Problems 2 - Minimax Approximation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Thursday 19th February**.
There will be a problem class on this chapter on Monday 16th February.
I use † to indicate (what I consider to be) trickier problems.

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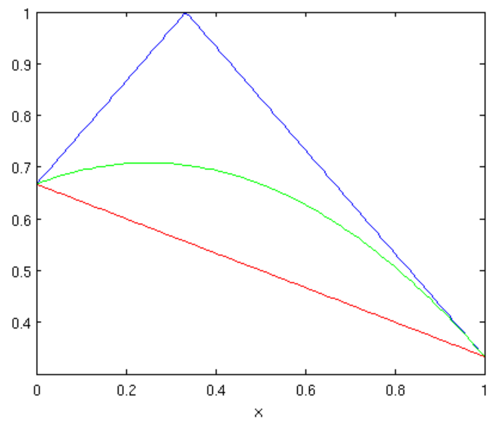
16. *Bernstein polynomial approximation.* Compute the approximations using Bernstein polynomials of degree $n = 1$ and $n = 2$ to the function $f(x) = 1 - |x - \frac{1}{3}|$ on $[0, 1]$. Verify that the approximation is converging in the ∞-norm.

Solution: We have

$$B_1(f,x) = f(0)\binom{1}{0}(1-x) + f(1)\binom{1}{1}x = f(0)(1-x) + f(1)x = \tfrac{2}{3}(1-x) + \tfrac{1}{3}x = \tfrac{2}{3} - \tfrac{1}{3}x,$$

$$\begin{aligned} B_2(f,x) &= f(0)\binom{2}{0}(1-x)^2 + f(\tfrac{1}{2})\binom{2}{1}x(1-x) + f(1)\binom{2}{2}x^2 \\ &= f(0)(1-x)^2 + 2f(\tfrac{1}{2})x(1-x) + f(1)x^2 = \tfrac{2}{3}(1-x)^2 + \tfrac{5}{3}x(1-x) + \tfrac{1}{3}x^2 = -\tfrac{2}{3}x^2 + \tfrac{1}{3}x + \tfrac{2}{3}. \end{aligned}$$

In pictures,



To verify convergence, we compute $\|f - B_1(f,x)\|_\infty$ and $\|f - B_2(f,x)\|_\infty$.
From the picture, we see that the maximum of $|f(x) - B_1(f,x)|$ occurs at $x = \frac{1}{3}$, so

$$\|f - B_1(f,x)\|_\infty = 1 - (\tfrac{2}{3} - \tfrac{1}{9}) = \tfrac{4}{9}.$$

To find $\|f - B_2(f,x)\|_\infty$, we check each subinterval. In $[0, \frac{1}{3}]$, we have

$$f(x) - B_2(f,x) = \tfrac{2}{3} + x - (-\tfrac{2}{3}x^2 + \tfrac{1}{3}x + \tfrac{2}{3}) = \tfrac{2}{3}x + \tfrac{2}{3}x^2 \quad \Longrightarrow \quad \frac{d}{dx}\big(f(x) - B_2(f,x)\big) = \tfrac{2}{3} + \tfrac{4}{3}x.$$

Hence $|f(x) - B_2(f,x)|$ is largest at $x = \frac{1}{3}$, where $f(x) - B_2(f,x) = \frac{8}{27}$. On the other hand, in $[\frac{1}{3}, 1]$, we have

$$f(x) - B_2(f,x) = \tfrac{4}{3} - x - (-\tfrac{2}{3}x^2 + \tfrac{1}{3}x + \tfrac{2}{3}) = \tfrac{2}{3} - \tfrac{4}{3}x + \tfrac{2}{3}x^2 \quad \Longrightarrow \quad \frac{d}{dx}\big(f(x) - B_2(f,x)\big) = -\tfrac{4}{3} + \tfrac{4}{3}x.$$

We conclude that the largest value of $|f(x) - B_2(f,x)|$ in this subinterval is also at $x = \frac{1}{3}$. Hence $\|f - B_2(f,x)\|_\infty = \frac{8}{27}$. This is less than $\|f - B_1(f,x)\|_\infty = \frac{4}{9} = \frac{12}{27}$, so we are indeed seeing convergence as n increases.

