A feeling for Khovanov homology

Andrew Lobb *

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One of the pleasures of studying knot and links is that, at some level, everybody understands what you are thinking about.

Formally speaking, a link is a smooth compact oriented 1-dimensional submanifold of the 3-sphere $\mathbb{S}^3$ or of 3-space $\mathbb{R}^3$, and a knot is a link of only one component. Some pictures of knots appear just below in Figures 1–6.

Informally speaking, a knot is just a knotted up piece of string with its ends glued together. Knots and links are most often considered up to isotopy, meaning up to smooth deformations that do not allow the passing of strands through one another. A fundamental question of knot theory is how to tell when two knots are different or really the same – given this knotted up piece of string and that knotted up piece of string can we, without getting out the scissors, arrange this one to look exactly like that one?

Another pleasure open to knot theorists is that their specialism is frequently enriched by contributions from outsiders. Perhaps this is not so surprising: knots give the simplest non-trivial examples of isotopy classes of submanifolds, so concepts from knot theory are encountered in other areas of mathematics and physics that have manifolds and submanifolds as underlying objects of interest. Similarly, concepts originating outside knot theory can often find a testing ground among knots and links.

To a knot theorist, this is all money for jam. A concept in knot theory is understood by physicists or algebraists, they come up with generalizations, these get fed back to you, and all of a sudden you have a whole new family of ideas to play with. Khovanov homology is the best example of such a concept.

At first sight, Khovanov homology seems a fairly innocuous link invariant – perhaps of use to those of us engaged in classifying isotopy classes of links, but with no obvious potential relative to other such invariants to break its bounds. However, in the quarter century or so since it appeared on the scene, Khovanov homology has infiltrated other areas of mathematics and physics in an unprecedented fashion, eclipsing in this way even its own direct forebear – the Jones polynomial.

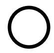
A feeling for Khovanov homology

Anyone who has seen Holbein's painting *The Ambassadors* will know that the same object can present very differently when viewed from an alternative perspective. Khovanov homology is not an exception to this, and it is a measure of the ubiquity of the theory that it admits understanding from several disparate vantage points.

The purpose of this article is to give the casual reader, who may have a sandwich in one hand as they read, a feeling for Khovanov homology, so that they finish the article and the sandwich with the sense that they know *what kind of thing Khovanov homology is.* To achieve our ends we will approach Khovanov homology from several different directions and record what we see.

To each link L, Khovanov homology associates a finitely-generated abelian group, $\mathrm{Kh}(L)$, invariant

*The author is a professor of mathematics at Durham University. His email address is andrew.lobb@durham.ac.uk.



$j \backslash i$	-1	0	1
3			
1		$\mathbb{Z}$	
-1		$\mathbb{Z}$	
-3			

Figure 1: The unknot.

$j \backslash i$	-3	-2	-1	0
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}_2$		
-9	$\mathbb{Z}$			

Figure 2: The left-handed trefoil.

up to isotopy of the link. This splits as a direct sum along a bigrading $(i, j) \in \mathbb{Z} \times \mathbb{Z}$

$$\mathrm{Kh}(L) = \bigoplus_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} \mathrm{Kh}^{i,j}(L).$$

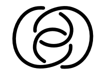
The simplest knots are the unknot, the left- and right-handed trefoils, and the figure eight knot – these are the first four knots K whose homologies $\mathrm{Kh}^{i,j}(K)$ we have given in Figures 1–4, plotted in the (i, j)-plane. The i-grading is called the *homological grading*, while the j-grading is called the *quantum grading*. We write $\mathbb{Z}_n$ for the n-element cyclic group. So that the reader may get their bearings quickly, we have included a thick dot in bidegree $(i, j) = (0, 0)$.

The forty-second nine-crossing knot to appear in Rolfsen's knot table, 9_{42}, is well-known in certain quarters as being the first knot in the table which nevertheless exhibits generic characteristics from various points of view. We have given the homology of 9_{42} in Figure 5.

Each of these five knots only admits one orientation up to isotopy, so we have omitted to orient the knots.

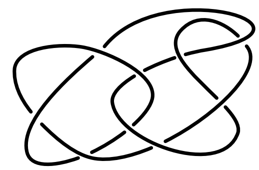
$j \backslash i$	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}_2$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

Figure 3: The right-handed trefoil.



$j \backslash i$	-2	-1	0	1	2
5					$\mathbb{Z}$
3					$\mathbb{Z}_2$
1			$\mathbb{Z}$	$\mathbb{Z}$	
-1		$\mathbb{Z}$	$\mathbb{Z}$		
-3		$\mathbb{Z}_2$			
-5	$\mathbb{Z}$				

Figure 4: The figure eight knot.



$j \backslash i$	-4	-3	-2	-1	0	1	2
7							$\mathbb{Z}$
5							$\mathbb{Z}_2$
3					$\mathbb{Z}$	$\mathbb{Z}$	
1				$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$		
-1				$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$		
-3		$\mathbb{Z}$	$\mathbb{Z}$				
-5		$\mathbb{Z}_2$					
-7	$\mathbb{Z}$						

Figure 5: The knot 9_{42}.

Phenomenology

Most mathematicians are phenomenologists. When trying to understand something, we often think about a toy case, then a toy case with bells on, then a toy case with bells and whistles, before we finally allow ourselves to think generally.

Above we see several examples of homology groups, and just by looking at them the reader will have started to get a feeling for Khovanov homology. What strikes us about these examples? Let's go through them.

Firstly, looking at Figure 1, we see that the homology of the unknot is 2-dimensional and is supported in bidegrees $(0, -1)$ and $(0, 1)$. If we were making a guess of what the homology of the unknot ought to be, we might guess that it should be 1-dimensional and supported in bidegree $(0, 0)$. This guess is correct for the variant of Khovanov homology called *reduced* Khovanov homology, but we are going to stick with vanilla flavored Khovanov homology for this exposition.

So much for the unknot, let's take a look at the trefoils in Figures 2 and 3. If you were to commission the carving of the following woodblock stamps, then you would be able to use them to produce the homology tables of the unknot, of the left-handed trefoil, and of its mirror image the right-handed trefoil.

$\mathbb{Z}$
$\mathbb{Z}$

	$\mathbb{Z}$
	$\mathbb{Z}_2$
$\mathbb{Z}$	

The block on the left is all that you would need to make the homology of the unknot, whereas you would also need the block on the right for the trefoils. This latter is sometimes called the *knight's move* block by chess-playing mathematicians.

