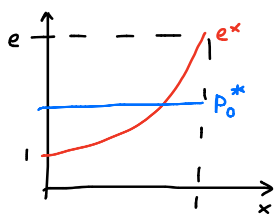
2.3). MINIMAX APPROXIMATIONS

idea goes back to Poncelet (1835) and Chebyshev (1850s).

Aim:- Find a polynomial $p_n \in \mathcal{P}_n$ that minimises the maximum error $\|f - p_n\|_\infty$.

Note: Thm 2.1 tells us that $\|f-p_n\|_\infty$ can be made abitrarily small for n large enough, but in practice we must work with limited n.

Example:- Find the minimax constant approximation $p_0^* \in \mathcal{P}_0$ to e^x on $[0,1]$.



We want to minimise $\|e^x - p_0\|_\infty$ over all constants p_0. Now

$$\|e^x - p_0\|_\infty = \max_{x\in[0,1]} |e^x - p_0| = \max\{|e-p_0|, |1-p_0|\}.$$

Notice that the error changes sign so we minimise the maximum error by balancing the error at both ends, i.e.

$$e - p_0^* = -(1-p_0^*) \quad \Rightarrow \quad p_0^* = \frac{e+1}{2}.$$

Clearly p_0^* is unique.

Example:- Find the minimax linear approximation $p_1^* \in \mathcal{P}_1$ to e^x on $[0,1]$.

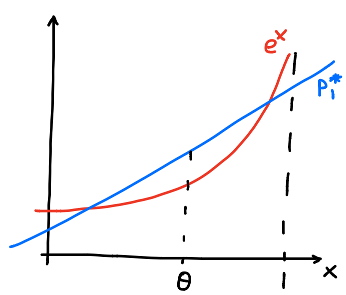
Let $p_1(x) = a + bx$.

We want to minimise $\max_{x\in[0,1]} |e^x - (a+bx)|$.

Local maxima occur at three places,

$x=0$, $x=\theta$, $x=1$.

to be determined



The maximum error will be minimised if the 3 maxima are equal:

$x=0 \to e^0 - (a+0) = E$ ①

$x=\theta \to e^\theta - (a+b\theta) = -E$ ②

$x=1 \to e - (a+b) = E.$ ③

We get a fourth condition from the fact that the error has a turning point at $x=\theta$, so

$$\frac{d}{dx}\left(e^x - (a+bx)\right)\Big|_{x=\theta} = 0 \quad \Rightarrow \quad e^\theta = b. \quad ④$$

Then

① and ③ $\Rightarrow$ $1-a = e-a-b$ $\Rightarrow$ $b = e-1 \simeq 1.718$

$\Rightarrow$ $\theta = \log b.$

② and ③ $\Rightarrow$ $e^\theta - a - b\theta = -e + a + b$ $\Rightarrow$ $a = \frac{1}{2}(e - b\theta) \simeq 0.894.$

So the minimax straight line is

$$p_1^*(x) = 0.894 + 1.718\,x.$$

We will see how to prove that the error in p_n^* has to alternate in sign like this, as well as how to calculate p_n^* more efficiently.

Actually we can already do it when f is a polynomial...

Recall that the Chebyshev polynomials $T_n(x) = \cos(n \arccos(x))$ have the property that

$$\left\| \tfrac{1}{2^n} T_{n+1} \right\|_\infty \leq \| w_{n+1} \|_\infty \qquad \text{for } x \in [-1, 1]$$

kills leading coeff.

Where w_{n+1} is any monic degree $n+1$ polynomial (Thm 0.3).

How can we use this? Let $f(x) = x^{n+1} + a_n x^n + \ldots + a_0$ be a monic polynomial of degree $n+1$.

Let

$$p_n^*(x) = f(x) - \tfrac{1}{2^n} T_{n+1}(x).$$

note: x^{n+1} cancels.

Then

$$\| f - p_n^* \|_\infty = \left\| \tfrac{1}{2^n} T_{n+1} \right\|_\infty \leq \| w_{n+1} \|_\infty$$

for any monic polynomial w_{n+1}.

Since f is monic, this implies that $f - w_{n+1} := p_n$

$$\| f - p_n^* \|_\infty \leq \| f - p_n \| \quad \text{for any } p_n \in \mathcal{P}_n.$$

Example:- Find the minimax polynomial $p_2^* \in \mathcal{P}_n$ for $f(x) = x^3 + ax^2 + bx + c$.

We compute $T_3(x)$ from the recurrence:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$
$$T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x.$$

So the minimax approximation of degree 2 to f is

$$\begin{aligned} p_2^*(x) &= f(x) - \tfrac{1}{2^2} T_3(x) \\ &= x^3 + ax^2 + bx + c - x^3 + \tfrac{3}{4}x \\ &= ax^2 + (b + \tfrac{3}{4})x + c. \end{aligned}$$

not just monic.

This can be extended to any $f \in \mathcal{P}_{n+1}$ using the fact that $\alpha p_n^*(x)$ is the minimax approximation to $\alpha f(x)$.