Problems 3 - Trigonometric Interpolation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Monday 9th March**. There will be a problem class on this chapter on Monday 2nd March.
I use † to indicate (what I consider to be) trickier problems.

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29. *Discrete Fourier transforms.* Compute the discrete Fourier transform of the following vectors, and interpret your results: (a) $\mathbf{x} = (1, 1, 1, 1)^\top$; (b) $\mathbf{x} = (0, 1, 0, -1, 0, 1, 0, -1)^\top$.

30. *Real entries.* Suppose that the entries of $\mathbf{f}$ are all real. If $\mathbf{c} = F_n^{-1}\mathbf{f}$ is the discrete Fourier transform of $\mathbf{f}$, show that $\bar{c}_{n-k} = c_k$ for $k = 0, \ldots, n-1$.

$\star$ **31.** *Trigonometric interpolation.* Consider the periodic function $f(x) = \sin(x) + 2\cos(2x)$.

(a) Write down the Fourier matrix F_3, and its inverse F_3^{-1}.

(b) Use this to find a real trigonometric polynomial of the form

$$p_3(x) = \sum_{k=0}^{2} \Big(a_k \cos(kx) - b_k \sin(kx)\Big)$$

that interpolates f at three equally-spaced nodes on $[0, 2\pi)$.

(c) Explain how it can be that the interpolant p_3 you found in part (b) does not reproduce the original function f exactly.

(d) Find a trigonometric polynomial of lower degree that interpolates the same data.

32. *Splitting.* Let $\mathbf{h} = \mathbf{f} + i\mathbf{g}$, where $\mathbf{f}$ and $\mathbf{g}$ are real vectors, and let $\mathbf{b}$ be the DFT of $\mathbf{h}$. Show that the DFTs of $\mathbf{f}$ and $\mathbf{g}$ are

$$c_k = \frac{1}{2}\big(b_k + \bar{b}_{n-k}\big), \qquad d_k = \frac{i}{2}\big(\bar{b}_{n-k} - b_k\big).$$

Remark: One can speed up the DFT of a real vector $\mathbf{f}$ *by splitting into* $\mathbf{f}_{\text{even}}$ *and* $\mathbf{f}_{\text{odd}}$ *and finding the size* $n/2$ *transform of* $\mathbf{h} = \mathbf{f}_{\text{even}} + i\mathbf{f}_{\text{odd}}$.

† 33. *Eigenvalues of* F_4.

(a) Find the 4×4 matrix P such that $F_4 = P\bar{F}_4$, and verify that $P^2 = I_4$.

(b) Show that $P = \frac{1}{4}F_4^2$.

(c) Hence show that $F_4^4 = 16I_4$, and deduce that the eigenvalues of F_4 must be either ± 2 or $\pm 2i$.

Remark: In fact, for any n, *we have* $F_n^4 = n^2 I_n$.

34. *The columns of* F_n *as eigenvectors.* Show that the columns of the Fourier matrix F_n are the eigenvectors of the cyclic permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & & & & & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

35. *Inverse Fast Fourier Transform.* Find $\frac{n}{2} \times \frac{n}{2}$ matrices A, B, C, D such that

$$F_n^{-1} = \frac{1}{n}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F_{n/2}^{-1} & 0 \\ 0 & F_{n/2}^{-1} \end{pmatrix} P_n,$$

where F_n is the $n \times n$ Fourier matrix and P_n is the odd-even permutation matrix (as defined in the lecture). This shows that the FFT algorithm works in both directions!

† 36. *Applying the FFT.* Compute $F_8\mathbf{x}$ using the recursive FFT algorithm for $\mathbf{x} = (1, 0, 1, 0, 1, 0, 1, 0)^\top$.

37. *Radix-3 FFT.* The FFT may be applied with more general splittings, instead of the radix-2 algorithm presented in the lecture. Suppose $\mathbf{f} = F_n\mathbf{c}$ but now n is a power of 3.

    (a) Show that the entries in $\mathbf{f}$ may be written as

    $$f_j = \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k} + (\omega_n)^j \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k+1} + (\omega_n)^{2j} \sum_{k=0}^{n/3-1} (\omega_{n/3})^{jk} c_{3k+2}.$$

    † (b) For $n = 6$, write out the explicit factorisation of $\mathbf{f} = F_n\mathbf{c}$ in matrix form, including the necessary permutation matrix.

    Remark: This can be generalised to any radix, which was known already to Gauss.

38. *Discrete cosine transform.* Consider the data $(\frac{\pi}{8}, 2)$, $(\frac{3\pi}{8}, 0)$, $(\frac{5\pi}{8}, -2)$, $(\frac{7\pi}{8}, 0)$.

    (a) Use the DCT to find an interpolant $p_4(x)$ for these data.

    (b) Hence find the least-squares approximations of the same form with $m = 1$, $m = 2$, and $m = 3$ terms, for the same data.

39. *DCT-4.* An alternative version of the discrete cosine transform known as DCT-4 is used in sound compression. It is based on the $n \times n$ matrix E_n with entries

    $$(E_n)_{jk} = \sqrt{\frac{2}{n}} \cos\frac{\pi(j + \frac{1}{2})(k + \frac{1}{2})}{n}.$$

    By considering the circulant matrix

    $$\begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 3 \end{pmatrix},$$

    show that the matrix E_n is orthogonal.

40. *Two-dimensional DCT.* A very simple "image" is represented by the matrix

    $$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

    Compute the two-dimensional DCT of this matrix, and hence the corresponding interpolation function $p_2(x, y)$ for the nodes $(\frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{3\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{4}, \frac{3\pi}{4})$, $(\frac{3\pi}{4}, \frac{3\pi}{4})$.