We also note that the homological degrees i which support the homology run from -3 to 0 for the left-handed trefoil and from 0 to 3 for the right-handed trefoil.

If you choose orientations for the trefoils you will see that the left-handed trefoil has three negative crossings in the given diagram while the right-handed trefoil has three positive crossings, following the conventions below.

$-$ $\qquad\qquad$ $+$

To add weight to our musings, the unknot has 0 crossings of either sign and its homology is entirely supported in homological grading $i = 0$. So we might guess that the homological support has something to do with the numbers of positive and negative crossings in a minimal crossing-number diagram.

Finally, there seems to be some sort of duality going on between the left-handed and the right-handed trefoil. These knots themselves are dual in the sense that one is the mirror image of the other. In their homology tables we see that, up to a bit of messing about with the $\mathbb{Z}_2$ summand (which you might hope to explain by some sort of universal coefficient theorem), the homology of one looks like it is obtained from the homology of the other by rotation by π around the thick dot that marks the bidegree $(0, 0)$.

Moving now to the figure eight knot in Figure 4, we see that our woodblocks still stand us in good stead – we need to use the unknot block once and then the knight's move twice.

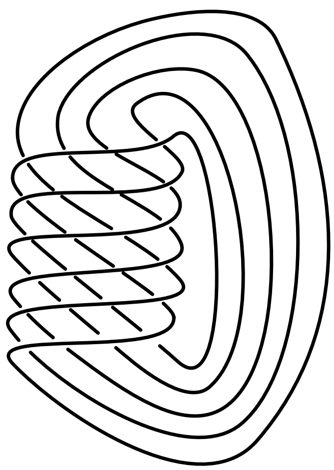
Moreover, the homology of the figure eight knot is supported between gradings $i = -2$ and $i = 2$, matching the two negative and two positive crossings in the diagram. And finally the homology has a self-duality around $(0, 0)$, which we would expect once we are told that the figure eight knot is its own mirror image.

Next we turn to the knot known as 9_{42}, pictured in Figure 5. It takes a little bit of scrutinizing, but we see that our woodblocks still suffice. At this point we notice that the unknot block is not getting as much use as the knight's move. For all the examples so far, the unknot block gets used exactly once while the rest of the homology is filled up with knight's moves. We further notice that the unknot block always appears in grading $i = 0$, but its quantum grading j can slide up or down.

These woodblocks, which we commissioned as a time-saving device, are starting to seem quite important. Do they remind us of anything? Relating the groups to the usual singular homology of a space, the unknot block has the total homology of a sphere, while the knight's move block has the total homology of $\mathbb{RP}^3$. We do not see yet what these spaces might have to do with knots or their Khovanov homology.

The homology of 9_{42} is supported between degrees $i = -4$ and $i = 2$, while the diagram given has four negative and five positive crossings. So the support of the homology cannot be telling us exactly how many negative and positive crossings there are in a minimal crossing-number diagram of the knot, but maybe it is giving us lower bounds on these quantities?

Let's end our phenomological tour by looking at a larger knot – the $(5, 6)$ torus knot $T_{5,6}$ – which has 24 positive and zero negative crossings in its minimal diagram $D_{5,6}$ given in Figure 6.



	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
43															$\mathbb{Z}_3$
41													$\mathbb{Z}$		
39													$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}$	
37											$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	
35										$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}$		
33								$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$				
31						$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$						
29						$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$						
27				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$								
25				$\mathbb{Z}_2$	$\mathbb{Z}$										
23			$\mathbb{Z}$												
21	$\mathbb{Z}$														
19	$\mathbb{Z}$														

Figure 6: The diagram $D_{5,6}$ of the $(5,6)$ torus knot $T_{5,6}$ and the Khovanov homology of $T_{5,6}$.

We have not included a thick dot in bidegree $(0,0)$ in the interests of space. The homology is supported between degrees $i = 0$ and $i = 14$, so our guess about the support possibly providing a lower bound on the numbers of positive and negative crossings in a diagram of the knot is not yet contradicted.

Unfortunately, the reader will notice the groups $\mathbb{Z}_3$ and $\mathbb{Z}_5$ appearing as summands in the higher homological degrees. Before we destroy our woodblocks in frustration, let us also notice that there is still an unknot block summand appearing in degree $i = 0$ and there are also several instances of the knight's move appearing among more general configurations.

And what else? The middle of the unknot block in degree $i = 0$ is placed at quantum degree $j = 20$. Since every second j-degree seems to support no homology, this is morally a jump of 10 above $j = 0$. If we look up $T_{5,6}$, the first topological measure of its complexity is its *genus* which is $g(T_{5,6}) = 10$. The genus of a knot is the minimal genus of a compact orientable surface in $\mathbb{S}^3$ whose boundary is the knot. Is this just a coincidence? Looking at the homologies of the earlier knots we have studied tells us that the height of the unknot block in degree $i = 0$ is *not* just recording the genus of the knot, although this does seem to be the case for the genus and homology of any other positive torus knot that we care to look up.

Finally, we should record that in all examples there seems to be a certain *diagonalness* to Khovanov homology, with the support of the homology very roughly lining up along a line from bottom left to top right. There does not seem to be an expected slope nor, in general, an expected y-intercept.

Now that we have started to get a feeling for Khovanov homology, let's ask for its historical meaning by discussing the context in which it was discovered.

History of the discovery

Mikhail Khovanov put a paper [Kho00] introducing his homology theory on the arXiv at the tail end of the 20th century, a couple of years after completing his PhD at Yale.

Khovanov homology may be viewed as an early exemplar of the success of the program known as *categorification*, a term coined by Louis Crane. Khovanov's PhD advisor Igor Frenkel was another originator and proponent of the nascent subject, and collaborated in a seminal paper with Crane [CF94]. Categorification was originally conceived with a somewhat precise meaning, but is now more often understood as an umbrella term used to refer to various lifts of structure. It is one of those things of which you may say, after having become acquainted with a few examples, that you know it when you see it.

The ideas around categorification included from quite early on the belief that one should be able to categorify the representation theory of quantum groups, a so-far successful enterprise that is still employing mathematicians today. The representation theory of quantum $\mathfrak{sl}_2$ yields up the Jones polynomial $V(L) \in \mathbb{Z}[q, q^{-1}]$, which is an invariant of links L whose discovery earned Vaughan Jones a Fields medal. So, it was believed that the Jones polynomial should eventually admit a 'lift' to some stronger invariant, but perhaps this lift would only be accessible a little further down the line once the groundwork had been laid by categorifying the underlying algebraic structure.

