1.3). B– SPLINES

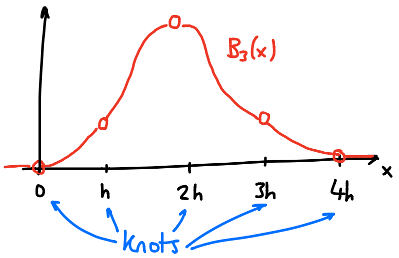
An alternative approach for computing cubic splines is to use basis functions called B-splines which are themselves cubic splines but with compact support.

B for "Basis" (coined by Schoenberg in 1946).

For simplicity, assume constant knot spacing h.

The cubic B-spline centered at $x = 2h$ has the form

$$B_3(x) = \begin{cases} x^3 = b_1(x) & \text{for } 0 \leq x \leq h, \\ x^3 - 4(x-h)^3 = b_2(x) & \text{for } h \leq x \leq 2h, \\ x^3 - 4(x-h)^3 + 6(x-2h)^3 = b_3(x) & \text{for } 2h \leq x \leq 3h, \\ (4h-x)^3 = b_4(x) & \text{for } 3h \leq x \leq 4h, \\ 0 & \text{elsewhere}. \end{cases}$$



We can check that this is really a cubic spline:

i). Continuity of B_3 at knots:

$b_1(0) = 0,\quad b_1(h) = h^3,\quad b_2(2h) = 4h^3,\quad b_3(3h) = h^3,\quad b_4(4h) = 0$

$b_2(h) = h^3\quad b_3(2h) = 4h^3\quad b_4(3h) = h^3$

ii). Continuity of B_3' at the knots:

$b_1'(0) = 0,\quad b_1'(h) = 3h^2\quad b_2'(2h) = 12h^2 - 12h^2 = 0\quad b_3'(3h) = 27h^2 - 48h^2 + 18h^2 = -3h^2\quad b_4'(4h) = 0.$

$b_2'(h) = 3h^2\quad b_3'(2h) = 0\quad b_4'(3h) = -3h^2$

iii). Continuity of B_3'' at the knots:

$b_1''(0) = 0,\quad b_1''(h) = 6h,\quad b_2''(2h) = 12h - 24h = -12h\quad b_3''(3h) = 18h - 48h + 36h = 6h\quad b_4''(4h) = 0.$

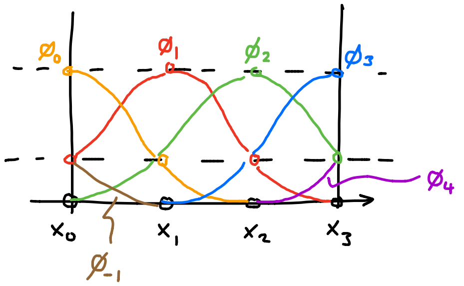
$b_2''(h) = 6h,\quad b_3''(2h) = -12h\quad b_4''(3h) = 6h$

Hence $B_3 \in C^2(-\infty, \infty)$ and so is a cubic spline.

The idea is that any cubic spline can be written as a linear combination

$$s(x) = \sum_{i=-1}^{n+1} \alpha_i \phi_i(x) \qquad \text{where} \qquad \phi_i(x) = \frac{1}{h^3} B_3(x - x_{i-2}).$$

e.g. $n = 3$



The extra basis functions ϕ_{-1}, ϕ_{n+1} are needed to incorporate boundary conditions.

Example:- Fit a natural cubic spline through the data (0,0), (1,1), (2,8).

Here $h=1$ and $n=2$, so we need $\phi_{-1}, \phi_0, \phi_1, \phi_2, \phi_3$. We have $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_{-1} = -1$, $x_{-2} = -2$, $x_{-3} = -3$, ...

The basis functions are: ↓ look at the picture!

$$\phi_{-1}(x) = B_3(x - x_{-3}) = \begin{cases} (4 - x - 3)^3 = (1-x)^3 & \text{in } [0,1] \\ 0 & \text{in } [0,2] \end{cases}$$

$$\phi_0(x) = B_3(x - x_{-2}) = \begin{cases} (x+2)^3 - 4(x+1)^3 + 6x^3 & \text{in } [0,1] \\ (2-x)^3 & \text{in } [1,2] \end{cases}$$

$$\phi_1(x) = B_3(x - x_{-1}) = \begin{cases} (x+1)^3 - 4x^3 & \text{in } [0,1] \\ (x+1)^3 - 4x^3 + 6(x-1)^3 & \text{in } [1,2] \end{cases}$$

$$\phi_2(x) = B_3(x - x_0) = \begin{cases} x^3 & \text{in } [0,1] \\ x^3 - 4(x-1)^3 & \text{in } [1,2]. \end{cases}$$

$$\phi_3(x) = B_3(x - x_1) = \begin{cases} 0 & \text{in } [0,1] \\ (x-1)^3 & \text{in } [1,2]. \end{cases}$$

In fact, to find the coefficients α_i, we just need to know $\phi_i(x_j)$ at the knots x_j.

