MATH4171:

Riemannian Geometry

Homework 4

Assigned on 1st November 2011

Starred problems due on Tuesday November 8th

Please submit solutions on or before the due date to Andrew Lobb's pigeonhole in the Mathematics Coffee Room on the 1st floor of the Mathematics Dept.

Problems:

1. (*) Let M be a smooth manifold and let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields on M, and let $a \in \mathbf{R}$. Prove the following identities concerning the Lie Bracket:

   (a) Linearity $[X + aY, Z] = [X, Z] + a[Y, Z]$.

   (b) Anti-symmetry $[Y, X] = -[X, Y]$,

   (c) The Jacobi Identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

2. (a) Let M and N be smooth manifolds. Using local coordinates, explain why $T_{(p,q)}(M \times N) = T_pM \oplus T_qN$ for $p \in M$ and $q \in N$.

   (b) Find vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ such that $\{X(p), Y(p), Z(p)\}$ is a basis for T_pS^3 for all $p \in S^3$.

   (c) *Tricky question.* Possibly using the first part of the question, find vector fields $X, Y, Z \in \mathfrak{X}(S^1 \times S^2)$ such that $\{X(p), Y(p), Z(p)\}$ is a basis for $T_p(S^1 \times S^2)$ for all $p \in S^1 \times S^2$.

3. Let X and Y be two vector fields on $\mathbf{R}^3$ defined by

$$\begin{aligned} X(x,y,z) &= z\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y} + (2y - x)\frac{\partial}{\partial z}, \\ Y(x,y,z) &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{aligned}$$

   And let S^2 sit inside $\mathbf{R}^3$ as the sphere of radius 1 centred at the origin.

(a) Compute the Lie bracket $[X, Y]$.

(b) Verify that the restrictions of the vector fields X and Y to S^2 are vector fields on S^2 (in other words, are everywhere tangent to S^2).

(c) Check that the restriction of $[X, Y]$ to S^2 is also a vector field on S^2.