17. *Recursive definition of Bernstein polynomials.* Let $b_{n,k}$ for $k = 0, \ldots, n$ be the Bernstein basis functions, as defined in the lecture. Show that these basis functions satisfy the recursion relation

$$b_{n,k}(x) = (1-x)b_{n-1,k}(x) + xb_{n-1,k-1}(x).$$

Remark: This is the basis of de Casteljau's fast algorithm for drawing Bézier curves.

Solution: This is just an exercise in algebra. We have

$$\begin{aligned}(1-x)b_{n-1,k}(x)+xb_{n-1,k-1}(x) &= (1-x)\binom{n-1}{k}x^k(1-x)^{n-1-k}+x\binom{n-1}{k-1}x^{k-1}(1-x)^{n-1-(k-1)},\\ &= \binom{n-1}{k}x^k(1-x)^{n-k}+\binom{n-1}{k-1}x^k(1-x)^{n-k},\\ &= \left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]x^k(1-x)^{n-k},\\ &= \left[\frac{(n-1)!}{k!(n-1-k)!}+\frac{(n-1)!}{(k-1)!(n-k)!}\right]x^k(1-x)^{n-k},\\ &= \left[\frac{(n-1)!(n-k)}{k!(n-k)!}+\frac{(n-1)!k}{(k)!(n-k)!}\right]x^k(1-x)^{n-k},\\ &= \binom{n}{k}x^k(1-x)^{n-k}=b_{n,k}(x).\end{aligned}$$

18. *Derivatives of Bernstein polynomials.* Show that the derivatives of the Bernstein basis functions $b_{n,k}(x)$ for $k = 0, \ldots, n$ satisfy

$$\frac{d}{dx}b_{n,k}(x) = n\Big(b_{n-1,k-1}(x) - b_{n-1,k}(x)\Big).$$

Solution: This can be shown by direct differentiation:

$$\begin{aligned}\frac{d}{dx}b_{n,k}(x) &= \binom{n}{k}\frac{d}{dx}\Big(x^k(1-x)^{n-k}\Big),\\ &= \frac{kn!}{k!(n-k)!}x^{k-1}(1-x)^{n-k}+\frac{(n-k)n!}{k!(n-k)!}x^k(1-x)^{n-k-1},\\ &= \frac{n(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}+\frac{n(n-1)!}{k!(n-k-1)!}x^k(1-x)^{n-k-1},\\ &= n\Big(b_{n-1,k-1}(x)-b_{n-1,k}(x)\Big).\end{aligned}$$

19. *Cubic Bézier curves.* Verify that the cubic Bézier curve $\mathbf{B}_3(t)$ with control points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is tangent (i) at $\mathbf{x}_0$ to the line joining $\mathbf{x}_0$ and $\mathbf{x}_1$, and (ii) at $\mathbf{x}_3$ to the line joining $\mathbf{x}_2$ and $\mathbf{x}_3$.

Solution: The cubic Bézier curve is

$$\mathbf{B}_3(t) = (1-t)^3\mathbf{x}_0 + 3t(1-t)^2\mathbf{x}_1 + 3t^2(1-t)\mathbf{x}_2 + t^3\mathbf{x}_3,$$

so

$$\mathbf{B}_3'(t) = -3(1-t)^2\mathbf{x}_0 + 3(1-t)(1-3t)\mathbf{x}_1 + 3t(2-3t)\mathbf{x}_2 + 3t^2\mathbf{x}_3.$$