The interpolation conditions give

$$\alpha_{-1}\underset{\overset{\shortparallel}{1}}{\phi_{-1}(x_0)} + \alpha_0\underset{\overset{\shortparallel}{4}}{\phi_0(x_0)} + \alpha_1\underset{\overset{\shortparallel}{1}}{\phi_1(x_0)} + \alpha_2\cancel{\phi_2(x_0)} + \alpha_3\cancel{\phi_3(x_0)} = f_0 = 0.$$

$$\alpha_{-1}\cancel{\phi_{-1}(x_1)} + \alpha_0\underset{\overset{\shortparallel}{1}}{\phi_0(x_1)} + \alpha_1\underset{\overset{\shortparallel}{4}}{\phi_1(x_1)} + \alpha_2\underset{\overset{\shortparallel}{1}}{\phi_2(x_1)} + \alpha_3\cancel{\phi_3(x_1)} = f_1 = 1$$

$$\alpha_{-1}\cancel{\phi_{-1}(x_2)} + \alpha_0\cancel{\phi_0(x_2)} + \alpha_1\underset{\overset{\shortparallel}{1}}{\phi_1(x_2)} + \alpha_2\underset{\overset{\shortparallel}{4}}{\phi_2(x_2)} + \alpha_3\underset{\overset{\shortparallel}{1}}{\phi_3(x_1)} = f_2 = 8.$$

← notice the system is tridiagonal

To close the system we have the natural boundary conditions:

$$s''(x_0) = 0. \Rightarrow \alpha_{-1}\underset{\overset{\shortparallel}{6}}{\phi''_{-1}(x_0)} + \alpha_0\underset{\overset{\shortparallel}{-12}}{\phi''_0(x_0)} + \alpha_1\underset{\overset{\shortparallel}{6}}{\phi''_1(x_0)} = 0 \qquad \longrightarrow \text{from above}$$

$$s''(x_2) = 0 \Rightarrow \alpha_1\underset{\overset{\shortparallel}{6}}{\phi''_1(x_2)} + \alpha_2\underset{\overset{\shortparallel}{-12}}{\phi''_2(x_2)} + \alpha_3\underset{\overset{\shortparallel}{6}}{\phi''_3(x_3)} = 0.$$

So the linear system is

$$\begin{bmatrix} 6 & -12 & 6 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 6 & -12 & 6 \end{bmatrix} \begin{bmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 8 \\ 0 \end{bmatrix} \Longrightarrow \begin{aligned} \alpha_{-1} &= \tfrac{1}{12} \\ \alpha_0 &= 0 \\ \alpha_1 &= -\tfrac{1}{12} \\ \alpha_2 &= \tfrac{4}{3} \\ \alpha_3 &= \tfrac{11}{4}. \end{aligned}$$

Thus

$$s(x) = \begin{cases} \frac{1}{12}(1-x)^3 - \frac{1}{12}(x+1)^3 + \frac{1}{3}x^3 + \frac{4}{3}x^3 = \frac{3}{2}x^3 - \frac{1}{2}x & \text{in } [0,1] \\ -\frac{1}{12}(x+1)^3 + \frac{1}{3}x^3 - \frac{1}{2}(x-1)^3 + \frac{4}{3}x^3 - \frac{16}{3}(x-1)^3 + \frac{11}{4}(x-1)^3 = -\frac{3}{2}x^3 + 9x^2 - \frac{19}{2}x + 3 & \text{in } [1,2]. \end{cases}$$

↙ agrees with our answer in previous lecture.

Why are B-splines important?

- There is a general formula defining them for different orders of spline. [see Problem sheet].
- There is a fast algorithm for evaluating them (de Boor, 1972 — see SIAM article).

They are widely used in computer design (e.g. cars, aircraft) as well as for interpolating data.