k=1 & k=2 \longrightarrow & k=n-1 \end{array} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\left(\frac{\pi}{n}\left(\frac{1}{2}\right)\right) & \cos\left(\frac{2\pi}{n}\left(\frac{1}{2}\right)\right) & \cdots & \cos\left(\frac{(n-1)\pi}{n}\left(\frac{1}{2}\right)\right) \\ \frac{1}{\sqrt{2}} & \cos\left(\frac{\pi}{n}\left(\frac{3}{2}\right)\right) & \cos\left(\frac{2\pi}{n}\left(\frac{3}{2}\right)\right) & & \cos\left(\frac{(n-1)\pi}{n}\left(\frac{3}{2}\right)\right) \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{1}{\sqrt{2}} & \cos\left(\frac{\pi}{n}\left(\frac{2n-1}{2}\right)\right) & \cdots & & \cos\left(\frac{(n-1)\pi}{n}\left(\frac{2n-1}{2}\right)\right) \end{bmatrix} \end{array} \begin{array}{l} \\ j=0 \\ j=1 \\ \downarrow \\ j=n-1 \end{array}$$

To find $\vec{a}$ we need to invert $C_n$, and the matrix $C_n^{-1}$ is called the discrete cosine transform (DCT).

again there are different definitions / normalisations used.

> **Thm 3.3** — $C_n$ is orthogonal, i.e. $C_n^{-1} = C_n^T$.

**Proof**:- We can avoid a nightmare of trigonometry with an indirect proof.

Let $A_n$ be the real symmetric circulant matrix

$$A_n = \begin{bmatrix} 1 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

We will show that the columns $\vec{v}^{(k)}$ of $C_n$ are the eigenvectors of $A_n$. This means that they are automatically orthogonal, since $A_n$ is real symmetric.

First, $\vec{v}^{(0)} = \frac{1}{\sqrt{2}}(1, 1, \dots, 1)^T$.

So

$$A_n\vec{v}^{(0)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \Rightarrow A_n\vec{v}^{(0)} = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{0}\,\vec{v}^{(0)}.$$

For every other column, the components take the form

$$v_\ell^{(k)} = \sqrt{\frac{2}{n}}\cos\Big(\underbrace{\frac{k\pi}{n}}_{\theta}\big(\ell+\tfrac{1}{2}\big)\Big).$$

For $j = 1, \dots, n-2$,

$$\begin{aligned}
\left(A_n\vec{v}^{(k)}\right)_j = \sum_{\ell=0}^{n-1}(A_n)_{j\ell}\,v_\ell^{(k)} &= -v_{j-1}^{(k)} + 2v_j^{(k)} - v_{j+1}^{(k)} \\
&= \sqrt{\tfrac{2}{n}}\left(-\cos\big(\theta(j-\tfrac{1}{2})\big) + 2\cos\big(\theta(j+\tfrac{1}{2})\big) - \cos\big(\theta(j+\tfrac{3}{2})\big)\right) \\
&= \sqrt{\tfrac{2}{n}}\left(-\cos\big(\overbrace{\theta(j+\tfrac{1}{2})}^{B}-\theta\big) + 2\cos\big(\theta(j+\tfrac{1}{2})\big) - \cos\big(\theta(j+\tfrac{1}{2})+\theta\big)\right) \\
&= \sqrt{\tfrac{2}{n}}\left(-\cos B\cos\theta - \sin B\sin\theta + 2\cos B - \cos B\cos\theta + \sin B\sin\theta\right) \\
&= \sqrt{\tfrac{2}{n}}\left(2-2\cos\theta\right)\cos B. \\
&= \underbrace{\left(2-2\cos\left(\tfrac{k\pi}{n}\right)\right)}_{\text{eigenvalues.}}v_j^{(k)}.
\end{aligned}$$

For $j=0$ we have

$$\begin{aligned}
\left(A_n\vec{v}^{(k)}\right)_0 = v_0^{(k)} - v_1^{(k)} &= \sqrt{\tfrac{2}{n}}\left(\cos\left(\tfrac{1}{2}\theta\right) - \cos\left(\tfrac{3}{2}\theta\right)\right) \\
&= \sqrt{\tfrac{2}{n}}\left(\cos\left(\tfrac{1}{2}\theta\right) - \cos\left(\tfrac{1}{2}\theta+\theta\right)\right) \\
&= \sqrt{\tfrac{2}{n}}\left(\cos\left(\tfrac{1}{2}\theta\right) - \cos\left(\tfrac{1}{2}\theta\right)\cos\theta + \sin\left(\tfrac{1}{2}\theta\right)\underbrace{\sin\theta}_{=2\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)}\right) \\
&= \sqrt{\tfrac{2}{n}}\cos\left(\tfrac{1}{2}\theta\right)\left(1-\cos\theta + 2\sin^2\left(\tfrac{1}{2}\theta\right)\right) \qquad 2\sin^2\left(\tfrac{1}{2}\theta\right) = 1-\cos\theta. \\
&= \sqrt{\tfrac{2}{n}}\cos\left(\tfrac{1}{2}\theta\right)\left(2-2\cos\theta\right) \\
&= \left(2-2\cos\left(\tfrac{k\pi}{n}\right)\right)v_0^{(k)}.
\end{aligned}$$

Similarly,

$$\left(A_n\vec{v}^{(k)}\right)_{n-1} = v_{n-1}^{(k)} - v_{n-2}^{(k)} = \sqrt{\tfrac{2}{n}}\left(\cos\big(\theta(n-\tfrac{1}{2})\big) - \cos\big(\theta(n-\tfrac{3}{2})\big)\right) = \underset{\text{exercise!}}{\dots} = \left(2-2\cos\left(\tfrac{k\pi}{n}\right)\right)v_{n-1}^{(k)}.$$

$\blacksquare$

To understand where this comes from, note that the columns $\vec{v}^{(k)}$ are discrete approximations of $\cos(kx)$ at the points $x_j$.

We know that $u(x) = \cos(kx)$ satisfies the differential equation $-\frac{d^2u}{dx^2} = k^2u$, ie. $\cos(kx)$ is an eigenvalue of the differential operator $-\frac{d^2}{dx^2}$, with eigenvalue $k^2$.

The matrix $A_n$ is a finite-difference approximation to $\frac{d^2}{dx^2}$ at equally-spaced points,

$$(A_n\vec{u})_j = -u_{j-1} + 2u_j - u_{j+1}$$
$$= -\left[(u_{j+1} - u_j) - (u_j - u_{j-1})\right]$$

so its eigenvectors are discrete cosines.

e.g. the columns $\vec{v}^{(0)}, \ldots, \vec{v}^{(3)}$ of $C_4$ :