There are various ways to avoid the Runge phenomenon when approximating a function by polynomials:

1). choose the nodes carefully (revise today).
2). use splines ⟶ good if you aren't free to choose nodes.
3). use minimax or Least-squares instead of interpolation.

0.1). CHEBYSHEV INTERPOLATION (still revision of 2H Numerical Analysis).

Recall that the error in polynomial interpolation satisfies

$$\|f - p_n\|_\infty \le \frac{\|w_{n+1}\|_\infty \, \|f^{(n+1)}\|_\infty}{(n+1)!}$$

see that smoother functions will be easier to interpolate, in general.

We can't control $f^{(n+1)}$ (derivatives of the original function) but we can change the error polynomial

$$w_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$$

by moving the nodes x_j.

We will show that a good set of nodes are the Chebyshev nodes, defined as roots of the Chebyshev polynomial

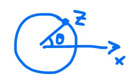
$$T_n(x) = \cos\big(n \arccos(x)\big).$$

from French transliteration "Tchebischeff" (1850s).

To see that this is a polynomial, let $\theta = \arccos(x)$, so

$$T_n(x) = \cos(n\theta) = \mathrm{Re}\big(e^{in\theta}\big) = \tfrac{1}{2}\big(e^{in\theta} + e^{-in\theta}\big) = \tfrac{1}{2}\big(z^n + z^{-n}\big) \quad \text{where } z = e^{i\theta}.$$

← complex number on unit circle



Now

$$\tfrac{1}{2}(z + z^{-1})(z^n + z^{-n}) = \tfrac{1}{2}\big(z^{n+1} + z^{-n-1}\big) + \tfrac{1}{2}\big(z^{n-1} + z^{-n+1}\big) \quad \text{for } n \ge 1$$

i.e.

$$2x\,T_n(x) = T_{n+1}(x) + T_{n-1}(x).$$

$$\Leftrightarrow \quad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x).$$

By induction, each $T_n(x)$ with $n \ge 1$ is a polynomial of degree exactly n, with leading coefficient 2^{n-1}.

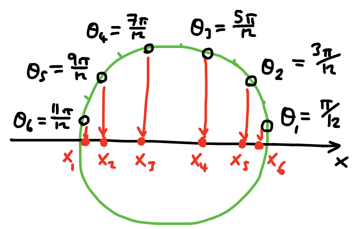
The roots of $T_n(x)$ are

$$n\theta_j = (j - \tfrac{1}{2})\pi \quad \text{for } j = 1, \dots, n$$

$$\Rightarrow \quad \tilde{x}_j = \cos\left(\pi - \frac{(j-\frac{1}{2})\pi}{n}\right) \quad \text{for } j = 1, \dots, n.$$

reverse order

e.g. $n = 6$



Notice that the nodes cluster near the end-points $x = \pm 1$.

Lemma 0.2:- Suppose $p_n \in \mathcal{P}_n$ interpolates $f \in C^{(n+1)}[-1,1]$ at the Chebyshev nodes $\tilde{x}_0, \dots, \tilde{x}_n$. Let $\tilde{w}_{n+1}$ be the error polynomial for these nodes. Then

$$\tilde{w}_{n+1}(x) = \frac{1}{2^n} T_{n+1}(x).$$

Proof:- Since T_{n+1} and $\tilde{w}_{n+1}$ both belong to $\mathcal{P}_{n+1}$ and have $n+1$ roots (at $\tilde{x}_0, \dots, \tilde{x}_n$), there is a $c \in \mathbb{R}$ such that $T_{n+1} = c\,\tilde{w}_{n+1}$.

But the leading coefficient of T_{n+1} is 2^n and of $\tilde{w}_{n+1}$ is 1, so $c = 2^n$. ▨

Now we can show that the Chebyshev nodes are a good choice.

Thm 0.3: Suppose $p_n \in \mathcal{P}_n$ interpolates $f \in C^{n+1}[-1,1]$ at the Chebyshev nodes $\tilde{x}_0, \dots, \tilde{x}_n$. Let $\tilde{w}_{n+1}$ be the error polynomial for these nodes. Let $x_0, \dots, x_n$ be $n+1$ arbitrary, distinct nodes with error polynomial w_{n+1}. Then

$$\|\tilde{w}_{n+1}\|_\infty \leq \|w_{n+1}\|_\infty.$$

i.e. interpolating at Chebyshev nodes will minimise $\|w_{n+1}\|_\infty$. ← not the same as minimising $\|f - p_n\|_\infty$ since different x's are weighted by the $f^{(n+1)}(\xi)$ term. (later).

Proof: Suppose there exist nodes $x_0, \dots, x_n$ such that $\|\tilde{w}_{n+1}\|_\infty > \|w_{n+1}\|_\infty$.

Then $q := \tilde{w}_{n+1} - w_{n+1}$ belongs to $\mathcal{P}_n$.

Now the extreme values of T_{n+1} occur at

$$\tilde{y}_j = \cos\left(\frac{j\pi}{n+1}\right) \qquad j = 0, \dots, n+1$$

and $T_{n+1}(\tilde{y}_j) = (-1)^j$.

Moreover, $\tilde{w}_{n+1} = \frac{1}{2^n} T_{n+1}$, so the maxima of $|\tilde{w}_{n+1}|$ occur at these points, with

$$\tilde{w}_{n+1}(y_j) = \frac{1}{2^n}(-1)^j.$$

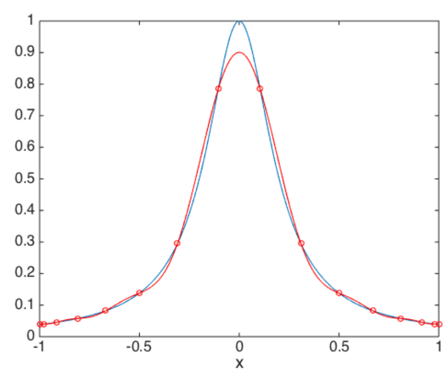
By assumption we must have $|w_{n+1}| < |\tilde{w}_{n+1}|$ at these points, so

$$\left.\begin{aligned} q(\tilde{y}_j) &> 0 \text{ if } j \text{ even} \\ &< 0 \text{ if } j \text{ odd.} \end{aligned}\right\} \text{ n+1 sign changes}$$

By the Intermediate Value Thm, q must have $n+1$ roots. Since $q \in \mathcal{P}_n$, we must have $q = 0$. ∎

Example: $f(x) = \dfrac{1}{1+25x^2}$, $x \in [-1, 1]$.

With Chebyshev points, the Runge phenomenon is absent.



Remark: In fact, polynomial interpolants in Chebyshev points converge even if f is not in C^{n+1} — for example if f is only Lipschitz continuous (Trefethen).

But smoother f leads to more rapid convergence. (by Thm 0.1).

↳ e.g. $f(x) = |x|$.