On the other hand, as well as arising from quantum representation theory, the Jones polynomial has a particularly simple definition provided by Kauffman [Kau87]. In this reformulation, given a diagram of the link, its Jones polynomial is expressed as an alternating sum of Laurent polynomials in q, each of which has non-negative integer coefficients.

Once you know to listen for it, this definition strongly echoes the notion of Euler characteristic. The Euler characteristic is a topological invariant expressed as an alternating sum of non-negative integers (the numbers of simplices or cells of each dimension) which are themselves not topological invariants. In what might now be described by the more excitable at the departmental teatime as one of the earliest examples of categorification, one realizes that these numbers are non-negative because they ought to be interpreted as the dimension of a vector space (or as the rank of an abelian group). These vector spaces or groups fit into a chain complex, of which the homology groups turn out to be strong topological invariants.

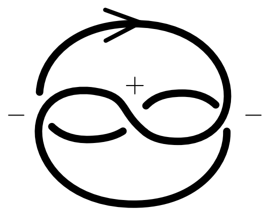
Khovanov saw that one could attempt to play the same game with Kauffman's definition of the Jones polynomial. The plan was to leap-frog the business of categorifying quantum representation theory and to jump straight to the answer – a homology theory whose Euler characteristic, appropriately defined, should be the Jones polynomial.

The quantum world

The philosophy of categorification would suggest that such a putative homology theory would have something to tell us about 4-dimensional topology and might be a quantum competitor or counterpart to the gauge and Floer theoretic invariants that we shall shortly come to discuss.

When we use the word *quantum* here, we are not thinking of quantum mechanics. Rather we use the term to refer to a collection of combinatorial topological invariants arising out of the representation theory of quantum groups. A quantum group is not a group, but rather a non-commutative algebra which may be thought of as a perturbation by an extra parameter $\hbar$ of the universal enveloping algebra of a Lie algebra. Specializing by setting $\hbar = 0$ recovers the universal enveloping algebra. So we see that the terminology arises by analogy with the fact that setting Planck's constant to zero recovers classical physics from quantum physics.

The reader intent on acquiring a feeling for Khovanov homology should think of two streams of invariants – the analytically defined gauge and Floer theoretic invariants, and the combinatorial and algebraic quantum invariants. The confluence of these two apparently distinct streams will form part of our narrative.



$n_- = 2,\ n_+ = 1,$

$w = n_+ - n_- = -1.$

Figure 7: The *trefail* diagram of the unknot.

Khovanov's chain complex

Khovanov's construction of a chain complex from a link diagram is so elegant that it has an air of inevitability about it. The original paper [Kho00] is very concrete and readable by anyone who has seen a modicum of homological algebra.

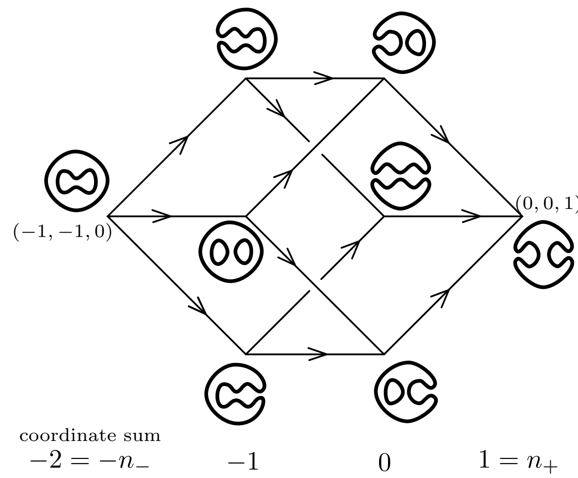
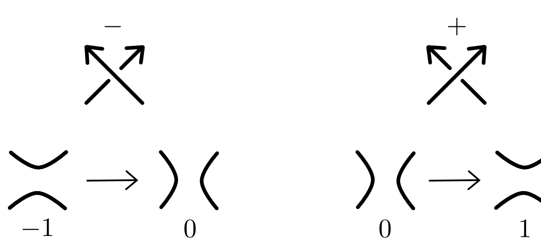


Figure 8: The smoothings of the trefail diagram decorate the vertices of a cube.

The Khovanov cube

We present Khovanov's construction in the case of a link diagram of sufficient complexity so that the reader can recover the general case. In fact, we start with a diagram of the unknot known to the cognoscenti as the *trefail*, pictured in Figure 7.

We have included an orientation so that we can collect some sign data from the crossings, following the conventions below. We write $n_- = 2$ for the number of negative crossings, and $n_+ = 1$ for the number of positive crossings. The quantity $w = n_+ - n_- = -1$ is known as the *writhe* of the diagram.



Below the positive and negative crossings you see two *smoothings* of these crossings. Each smoothing is associated with an integer – either −1, 0, or 1. We are going to be interested in smoothings of the diagram which arise from picking a smoothing at each crossing. Each diagram smoothing is nothing more than a collection of circles embedded in the plane.

Since the diagram has three crossings, there are $2^3 = 8$ possible diagram smoothings. Picking an ordering on the negative crossings enables us to associate each diagram smoothing with a vertex of the cube

$$[-1,0]^2 \times [0,1] = [-1,0]^{n_-} \times [0,1]^{n_+}.$$

In Figure 8 we have put the vertex $(-1,-1,0)$ on the far left and the opposite vertex $(0,0,1)$ on the far right, and the edges are oriented from left to right. Moving along any edge will increase exactly one coordinate of the cube by 1, leaving the others fixed. So we see that the coordinate sum of the vertices ranges between $-2 = -n_-$ and $1 = n_+$. The cube is arranged so that vertices with the same coordinate sum lie in a vertical line in the page.

To pass to algebra, our fundamental building block will be the free 2-dimensional $\mathbb{Z}$-module

$$V = \langle v_-, v_+ \rangle.$$

The module V comes graded by the quantum j-grading that we have already encountered. We think of v_-, v_+ as being homogeneous of gradings $j = -1, +1$ respectively. A useful shorthand to

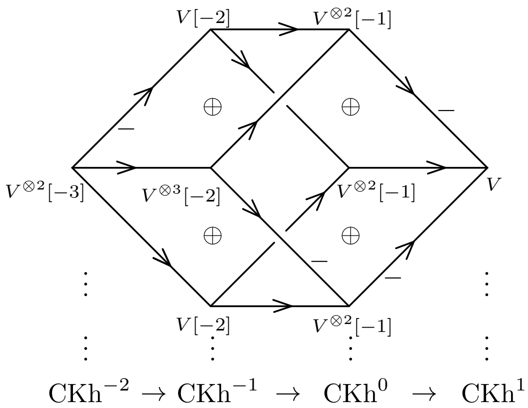


Figure 9: The chain groups arising from the trefail diagram.

record this is to write

$$\operatorname{qdim}(V) = q^{-1} + q$$

for the so-called *quantum dimension* of V in which the power of q records the quantum grading and the coefficient records the dimension. In other contexts this shorthand is referred to as the Poincaré polynomial.