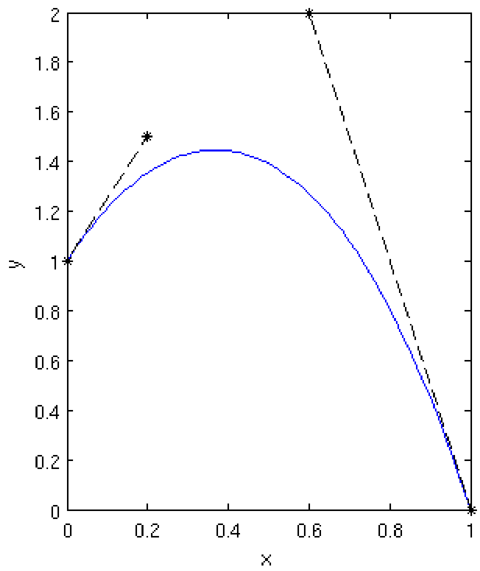
The tangent direction at $\mathbf{x}_0$ is $\mathbf{B}_3'(0) = -3(\mathbf{x}_1 - \mathbf{x}_0)$, while at $\mathbf{x}_3$ it is $\mathbf{B}_3'(1) = -3(\mathbf{x}_3 - \mathbf{x}_2)$. This shows that the required lines are indeed the tangents at these two points.

20. *A Bézier curve.* Find the parametric equations of the Bézier curve with control points $(0,1)$, $(\frac{1}{5},\frac{3}{2})$, $(\frac{3}{5},2)$ and $(1,0)$. Find the slope of the curve at each of its end-points and make a rough sketch of the curve.
As a check, you could try drawing the curve in Postscript.

Solution: We have

$$\begin{aligned}x(t) &= (1-t)^3(0)+3t(1-t)^2(\tfrac{1}{5})+3t^2(1-t)(\tfrac{3}{5})+t^3(1)=\tfrac{1}{5}t(3+3t-t^2),\\ y(t) &= (1-t)^3(1)+3t(1-t)^2(\tfrac{3}{2})+3t^2(1-t)(2)+t^3(0)=\tfrac{1}{2}(1-t)(2+5t+5t^2).\end{aligned}$$

For the slope, note that $x'(t) = \frac{3}{5} + \frac{6}{5}t - \frac{3}{5}t^2$ and $y'(t) = \frac{3}{2} - \frac{15}{2}t^2$. So at the endpoints $dy/dx(0) = y'(0)/x'(0) = \frac{5}{2}$ and $dy/dx(1) = y'(1)/x'(1) = -5$. Alternatively, you could get these from the slopes of the straight lines between the control points. Note that $x'(t)$ is always positive for $t \in [0,1]$, so that x is monotonically increasing. On the other hand, $y'(t)$ changes sign, so there is a maximum in y. The curve and its control points are shown below:



21. *Minimax approximation.* Find the minimax linear approximation to $f(x) = \sinh(x)$ on $[0,1]$.

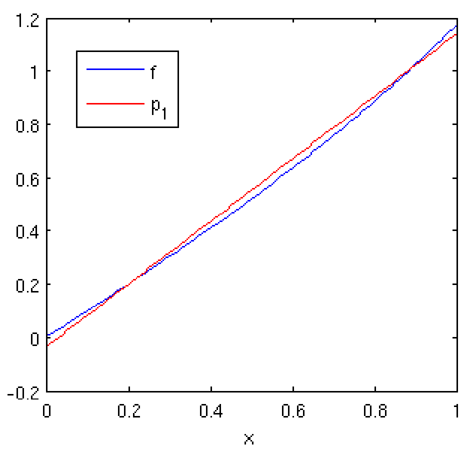
Solution: We look for a straight line $p_1^*(x) = a + bx$ such that f, p_1^* have an alternating set $\{0, \theta, 1\}$. We require

$$f(0) - p_1^*(0) = 0 - a = E, \tag{1}$$
$$f(\theta) - p_1^*(\theta) = \sinh(\theta) - a - b\theta = -E, \tag{2}$$
$$f(1) - p_1^*(1) = \sinh(1) - a - b = E. \tag{3}$$

There are four unknowns (a, b, θ, E) but only three equations - we get a fourth equation by requiring that the error has a turning point at $x = \theta$. This gives

$$\cosh(\theta) - b = 0. \tag{4}$$

Eliminating E from (3) gives $b = \sinh(1) = 1.1752$, and from (2) gives $a = \frac{1}{2}\big(\sinh(\theta) - \sinh(1)\theta\big) \approx -0.0343$, where θ is given by $\cosh(\theta) = \sinh(1)$ [from (4)]. The solution looks like this:



22. *Minimax approximation to a polynomial.* Find the minimax approximation of degree 4 to the polynomial $f(x) = x^5 + 2x^2 - x$.

