Problems 0 - Polynomial interpolation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Thursday 22nd January**. I use † to indicate (what I consider to be) trickier problems.

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1. *Interpolation error.*

   (a) Write down the Lagrange interpolation polynomial of degree 1 for the function $f(x) = x^3$, using the points $x_0 = 0$, $x_1 = a$.

   (b) Verify Theorem 0.1 by direct calculation, and show that in this case ξ is unique and has the value $\xi = \frac{1}{3}(x + a)$.

   Solution: (a) We have

$$p_1(x) \quad = \quad \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \quad = \quad \frac{x-a}{-a}(0)^3 + \frac{x-0}{a-0}(a)^3 \quad = \quad a^2 x.$$

   (b) Here $n = 1$, so

$$w_{n+1}(x) = (x - 0)(x - a) = x(x - a), \qquad f^{n+1}(x) = f''(x) = 6x.$$

   From calculating p_1, we know that $f - p_1 = x^3 - a^2 x = x(x - a^2)$, so to verify Theorem 0.1 we need to find $\xi \in [0, a]$ such that

$$\frac{w_{n+1}(x) f''(\xi)}{2!} = x(x - a) \qquad \Longleftrightarrow \qquad \frac{x(x-a)6\xi}{2} = x(x - a^2) \qquad \Longleftrightarrow \qquad \xi = \tfrac{1}{3}(x + a).$$

   This lies in the required interval (in this case, the same ξ is true for all x in $[0, a]$).

2. *Equally-spaced versus Chebyshev nodes.* Consider the problem of finding a degree 4 polynomial interpolant $p_4(x)$ for the function $f(x) = e^x$ on the interval $[-1, 1]$.

   (a) Suppose we choose the equally-spaced nodes $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Find an upper bound for the difference between f and p_4 at (i) $x = \frac{1}{4}$ and (ii) $x = \frac{3}{4}$.

   (b) Without computing the locations of the Chebyshev nodes, find an upper bound on the error in the Chebyshev interpolant q_4, and comment on how this compares to part (a).

   Solution: (a) We can use Theorem 0.1, which gives

$$\Big| f(x) - p_4(x) \Big| \leq \frac{|w_5(x)| \|f^{(5)}\|_\infty}{5!},$$

   where $w_5(x) = (x + 1)(x + \frac{1}{2})(x - 0)(x - \frac{1}{2})(x - 1)$, $\|f^{(5)}\|_\infty = e$ (maximum is at $x = 1$), and $5! = 120$.

   i. For $x = \frac{1}{4}$, we have $|w_5(\frac{1}{4})| = \frac{45}{1024}$, so

$$\Big| f(\tfrac{1}{4}) - p_4(\tfrac{1}{4}) \Big| \leq \frac{\frac{45}{1024} e}{120} = 9.96 \times 10^{-4}.$$

   ii. For $x = \frac{3}{4}$, we have $|w_5(\frac{1}{4})| = \frac{105}{1024}$, so

$$\Big| f(\tfrac{3}{4}) - p_4(\tfrac{3}{4}) \Big| \leq \frac{\frac{105}{1024} e}{120} = 2.3 \times 10^{-3}.$$

   (b) For the Chebyshev nodes, Lemma 0.2 gives $w_5(x) = T_5(x)/2^5$. Since $|T_k(x)| \leq 1$ for all of the Chebyshev polynomials, we have

$$|w_5(x)| = \left| \frac{1}{32} T_5(x) \right| \leq \tfrac{1}{32}$$

for any $x \in [-1,1]$. Therefore

$$\left|f(\tfrac{1}{4}) - q_4(\tfrac{1}{4})\right| \leq \frac{\frac{1}{32}e}{120} = 7.1 \times 10^{-4},$$
$$\left|f(\tfrac{3}{4}) - q_4(\tfrac{3}{4})\right| \leq \frac{\frac{1}{32}e}{120} = 7.1 \times 10^{-4}.$$

Notice that (i) both of these error bounds are lower than the equally-spaced interpolant, and (ii) we get the same error bound at any x (although the error itself will be different, of course).