To create each chain group we shall be applying three algebraic operations to copies of V; these operations are tensor product, direct sum, and quantum grading shift. The quantum grading shift is exactly as it sounds, and is written by appending a square bracket with the degree of the shift. Concretely, we have for each free j-graded module W

$$\operatorname{qdim}(W[n]) = q^n \operatorname{qdim}(W).$$

The chain groups CKh^i of the Khovanov chain complex coming from the trefail diagram of the unknot are given in Figure 9.

Recall that we started by decorating each corner of the cube by a smoothing of the trefail diagram. Now each such smoothing has been replaced by a tensor power of V together with some shift. The tensor power corresponds to the number of components of the smoothing, while the shift is by $i + w$ where i is the coordinate sum and $w = -1$ is the writhe. Finally, the chain group in homological degree i is given by taking the direct sum of the groups on corners with coordinate sum i.

This decorated cube is known as the *Khovanov cube*, which is a name also applied to the earlier cube of smoothings. It is useful in thinking about the chain complex to bear both in mind.

The Jones polynomial

At this point we pause to take stock. We do not yet have a chain complex because we have not given the chain maps between the chain groups

$$\partial^i_{\mathrm{Kh}} \colon \mathrm{CKh}^i \longrightarrow \mathrm{CKh}^{i+1}.$$

But no matter what chain maps we give, it will of course not affect the Euler characteristic, since this should just be the alternating sum of the dimensions of the chain groups. We note that each chain group is quantum graded. If we give chain maps that preserve the quantum grading then the chain complex will split as a direct sum of chain complexes, one complex for each quantum grading.

$$\mathrm{CKh}^i = \bigoplus_j \mathrm{CKh}^{i,j},$$

$$\partial^i_{\mathrm{Kh}} \colon \mathrm{CKh}^{i,j} \longrightarrow \mathrm{CKh}^{i+1,j}.$$

If we compute

$$\sum_i (-1)^i \operatorname{qdim}(\mathrm{CKh}^i) = \sum_{i,j} (-1)^i q^j \dim(\mathrm{CKh}^{i,j})$$

then we will get a Laurent polynomial in q in which the coefficient of q^j will just be the Euler characteristic of the chain complex summand in quantum degree j.

This Laurent polynomial is in fact the *Jones polynomial* of the link, and its expression as an alternating sum of Laurent polynomials in q with non-negative integer coefficients is exactly Kauffman's reformulation of the Jones polynomial.

Now it is clear where the construction up to this point has come from. If you believe that Kauffman's alternating sum is actually computing the Euler characteristic of a chain complex and you want to guess the chain groups, then a natural choice would be the one that we have given above.

The differential

Of course, it is well and good to imagine that we can make a chain complex this way, but to do so we still need to give the differential. We cannot cheat by just using the zero differential, because the aim is to produce homology groups that do not depend on the chosen diagram.

To specify a differential $\partial^i_{\mathrm{Kh}} \colon \mathrm{CKh}^i \to \mathrm{CKh}^{i+1}$, we decorate each edge of the Khovanov cube with a map, and then ∂^i_{Kh} is the direct sum of all those maps decorating edges connecting the summands of CKh^i to those of CKh^{i+1}.

Looking back at the cube of smoothings, we recall that as we travel along an edge either two circles of a smoothing are being joined into one, or a single circle is being split into two. Each circle of a smoothing corresponds to a tensor factor V. So in order to give the edge maps we first write down maps $m \colon V \otimes V \to V$ and $\Delta \colon V \to V \otimes V$ and extend by the identity map on those tensor factors V corresponding to circles that do not get merged or split as we move along an edge.

$$\begin{aligned}
m \colon V \otimes V \to V \colon\ & v_- \otimes v_- \mapsto 0, \\
& v_- \otimes v_+ \mapsto v_-, \\
& v_+ \otimes v_- \mapsto v_-, \\
& v_+ \otimes v_+ \mapsto v_+; \\
\Delta \colon V \to V \otimes V \colon\ & v_- \mapsto v_- \otimes v_-, \\
& v_+ \mapsto v_- \otimes v_+ + v_+ \otimes v_-.
\end{aligned}$$

These edge maps that we have now specified make each square face of the cube *commute* (this should not be obvious, but it is a relatively easy check). In fact, m and Δ are one of the very few non-trivial choices that achieve this, which may explain how they were chosen. In order that the differential squares to zero, we want rather that each face of the cube should *anti-commute*, so that travelling one way around any face gives the negative of travelling the other way.

To turn the commutative faces into anti-commutative faces we replace some (in this case, four) of the edge maps with their negative, as indicated by the minus signs in the algebraic cube diagram. It can be checked that any selection of negative edges achieving anti-commutativity of faces results in a chain homotopic complex.

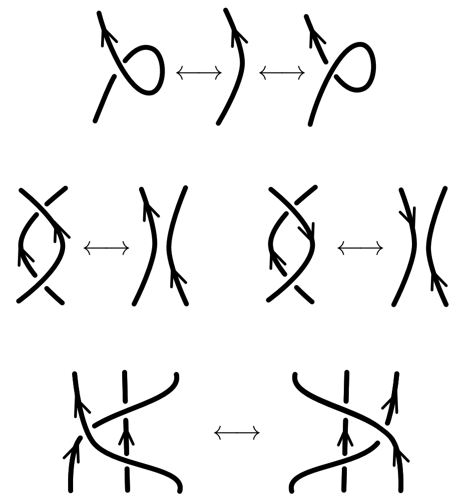


Figure 10: The oriented Reidemeister moves.

Finally, note that both m and Δ are quantum j-graded of degree -1. Since there is a quantum shift by $i + w$ on the chain group summands, it follows that ∂^i_{Kh} preserves j, as we wanted.