Solution: As shown in the lecture,

$$p_4^*(x) = f(x) - \frac{1}{2^4}T_5(x).$$

We use the recurrence relation to compute the Chebyshev polynomial $T_5(x)$:

$$\begin{aligned}
&T_0(x) = 1, \qquad T_1(x) = x,\\
&T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1,\\
&T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x,\\
&T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1,\\
&T_5(x) = 2xT_4(x) - T_3(x) = 16x^5 - 20x^3 + 5x.
\end{aligned}$$

Therefore

$$p_4^*(x) = x^5 + 2x^2 - x - x^5 + \tfrac{5}{4}x^3 - \tfrac{5}{16}x = \tfrac{5}{4}x^3 + 2x^2 - \tfrac{21}{16}x.$$

23. *Non-monic polynomials.* Prove that, if p_m^* is the minimax polynomial of degree m for a polynomial $f \in \mathcal{P}_{m+1}$, then αp_m^* is the minimax approximation for αf.

Solution: We need to show that

$$\|\alpha f - \alpha p_m^*\|_\infty \leq \|\alpha f - p_m\|_\infty$$

for all $p_m \in \mathcal{P}_m$. For a scalar α, all norms satisfy $\|\alpha g\| = |\alpha|.\|g\|$, so for any $q_m \in \mathcal{P}_m$ we have

$$\|\alpha f - \alpha p_m^*\|_\infty = |\alpha|.\|f - p_m^*\|_\infty \quad \leq \quad |\alpha|.\|f - q_m\|_\infty \quad = \quad \|\alpha f - \alpha q_m\|_\infty.$$

Writing $q_m = p_m/\alpha$ gives the result (unless $\alpha = 0$ in which case it is trivial).

⋆ **24.** *De la Vallée Poussin Theorem.* Let $f(x) = -\cos(x)$ and $q_1(x) = 0.5x - 1.1$.

(a) Show that $\{0, \frac{1}{2}, 1\}$ is a non-uniform alternating set for f and q_1 on $[0, 1]$.

(b) Use the De la Vallée Poussin Theorem with these points to find a lower bound for $\|f - p_1^*\|_\infty$, where p_1^* is the minimax degree 1 polynomial for f on $[0, 1]$.

(c) Use q_1 to find an upper bound for $\|f - p_1^*\|_\infty$.

(d) By postulating a suitable alternating set, or otherwise, find p_1^*.

Solution: (a) We have

$$\begin{aligned}
f(0) - q_1(0) &= 0.1 := e_0,\\
f(\tfrac{1}{2}) - q_1(\tfrac{1}{2}) &= -0.0276 := e_1,\\
f(1) - q_1(1) &= 0.0597 := e_2.
\end{aligned}$$

The points are ordered and the successive e_i alternate in sign, so this is a non-uniform alternating set for f and q_1.

(b) By the DLVP Theorem, it follows from (a) that $\|f - p_1^*\|_\infty > 0.0276$.

(c) To find an upper bound, we can use $\|f - q_1\|_\infty$. To find this, consider the derivative

$$f'(x) - q_1'(x) = \sin(x) - 0.5.$$

Thus the error has a turning point at $\sin(x) = 0.5$, or $x = \frac{\pi}{6}$. At this point $f(\frac{\pi}{6}) - q_1(\frac{\pi}{6}) = -0.0278$. Thus the maximum on $[0, 1]$ is $\|f - q_1\|_\infty = 0.1$ (at the left end). Hence our upper bound is

$$\|f - p_1^*\|_\infty \leq 0.1.$$

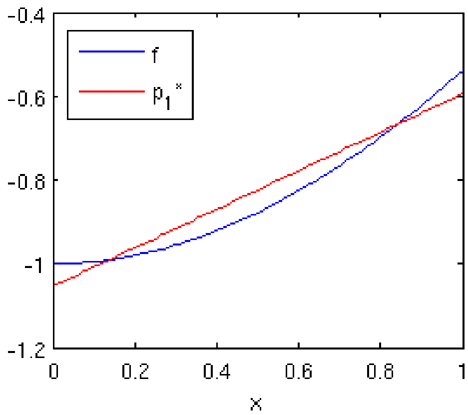
(d) We look for a straight line $p_1^*(x) = a + bx$ such that f, p_1^* have an alternating set $\{0, \theta, 1\}$. We require

$$f(0) - p_1^*(0) = -1 - a = E, \quad (5)$$
$$f(\theta) - p_1^*(\theta) = -\cos(\theta) - a - b\theta = -E, \quad (6)$$
$$f(1) - p_1^*(1) = -\cos(1) - a - b = E. \quad (7)$$