$\star$ **3.** *Chebyshev nodes.* Consider the problem of fitting a linear polynomial to the function $f(x) = x^2$ on $[-1,1]$.

(a) Find the Chebyshev nodes and the linear polynomial p_1 that interpolates f at these nodes.

(b) Compute $\|f - p_1\|_\infty$. Is it possible to find a linear polynomial q_1 (not necessarily interpolating f at the same nodes) such that $\|f - q_1\|_\infty$ is any smaller?

(c) Find the linear polynomial that interpolates $g(x) = x^3$ at the same Chebyshev nodes. Is this the polynomial that minimises $\|g - p_1\|_\infty$ among all $p_1 \in \mathcal{P}_1$? Explain why or why not.

Solution: (a) We have $n = 1$ so need two Chebyshev nodes. These are

$$\tilde{x}_1 = \cos\left(\pi - \frac{\frac{1}{2}\pi}{2}\right) = \cos\left(\frac{3\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}, \qquad \tilde{x}_2 = \cos\left(\pi - \frac{\frac{3}{2}\pi}{2}\right) = \cos\left(\frac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}.$$

The linear interpolating polynomial with these nodes is

$$p_1(x) \quad = \quad \frac{x - \tilde{x}_2}{\tilde{x}_1 - \tilde{x}_2} f_0 + \frac{x - \tilde{x}_1}{\tilde{x}_2 - \tilde{x}_1} f_1 \quad = \quad \frac{x - 1/\sqrt{2}}{-2/\sqrt{2}}(\tfrac{1}{2}) + \frac{x + 1/\sqrt{2}}{2/\sqrt{2}}(\tfrac{1}{2}) \quad = \quad \tfrac{1}{2}.$$

So in fact it is a horizontal line.

(b) Note that $f - p_1 = x^2 - \frac{1}{2}$. The extremum of this function occurs either at $x = -1$ or $x = 1$ (the boundary points), or at the turning point $x = 0$. In fact, at all of these points we have $|f - p_1| = \frac{1}{2}$, so $\|f - p_1\|_\infty = \frac{1}{2}$.
To decide whether any other straight line could give a smaller error, consider the graphs. Moving the horizontal line up or down would lead to larger $\|f - p_1\|_\infty$, as would changing the slope. So in fact p_1 has the minimum possible $\|f - p_1\|_\infty$ on this interval.

Remark: Once we do Chapter 2 on minimax approximation, you will know how to prove this rigorously, using the fact that $\{-1, 0, 1\}$ are an "alternating set" for f, p_1.

(c) Now we have

$$p_1(x) \quad = \quad \frac{x - \frac{1}{\sqrt{2}}}{-\frac{2}{\sqrt{2}}}\left(-\tfrac{1}{2\sqrt{2}}\right) + \frac{x + \frac{1}{\sqrt{2}}}{\frac{2}{\sqrt{2}}}\left(\tfrac{1}{2\sqrt{2}}\right) \quad = \quad \tfrac{1}{2}x.$$

Now $g - p_1 = x^3 - \frac{1}{2}x$. Now

$$\frac{d}{dx}(g - p_1) = 3x^2 - \tfrac{1}{2},$$

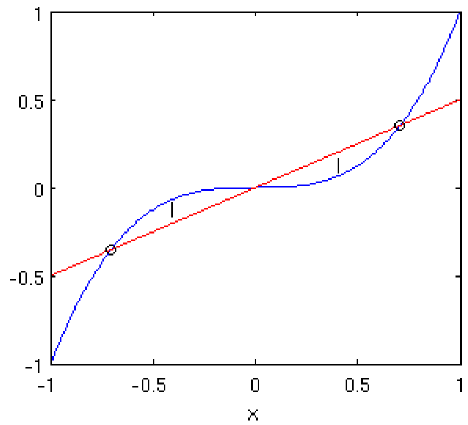
so there are turning points at $x = \pm 1/\sqrt{6}$. We have