It remains to verify that the resulting homology does not depend on the diagram of the link chosen. Since any two link diagrams of the same link are related by a finite sequence of Reidemeister moves, this amounts to finding a chain homotopy equivalence between two diagrams differing by such a move.

More precisely, the *oriented Reidemeister moves* are pictured in Figure 10. To each double-headed arrow Khovanov associated a chain homotopy equivalence between the chain complexes of diagrams differing locally by the moves.

Theorem 1 ([Kho00]). *Suppose that D is a diagram of a link L. The chain complex $\mathrm{CKh}^{i,j}(D)$ associated to D is invariant under Reidemeister moves up to chain homotopy equivalence. Consequently, the homology groups $\mathrm{Kh}^{i,j}(L)$ are an invariant of L. The graded Euler characteristic*

$$\sum_{i,j} (-1)^i q^j \mathrm{rk}(\mathrm{Kh}^{i,j}(L))$$

coincides with the Jones polynomial of L.

If the aim of Khovanov's paper were simply to produce a bigraded abelian group whose graded Euler characteristic coincides with the Jones polynomial then he could have simply started with the Jones polynomial, forgotten about the link, and defined a bigraded group from there. However, Khovanov homology is strictly stronger than the Jones polynomial (for example the knots 5_1 and 10_{132} have the same Jones polynomial but different Khovanov homologies). And, as we shall soon see, Khovanov homology is more than just an improvement on the Jones polynomial as an isotopy invariant.

Observations

Now that we have a definition, let's see whether we can explain some of the phenomenological features of Khovanov homology that we recorded earlier.

From the definition we can immediately see that if the homology of a link is non-zero in degree $i = m$ then any diagram of the link must have at least $-m$ negative crossings if $m < 0$, and at least m positive crossings if $m > 0$. So this bears out some of our observations on the homological support of the homology.

With a little bit of work, the reader would be able to convince themselves that the complex associated to the mirror of a link diagram is isomorphic to the dual of the complex associated to the unmirrored diagram. This explains the duality in the homology that we observed between the left- and right-handed trefoils, and the self-duality in the homology of the figure eight knot.

What about the copy of the homology of the unknot that seems to appear in homological degree $i = 0$? Let's think about the complex associated to the diagram $D_{5,6}$ that we drew above of the torus knot $T_{5,6}$.

Firstly, $D_{5,6}$ has 24 positive crossings, meaning that the complex $\mathrm{CKh}^i(D_{5,6})$ runs from degree $i = 0$ to degree $i = 24$. We can identify the chain group in degree $i = 0$ very quickly. It is just the group that decorates the vertex $(0, 0, \ldots, 0)$ of the Khovanov cube. To construct this group we first take the 0-smoothing of every crossing of $D_{5,6}$. This gives us 5 nested circles in $\mathbb{R}^2$. Remembering to shift by $i + w = 0 + 24 = 24$ we see that

$$\mathrm{CKh}^0(D_{5,6}) = V^{\otimes 5}[24].$$

Since $\mathrm{CKh}^{-1}(D_{5,6}) = 0$, we have

$$\mathrm{Kh}^0(T_{5,6}) = \ker(\partial^0_{\mathrm{Kh}} \colon \mathrm{CKh}^0(D_{5,6}) \to \mathrm{CKh}^1(D_{5,6})).$$

Thinking about the vertices of the Khovanov cube in degree $i = 1$, we note that each of them is decorated by 4 circles, arising from merging a pair of the 5 nested circles on the vertex $(0, 0, \cdots, 0)$. Looking back at the definition of the map m associated to edges of the Khovanov cube along which circles merge, we see that the element

$$v_-^{\otimes 5} \in V^{\otimes 5}[24] = \mathrm{CKh}^0(D_{5,6})$$

must be in the kernel of ∂^0_{Kh} and hence represents a non-trivial homology class in the 0th homology group.

The quantum j-grading of this element is $24 - 5 = 19$, so this corresponds to the copy of $\mathbb{Z}$ that appears at $(0, 19)$ in the table of $\mathrm{Kh}^{i,j}(T_{5,6})$. The one task that this article will leave to the reader, in between bites of their sandwich, is to find the generator of the copy of $\mathbb{Z}$ at $(0, 21)$.

Of course, there was nothing particularly special about $T_{5,6}$ among all torus knots. If we were to repeat the arguments for the torus knot $T_{p,q}$ where $0 < p < q$ with p, q coprime, we would see copies of $\mathbb{Z}$ in bidegrees $(0, (p-1)(q-1) \pm 1)$. In other words we would find the unknot homology but shifted up by $(p-1)(q-1)$. This matches twice the knot genus

$$g(T_{p,q}) = \frac{(p-1)(q-1)}{2},$$

although we might make the mistake, at this point in our tour of Khovanov homology, of discounting this as a coincidence rather than recognizing it as a foreshadowing.

Floer homology

It was recognized almost immediately that Khovanov homology had some similarities with homological invariants of 3-manifolds known as *Floer* homologies. Although combinatorial formulations of several Floer homologies have appeared, these invariants are, at heart, analytic: the differentials of their chain complexes count solutions to differential equations, even if combinatorial means might be found to make the count.

Floer homologies of 3-manifolds arose in the last two decades of the twentieth century from the study of gauge-theoretic invariants of smooth closed 4-manifolds. These gauge theoretic invariants notably include the Donaldson invariant from instanton gauge theory or the Seiberg-Witten invariant from monopole gauge theory.

Removing two open balls from a closed 4-manifold X turns it into a cobordism $\mathring{X}$ from $\mathbb{S}^3$ to $\mathbb{S}^3$. It was realized that the gauge theoretic invariants of the closed manifold are determined by a map on the relevant Floer homology of $\mathbb{S}^3$ (which is always a very simple algebraic structure such as a polynomial ring) induced by the cobordism

$$\mathring{X}_*\colon H_*^{\mathrm{Floer}}(\mathbb{S}^3) \longrightarrow H_*^{\mathrm{Floer}}(\mathbb{S}^3).$$

Essentially one reproduces the invariants of closed 4-manifolds by using the functoriality of Floer homology.

It is interesting to note the progression in this case was by jumping down from dimension 4 to dimension 3. There is much effort at the moment in trying to jump further down the dimensions by associating invariants of some kind to 2–, 1–, and eventually to 0–manifolds. The idea is to attempt to understand known examples of $(3+1)$-dimensional topological quantum field theories in the spirit of the cobordism hypothesis laid out by Baez and Dolan [BD95].