There are four unknowns (a, b, θ, E) but only three equations - we get a fourth equation by requiring that the error has a turning point at $x = \theta$. This gives

$$\sin(\theta) - b = 0. \quad (8)$$

Eliminating E from (7) gives $b = 1 - \cos(1) = 0.4597$, and from (6) gives $a = \frac{1}{2}\big(-1 - \cos(\theta) - [1 - \cos(1)]\theta\big) \approx -1.0538$, where θ is given by $\sin(\theta) = 1 - \cos(1)$ [from (8)]. The solution looks like:



25. *The Equioscillation Theorem.* In light of the Chebyshev Equioscillation Theorem, explain why the function $q_1(x)$ in Problem 24 could not possibly be the minimax degree 1 polynomial.

Solution: In the solution to Problem 24(c), we found that the local extrema of the error $f - q_1$ were 0.1, -0.0278, 0.0597. Therefore it is impossible to find an alternating set of length 3 for f and q_1 (remember that alternating sets must attain $\pm\|f - q_1\|_\infty$ at each point), meaning that q_1 cannot possibly be the minimax polynomial (by the Equioscillation Theorem).

26. *Every minimax polynomial is an interpolant.* Let $p_n^* \in \mathcal{P}_n$ be a minimax approximation to $f \in C[a, b]$. Show that there exist $n + 1$ distinct points $a < x_0 < x_1 < \ldots < x_n < b$ such that p_n^* is the polynomial interpolant in $\mathcal{P}_n$ to f at these $n + 1$ points.

Solution: We know from the Equioscillation Theorem that f and p_n^* have an alternating set of length $n+2$. Therefore, $f - p_n^*$ changes sign at $n + 1$ distinct points, which are the required interpolation points.

† 27. *Minimax polynomials of even functions.* Let $f \in C[-1, 1]$ be even, i.e. $f(-x) = f(x)$.

(a) Use the Equioscillation Theorem to prove that the minimax polynomial p_n^* is even for any $n \geq 0$.

(b) Prove that for any $n \geq 0$, $p_{2n}^* = p_{2n+1}^*$.

(c) Find the minimax polynomial of degree 1 for $f(x) = |x|$ on $[-1, 1]$.

Solution: (a) Since p_n^* is the minimax polynomial for f, these two functions have an alternating set $\{x_i\}$ of length $n + 2$ such that

$$f(x_i) - p_n^*(x_i) = (-1)^i E, \quad \text{for } i = 0, \ldots, n + 1, \text{ where } E = \|f - p_n^*\|_\infty.$$

Now let $g(x) = f(-x)$. We have

$$g(-x_i) - p_n^*(x_i) = (-1)^i E, \quad \text{for } i = 0, \ldots, n+1,$$

so $\{-x_i\}$ are an alternating set for $g(x)$, $p_n^*(-x)$. Thus $p_n^*(-x)$ is a minimax polynomial for $g(x)$. But f is even so $g = f$. Therefore $p_n^*(-x)$ is also a minimax polynomial for f. Since the minimax polynomial is unique (Corollary 2.4), it follows that $p_n^*(-x) = p_n^*(x)$, i.e. p_n^* is even.

(b) This follows from part (a). Since the minimax polynomial is even for any n, it cannot have any odd powers of x, so the coefficient of x^{2n+1} must be zero.

(c) Using the above, we know that since $f(x) = |x|$ is even, we must have $p_1 = p_0$. By symmetry, $p_0 = \frac{1}{2}$. Hence $p_1(x) = \frac{1}{2}$.