$$g(\pm\tfrac{1}{\sqrt{6}}) - p_1(\tfrac{1}{\sqrt{6}}) = \left(\pm\tfrac{1}{\sqrt{6}}\right)^3 \mp \tfrac{1}{2}\left(\tfrac{1}{\sqrt{6}}\right) = \mp\tfrac{1}{3}\left(\tfrac{1}{\sqrt{6}}\right) \approx 0.136.$$

while the values at the endpoints $x = \pm 1$ are

$$g(\pm 1) - p_1(\pm 1) = \tfrac{1}{2}.$$

Sketching the graph shows that the distance between p_1 and $g = x^3$ is different at the turning points of $g - p_1$ from at the end-points. Therefore you could reduce $\|g - p_1\|_\infty$ by increasing the slope of the line p_1 to equalize the four extreme distances. The figure below shows $g = x^3$ (in blue) and p_1 (in red), with the interpolation nodes and the points $\pm\frac{1}{\sqrt{6}}$ marked.



Remark: Once we do Chapter 2 on minimax approximation, you will know how to prove rigorously that this Chebyshev interpolant is not the minimax polynomial.

4. *Barycentric formulae.*

   (a) Let $\lambda_i = 1/w'_{n+1}(x_i)$, where w_{n+1} is the usual error polynomial from Theorem 0.1 and x_i for $i = 0, \ldots, n$ are the interpolation nodes. Show that the polynomial interpolating f may be written as

$$p_n(x) = w_{n+1}(x) \sum_{i=0}^{n} \frac{\lambda_i}{x - x_i} f_i.$$

   Remark: This was derived by Jacobi in his 1825 PhD thesis. Once the weights λ_i are known, it allows you to compute each value $p_n(x)$ with only $O(n)$ operations.

   † (b) Show that $p_n(x)$ may alternatively be written as the *barycentric interpolation formula*

$$p_n(x) = \sum_{i=0}^{n} \frac{\lambda_i f_i}{x - x_i} \Bigg/ \sum_{i=0}^{n} \frac{\lambda_i}{x - x_i}.$$

   Remark: This formula is an efficient and stable way to compute Chebyshev interpolants (although it should not be used at the nodes themselves because both numerator and denominator are infinite).

Solution: (a) Firstly, note that

$$w'_{n+1}(x) = \prod_{\substack{j=0\\j\neq 0}}^{n} (x - x_j) + \prod_{\substack{j=0\\j\neq 1}}^{n} (x - x_j) + \ldots + \prod_{\substack{j=0\\j\neq n}}^{n} (x - x_j),$$

so at the nodes x_i, we have

$$w'_{n+1}(x_i) = \prod_{\substack{j=0\\j\neq i}}^{n} (x_i - x_j).$$

From the first lecture, the Lagrange polynomials are

$$l_i(x) = \prod_{\substack{j=0\\j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{1}{w'_{n+1}(x_i)} \prod_{\substack{j=0\\j\neq i}}^{n} (x - x_j) = \frac{\lambda_i}{x - x_i} w_{n+1}(x).$$

and the result follows from $p_n(x) = \sum_{i=0}^{n} f_i l_i(x)$.

(b) The trick is to use the fact that

$$\sum_{j=0}^{n} l_j(x) = 1.$$

You can see this from the fact that this is just the interpolant to a function $f(x) = 1$. Then, we divide $l_i(x)$ from part (a) by this expression:

$$l_i(x) = \frac{\lambda_i w_{n+1}(x)}{x - x_i} \Bigg/ \sum_{j=0}^{n} l_j(x) = \frac{\lambda_i w_{n+1}(x)}{x - x_i} \Bigg/ \sum_{j=0}^{n} \frac{\lambda_j w_{n+1}(x)}{x - x_j} = \frac{\lambda_i}{x - x_i} \Bigg/ \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}.$$

Substituting this into $p_n(x) = \sum_{i=0}^{n} f_i l_i(x)$ and relabelling the index $j \to i$ gives the required formula.