From its inception it was expected, in line with the program of categorification, that Khovanov homology should not just be a strengthening of the essentially 3-dimensional Jones polynomial, but should also give rise to 4-dimensional invariants. As we have already mentioned, the Jones polynomial being the Euler characteristic of Khovanov homology suggests singular homology as a useful parallel. Singular homology not only strengthens the Euler characteristic invariant of a space but is functorial for continuous maps. The correct analogue of continuous map turns out to be *link cobordism.*

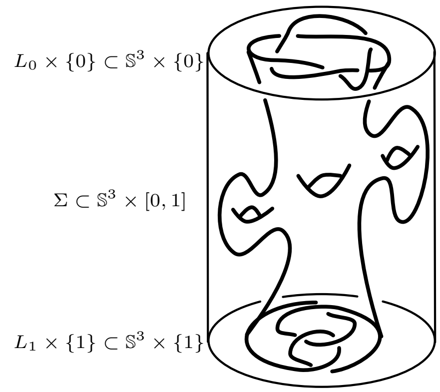


Figure 11: A smooth link cobordism Σ from L_0 to L_1.

A link cobordism between the links $L_0, L_1 \subset \mathbb{S}^3$ is a smooth embedding of a compact oriented surface $\Sigma \subset \mathbb{S}^3 \times [0,1]$ which satisfies $\partial\Sigma = L_0 \times \{0\} \cup L_1 \times \{1\}$. We give a picture of this in Figure 11. In fact, the object of interest is usually the isotopy class of a link cobordism, in which we allow smooth deformations of Σ while keeping the boundary links fixed. Such a cobordism induces a map

$$\Sigma_*\colon \mathrm{Kh}^{i,j}(L_0) \longrightarrow \mathrm{Kh}^{i,j+\chi(\Sigma)}(L_1)$$

which preserves the homological i-grading and shifts the quantum j-grading by the Euler characteristic of the surface.

The map Σ_* is constructed combinatorially, beginning with a suitable description of Σ. Any such cobordism Σ can be described by a so-called *movie*, which is a finite sequence of link diagrams (the *frames* of the movie), starting with a diagram D_0 for L_0 and ending with a diagram D_1 for L_1. Successive diagrams in the list differ either by an oriented Reidemeister move or by one of the three *Morse moves*, drawn in Figure 12, which represent a local maximum, a local minimum, or a saddle point of Σ.

To each Morse move there is an associated chain map of Khovanov chain complexes. The chain

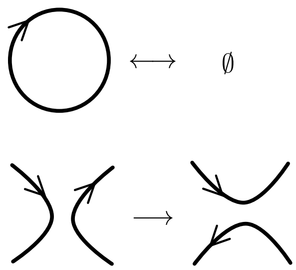


Figure 12: The three Morse moves.

maps corresponding to maxima and minima are quantum j-graded of degree $+1$, while the saddle chain map is quantum j-graded of degree -1. And in his original paper Khovanov already gave chain homotopy equivalences for each Reidemeister move. Composing all the chain maps coming from Reidemeister and Morse moves in a movie of Σ, one obtains a chain map

$$\Sigma_{\#}\colon \mathrm{CKh}^{i,j}(D_0) \longrightarrow \mathrm{CKh}^{i,j+\chi(\Sigma)}(D_1).$$

Theorem 2 ([Jac04, CMW09])**.** *The induced map*

$$\Sigma_*\colon \mathrm{Kh}^{i,j}(L_0) \longrightarrow \mathrm{Kh}^{i,j+\chi(\Sigma)}(L_1)$$

on homology is an invariant of Σ up to isotopy.

To prove this theorem, one essentially verifies that equivalent movies define chain homotopic chain maps.

Heegaard-Floer homology

From the start, then, 4-dimensional functoriality provided at least a superficial similarity between Khovanov homology and Floer homology. Back at the turn of the century, however, one could still imagine a world in which quantum homological invariants (*i.e.* generalizations of Khovanov homology) and Floer homological invariants, although similar, would continue to develop in parallel but were essentially unrelated, in the same way that the Jones polynomial and the Alexander polynomial were thought of as being very distinct. This is not how it has turned out.

The meeting of quantum and Floer homological invariants was first presaged by a result relating the Khovanov homology of a link L to the *Heegaard-Floer homology* of its *branched double cover* $M(L)$.

Heegaard-Floer homology is a package of homological invariants; in its simplest incarnation it is an invariant of a closed 3-manifold M that takes the form of a singly-graded $\mathbb{Z}_2$-vector space $\widehat{\mathrm{HF}}(M)$.

Avoiding a formal definition, the branched double cover $M(L)$ is a closed 3-manifold that admits an everywhere 2-to-1 map $M(L) \to \mathbb{S}^3$ apart from at points of $L \subset \mathbb{S}^3$ where it is 1-to-1. Its topology is intimately connected with that of L. Ozsváth and Szabó, the progenitors of Heegaard-Floer homology, proved the following result.

Theorem 3 ([OS05])**.** *There is a* spectral sequence *from the Khovanov homology of L with $\mathbb{Z}_2$-coefficients to* $\widehat{\mathrm{HF}}(-M(L))$.

Here we have written $-M(L)$ for the branched double cover but with reversed orientation. For readers unfamiliar with spectral sequences, the important content for us is that it means that there is an extra differential on $\mathrm{Kh}(L;\mathbb{Z}_2)$ whose homology gives $\widehat{\mathrm{HF}}(-M(L))$.

This theorem came early on in an era of results that established the existence of spectral sequences between various quantum homological invariants, and from quantum homological invariants to Floer theoretic invariants. Often these spectral sequences were more structural rather than being proved with any concrete topological application in mind. The result is a web of interdependency that ties Khovanov homology and Floer homology tightly together.

What we would like the reader to take from this is that Khovanov homology may be thought of as a first order combinatorial approximation to more analytic invariants.

Instanton floer homology

Instanton Floer homology is another invariant of a pair (M, L) where M is a 3-manifold and $L \subset M$ is a 1-dimensional submanifold, taking the form of a graded abelian group $\mathrm{I}(M, L)$. Kronheimer

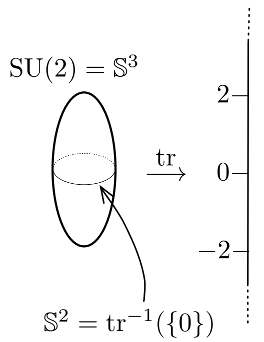


Figure 13: The trace function on $\mathbb{S}^3 = \mathrm{SU}(2)$.