28. *Remez algorithm.* Use the Remez Exchange algorithm to compute the linear minimax approximation to $f(x) = x^2$ on $[0, 3]$, using the initial reference set $\{0, 1, 3\}$. Comment on the convergence of the algorithm.

Solution: Let $p_1 = a_0 + a_1 x$.

Step 1: solve the linear system

$$\begin{aligned} a_0 + E &= 0^2 = 0, \\ a_0 + a_1 - E &= 1^2 = 1, \\ a_0 + 3a_1 + E &= 3^2 = 9. \end{aligned}$$

Solving this system gives $a_0 = -1$, $a_1 = 3$, $E = 1$, i.e. $p_1^{(1)} = -1 + 3x$.

Step 2: to update the reference set, we look for the point of maximum $|f - p_1^{(1)}|$. We have

$$f - p_1^{(1)} = x^2 - 3x + 1.$$

This has a turning point at $x = \frac{3}{2}$, where $f(\frac{3}{2}) - p_1^{(1)}(\frac{3}{2}) = -\frac{5}{4}$. At the end-points, $f(0) - p_1^{(1)}(0) = -1$ and $f(3) - p_1^{(1)}(3) = 1$, so $\|f - p_1^{(1)}\|_\infty = \frac{5}{4}$. At the middle point of the old reference set, $f(1) - p_1^{(1)}(1) = -1$. So we form the new reference set $\{0, \frac{3}{2}, 3\}$.

Step 1: now solve the linear system

$$\begin{aligned} a_0 + E &= 0^2 = 0, \\ a_0 + \tfrac{3}{2}a_1 - E &= (\tfrac{3}{2})^2 = \tfrac{9}{4}, \\ a_0 + 3a_1 + E &= 3^2 = 9. \end{aligned}$$

Solving this system gives $a_0 = -\frac{9}{8}$, $a_1 = 3$, $E = \frac{9}{8}$, i.e. $p_1^{(2)} = -\frac{9}{8} + 3x$.

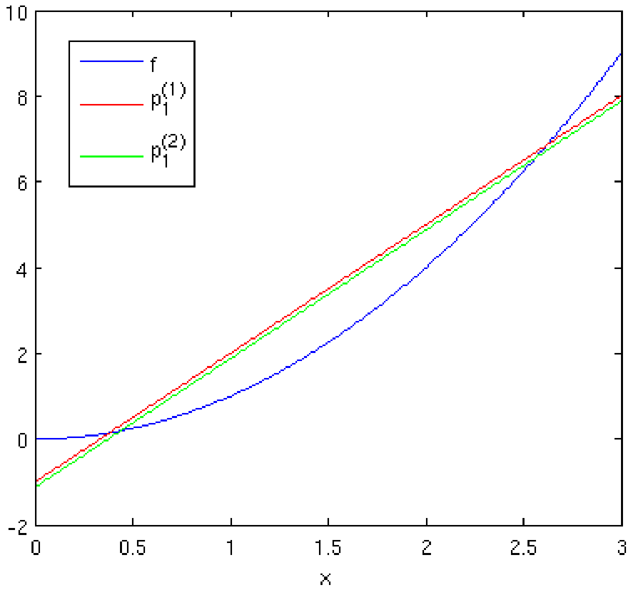
Step 2: Now we have

$$f - p_1^{(2)} = x^2 - 3x + \tfrac{9}{8}.$$

Again this has a turning point at $x = \frac{3}{2}$, but now $f(\frac{3}{2}) - p_1^{(2)}(\frac{3}{2}) = -\frac{9}{8}$. The end point values are now $f(0) - p_1^{(2)}(0) = \frac{9}{8}$ and $f(3) - p_1^{(2)}(3) = \frac{9}{8}$. Now the maximum $|f - p_1^{(2)}|$ is achieved with alternating signs at $\{0, \frac{3}{2}, 3\}$, so this is an alternating set. Hence (by Equioscillation Theorem) the minimax polynomial is

$$p_1^*(x) = p_1^{(2)} = 3x - \tfrac{9}{8}.$$

The algorithm has converged to the exact solution after two steps. See the illustration below:


f
$p_1^{(1)}$
$p_1^{(2)}$
10
8
6
4
2
0
-2
0
0.5
1
1.5
2
2.5
3
x