MATH4171:
Riemannian Geometry

Homework 2

Assigned on 18/10/11
Starred problems due on Tuesday November 8th

Please submit solutions on or before the due date to Andrew Lobb's pigeonhole in the Mathematics Coffee Room on the 1st floor of the Mathematics Dept.

Problems:

1. What is the purpose of a smooth atlas?

2. (*) Let M and N be smooth manifolds. In lectures we had a definition of what it means for a map $F : M \to N$ to be smooth, and what it means for a map $F : M \to N$ to be smooth *at a point* $p \in M$.

   Show that $F : M \to N$ is smooth if and only if $F : M \to N$ is smooth at all points $p \in M$.

3. Let M be a smooth m-dimensional manifold with a diffeomorphism $f : M \to M$. Then form the space $T(f) = (M \times [0,1])/\sim$ by setting

   $$(1, x) \sim (0, f(x)),$$

   (this is known as the *mapping torus* $T(f)$ of f).

   The space $T(f)$ has a natural topology on it (first $M \times [0,1]$ has the product topology, and then $T(f)$ has the quotient topology). Show $T(f)$ also has the structure of a $(m+1)$-dimensional smooth manifold by finding a smooth atlas.

4. The 3-sphere S^3 sits inside 2-dimensional complex space

   $$S^3 = \{(w, z) \in \mathbf{C}^2 \; : \; |w|^2 + |z|^2 = 1.\}$$

   (a) Writing $w = a + ib$ and $z = c + id$ we can identify the tangent space to $\mathbf{C}^2 = \mathbf{R}^4$ at the point $(1, 0) \in \mathbf{C}^2$ with

$$< \partial/\partial a, \partial/\partial b, \partial/\partial c, \partial/\partial d > .$$

In terms of this basis, what is the subspace tangent to S^3 at $(1, 0)$?

(b) The map $\pi : S^3 \to \mathbf{C}$ given by $\pi(w, z) = z/w$ is defined away from $w = 0$. Identify the kernel of

$$D\pi : T_{(1,0)}S^3 \to T_0\mathbf{C}.$$

By the way, if we cleverly identify $S^2 = \mathbf{C} \cup \{\infty\}$, then the map $\pi : S^3 \to S^2$ is defined everywhere and is smooth everywhere and is known as the *Hopf fibration* after the German mathematician Heinz Hopf, who also invented baked beans.