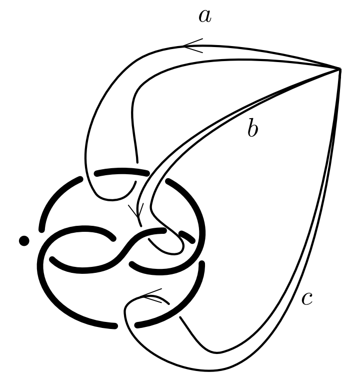


Figure 14: Generators for $\pi_1(\mathbb{S}^3 \setminus K)$.

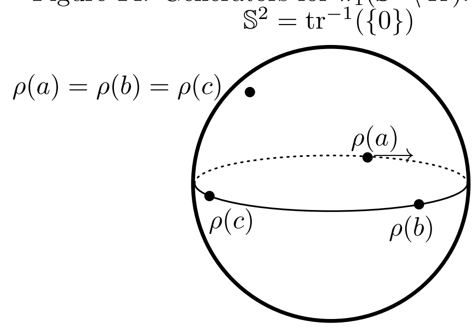


Figure 15: Computing the space $\mathrm{Rep}(K)$.

and Mrowka observed a curious coincidence which led them to suspect the existence of a spectral sequence from $\mathrm{Kh}(L)$ to $\mathrm{I}(\mathbb{S}^3, L)$ [KM11].

The chain group for $\mathrm{I}(M, L)$ is generated by a perturbed space of flat $\mathrm{SU}(2)$ connections on (M, L) which have a prescribed singularity along L. Since there is a correspondence between flat connections and representations of the fundamental group, to get a handle on the chain group for $\mathrm{I}(\mathbb{S}^3, L)$ we consider the space of homomorphisms

$$\mathrm{Rep}(L) := \{\rho\colon \pi_1(\mathbb{S}^3 \setminus L) \to \mathrm{SU}(2) \mid (*)\}$$

where $(*)$ is the condition that ρ should send every meridional element of $\pi_1(L)$ to an element of $\mathrm{SU}(2)$ of trace $\mathrm{tr} = 0$.

Topologically, $\mathrm{SU}(2)$ is diffeomorphic to the 3-sphere (see Figure 13). The trace function

$$\mathrm{tr}\colon \mathrm{SU}(2) \longrightarrow [-2, 2]$$

has a unique maximum at the identity matrix I_2 and a unique minimum at $-I_2$. The traceless matrices are the equatorial 2-sphere $\mathrm{tr}^{-1}(\{0\}) = \mathbb{S}^2$.

In this example, K will be the left-handed trefoil. In Figure 14 we give a diagram giving three elements $a, b, c \in \pi_1(\mathbb{S}^3 \setminus K)$. These are known as *meridional* elements since, forgetting the basepoint, each is homotopic in the knot complement to a small meridional loop to the knot. We have drawn one for each of the three arcs that form the knot diagram. Each of the arcs is incident to each of the crossings. We have labelled the leftmost crossing with a bullet $\bullet$. If the reader performs a mental isotopy and slides the generators a, b, c towards the crossing $\bullet$, they will be able to see that

$$b = c^{-1}ac.$$

There are similar relations coming from the other two crossings. In fact, these generators and relations give the group

$$\pi_1(\mathbb{S}^3 \setminus K) = \langle a, b, c | b = c^{-1}ac, c = a^{-1}ba, a = b^{-1}cb\rangle.$$

An element $\rho \in \mathrm{Rep}(K)$ is determined by the images $\rho(a), \rho(b), \rho(c) \in \mathbb{S}^2 = \mathrm{tr}^{-1}(0)$. This is not a completely free choice because we need to make sure that the three group relations are satisfied. Unpacking the equation $\rho(b) = \rho(c)^{-1}\rho(a)\rho(c)$, it turns out to be equivalent to requiring that $\rho(a), \rho(b), \rho(c)$ each lie on the same great circle geodesic of $\mathbb{S}^2$, with $\rho(c)$ lying halfway between $\rho(a)$ and $\rho(b)$.

Since there are two further relations, $\mathrm{Rep}(K)$ corresponds to choices of three points $\rho(a), \rho(b), \rho(c) \in \mathbb{S}^2$ which are equidistantly spaced around a great circle of $\mathbb{S}^2$ (see Figure 15).

It follows that $\mathrm{Rep}(K)$ falls into two connected components. One of the components parametrizes representations satisfying $\rho(a) = \rho(b) = \rho(c) \in \mathbb{S}^2$ and is topologically just $\mathbb{S}^2$. Points in the other component correspond to choosing a great circle and three non-equal but equidistant points along it. This is equivalent to the choice of the point $\rho(a) \in \mathbb{S}^2$ along with a unit tangent vector in the tangent space $T_{\rho(a)}\mathbb{S}^2$. Specifically, there is a unique great circle running in the direction of the tangent vector, and we place $\rho(b)$ a third of the way along the great circle in this direction, and then $\rho(c)$ two thirds of the way along. In other words, this second component is topologically just the unit tangent bundle to $\mathbb{S}^2$. This is a circle bundle over $\mathbb{S}^2$, and a characteristic class calculation shows that it is $\mathbb{RP}^3$. In conclusion we have

$$\mathrm{Rep}(K) = \mathbb{S}^2 \sqcup \mathbb{RP}^3.$$

This reminds us of something – where did we put our woodblock stamps? The unknot block has the same total homology as $\mathbb{S}^2$, while the knight's move has the total homology of $\mathbb{RP}^3$.

Looking back at Figure 2, we see that

$$\bigoplus_i H_i(\mathrm{Rep}(K)) = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(K).$$

This is, and should be, both surprising and motivational. The definition that we have seen of Khovanov homology was entirely combinatorial and yet here is some relationship with tangible spaces parametrizing representations of the fundamental group of the knot complement.

The coincidence between the singular homology $\bigoplus_i H_i(\mathrm{Rep}(L))$ and $\bigoplus_{i,j} \mathrm{Kh}^{i,j}(L)$ does not hold for all links L, this coincidence rather being a shadow for small links of a more general algebraic relationship. Kronheimer and Mrowka proved the following result.

Theorem 4 ([KM11])**.** *There is a spectral sequence from* $\mathrm{Kh}^{i,j}(L)$ *to* $I(\mathbb{S}^3, L)$.

