MATH4171:
Riemannian Geometry

Homework 3

Assigned on 25th October 2011
Starred problems due on Tuesday November 8th

Please submit solutions on or before the due date to Andrew Lobb's pigeonhole in the Mathematics Coffee Room on the 1st floor of the Mathematics Dept.

Problems:

1. Let $S^3 \subset \mathbf{C}^2$ be defined as in Sheet 2, and let $c(t)$ be the circular curve $c(t) = (\cos(t), e^{it}\sin(t))$. Show that $c(t)$ is a curve on S^3.

   Define a map $f : S^3 \to \mathbf{R}$ by $f(w, z) = \mathrm{im}(w + z)$ where we write 'im' for the *imaginary part* of a complex number.

   Compute $c'(\pi/4)(f)$.

   In question 4a of Sheet 2, you found a basis for $T_{(1,0)=c(0)}S^3$. Express $c'(0)$ in terms of this basis.

2. (*) In class we saw that if we think of the Lie group $SL(n, \mathbf{R})$ as a manifold inside $M(n, \mathbf{R})$, then we can realize the tangent space to the identity $T_I(SL(n, \mathbf{R}))$ as being the linear space of trace-free $n \times n$ matrices.

   We can also include the Lie group $SO(n, \mathbf{R})$ inside $M(n, \mathbf{R})$ as a manifold of dimension $n(n-1)/2$ (as we saw earlier in the lecture course). Show that

   $$T_I(SO(n, \mathbf{R})) = \{A \in M(n, \mathbf{R}) | A^T = -A\},$$

   (in other words, the tangent space to the identity matrix in $SO(n, \mathbf{R})$ consists of the skew-symmetric matrices).

3. *Change of coordinates formula.* Let $p \in M$ and suppose there are two charts $\phi_\alpha : U_\alpha \to V_\alpha$ and $\phi_\beta : U_\beta \to V_\beta$ with $p \in U_\alpha \cap U_\beta$. Then there is a bijection

   $$\phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

given by restriction of $\phi_\beta \circ \phi_\alpha^{-1}$ to the relevant set.

This means that the coordinates on $\phi_\beta(U_\alpha \cap U_\beta) \subseteq V_\beta$ are given as functions on the coordinates of $\phi_\alpha(U_\alpha \cap U_\beta) \subseteq V_\alpha$,

$$x_j^\beta = x_j^\beta(x_1^\alpha, \ldots, x_n^\alpha) \text{ for } j = 1, \ldots, n.$$

Prove the change of coordinates formula for T_pM:

$$\frac{\partial}{\partial x_i^\alpha} = \sum_{j=1}^{n} \left(\frac{\partial x_j^\beta}{\partial x_i^\alpha} \right) \Bigg|_{\phi_\alpha(p)} \frac{\partial}{\partial x_j^\beta}.$$