If U is the unknot and K is a knot, then Kronheimer-Mrowka further showed that $I(\mathbb{S}^3, K) = I(\mathbb{S}^3, U)$ only if $K = U$. Their spectral sequence then allowed Kronheimer-Mrowka to co-opt this Floer homological result to conclude that $\mathrm{Kh}(K) = \mathrm{Kh}(U)$ only if $K = U$. In other words, Khovanov homology *detects the unknot.* The question of whether the Jones polynomial detects the unknot remains open.

Observations

We now have a partial explanation for the appearance of the unknot and knight's move blocks in the homology. The space $\mathrm{Rep}(K)$ always includes an $\mathbb{S}^2$ component corresponding to those $SU(2)$ representations of $\pi_1(\mathbb{S}^3 \setminus K)$ that factor through $H_1(\mathbb{S}^3 \setminus K) = \mathbb{Z}$. Roughly speaking, there should also be many $\mathbb{RP}^3$ components consisting of representations which are path-connected only to their own conjugates within $\mathrm{Rep}(K)$.

Furthermore, although we withhold the details, the internal gradings of various spectral sequences that start at Khovanov homology can give something of an explanation of the 'diagonalness' that we saw in the homology tables.

It remains to account for the quantum height of the unknot block summand and its apparent relationship with the genus of torus knots.

The 4-ball genus

As was mentioned in the previous section, there are many spectral sequences that start at Khovanov homology or its quantum homological relatives. Some of these limit to Floer homological invariants, while others limit to quantum invariants. The original example of a spectral sequence that remained in the quantum world was due to Lee and was studied and used by Rasmussen.

Theorem 5 ([Lee05, Ras10])**.** *For each knot K, there exists an even integer $s(K) \in 2\mathbb{Z}$ and a spectral sequence from* $\mathrm{Kh}^{i,j}(K)$ *to* $\mathrm{Kh}^{i,j-s(K)}(U)$ *where U is the unknot.*

(Lee and Rasmussen originally worked over the rationals $\mathbb{Q}$, although the result is now known over $\mathbb{Z}$ as well.) In other words, the quantum j-grading of this copy of the unknot that we see, in all our examples, appearing in homological degree $i = 0$, is this even integral knot invariant known as the Rasmussen invariant $s(K)$.

Using the functoriality of Khovanov homology for smooth knot cobordism, Rasmussen showed the following result.

Theorem 6. *Suppose that $\Sigma\colon K_0 \to K_1$ is a connected smooth knot cobordism between the knots K_0 and K_1. Then we have that*

$$2g(\Sigma) \geq |s(K_0) - s(K_1)|.$$

In other words, the difference in values of the s invariant between two knots provides a lower bound on the genus of a smooth connected knot cobordism between them. What has this got to do with torus knots?

For our example of the $(5,6)$-torus knot we see that $s(T_{5,6}) = 20$. Suppose that $T_{5,6} = \partial S$ where $S \subset B^4$ is a smooth, connected, oriented surface in the 4-ball. Such a surface is called a *slice surface* for $T_{5,6}$ and the minimal genus of such a surface is called the *slice genus* $g_*(T_{5,6})$.

Puncturing by removing a small ball centered at an interior point of S gives rise to a knot cobordism $\mathring{S}\colon T(5,6) \to U$ where U is the unknot. We then have

$$g(S) = g(\mathring{S}) \geq \frac{|s(T_{5,6}) - s(U)|}{2} = \frac{|20 - 0|}{2} = 10.$$

So we have $g_*(T_{5,6}) \geq 10$. Since there is a surface in the 3-sphere of genus 10 whose boundary is $T_{5,6}$, pushing the interior of this surface down into the 4-ball shows that

$$g_*(T_{5,6}) = 10 = g(T_{5,6}).$$

Similar arguments using $s(T_{p,q})$ work for all torus knots $T_{p.q}$ so that we can conclude the following result which is sometimes known as the Milnor Conjecture, and sometimes as the Local Thom Conjecture.

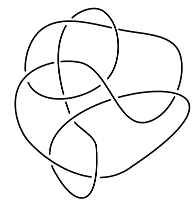


Figure 16: The Conway knot C.

Theorem 7. *For any torus knot $T_{p,q}$ we have*

$$g_*(T_{p,q}) = \frac{(p-1)(q-1)}{2} = g(T_{p,q}).$$

This result was first proved thirty years ago [KM94], and was one of the high points of gauge theory as applied to the understanding of smooth 4-manifolds. Rasmussen's purely combinatorial proof, which we have just outlined, was something of a surprise to the low-dimensional topology community. It was hitherto believed that very little of substance in smooth 4-dimensional topology could be proved while sidestepping serious analysis.

Of course the structure of the argument which produces from $s(K)$ a lower bound on $g_*(K)$ applies equally well to knots K other than torus knots. In one of the more celebrated results of recent years, Piccirillo used the s invariant to show that the Conway knot C (pictured in Figure 16) does not bound a smooth disk in the 4-ball [Pic20] (in other words $g_*(C) \neq 0$). Although it was known that the Conway knot satisfies $g_*(C) \in \{0,1\}$, it had hitherto stubbornly resisted attempts to determine its slice genus precisely.

Piccirillo's work was more than just the computation of $s(C)$; in fact $s(C) = 0$ so does not obstruct $g_*(C) = 0$. Piccirillo gave a knot $\overline{C}$ that she showed satisfies $g_*(\overline{C}) = 0$ if and only if $g_*(C) = 0$. Then an electronic computation gave $s(\overline{C}) = 2$.

Although we have seen that quantum and Floer theoretic invariants are becoming increasingly tightly woven together, at the time of writing there are no Floer theoretic invariants that can be combined with Piccirillo's topological arguments to demonstrate that $g_*(C) \neq 0$.

Further reading

Khovanov homology is so wide-ranging that one would need a book, rather than an article, to attempt to chase down its influence on all the fields that it has colonized. In particular, we are sorry to forgo consideration of the role of Khovanov homology in quantum representation theory and in mathematical physics; we refer the interested reader to elegant and comprehensible discussions of these in [LS22] and [GS16] respectively.

On the other hand, we hope that we have realized our objective of giving readers who have persevered to this point a feeling for Khovanov homology. We entrust those who now wish to read a little more to the seductive work of Bar-Natan [BN05]. In this paper Bar-Natan takes as his object of study the first, topological, Khovanov cube that we drew above in Figure 8. He shows how this cube itself, in a suitable category, already yields a link invariant. In this way he opens the door to a much subtler understanding of Khovanov homology, and one that is more amenable to abstraction and generalization.

Acknowledgements

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