MATH4171:
Riemannian Geometry

Homework 5

Assigned on 15th November 2011
Starred problems due on Tuesday December 6th

Please submit solutions on or before the due date to Andrew Lobb's pigeonhole in the Mathematics Coffee Room on the 1st floor of the Mathematics Dept.

Problems:

1. Let $B^2 \subset \mathbf{C}$ be the Poincare unit ball of hyperbolic 2-space. For real numbers R, r with $0 < r < R < 1$, compute the volume of the subset

$$A_{R,r} = \{z \in B^2 \subset \mathbf{C} \ : \ r < |z| < R\}.$$

2. (*) In class we established that the Poincare unit ball model of hyperbolic 2-space was isometric to the the upper half-plane model.

   Now let $W = \{x \in \mathbf{R}^3 \ : \ q(x, x) = -1, x_3 > 0\}$ with $q(x, y) = x_1y_1 + x_2y_2 - x_3y_3$ be the hyperboloid model of the hyperbolic plane (see your notes for how we define the Riemannian metric on this space - it is not just the restriction of the standard Euclidean metric on $\mathbf{R}^3$). Also let the Poincare unit ball model B^2 of hyperbolic 2-space sit inside $\mathbf{R}^3$ as $B^2 = \{x \in \mathbf{R}^3 \ : \ x_3 = 0, x_1^2 + x_2^2 < 1\}$.

   We define a map $f : W \to B$ by requiring that $p \in W$ and $f(p) \in B^2$ are collinear with the point $(0, 0, -1)$, for each $p \in W$.

   (a) Calculate explicitly the maps $f(X, Y, Z)$ for $(X, Y, Z) \in W$ and $f^{-1}(x, y, 0)$ for $(x, y, 0) \in B^2$.

   (b) An almost global coordinate chart $\phi : U \to V$ of W is given by

   $$\phi^{-1}(x_1, x_2) = (\cos(x_1)\sinh(x_2), \sin(x_1)\sinh(x_2), \cosh(x_2)),$$

   where $0 < x_1 < 2\pi$ and $0 < x_2 < \infty$. Let $\psi = \phi \circ f^{-1}$ be a coordinate chart for B^2 with coordinate functions y_1, y_2.
   Calculate ψ^{-1} explicitly.

(c) Explain why

$$Df_p(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},$$

for $i = 1, 2$ where $\frac{\partial}{\partial x_i} \in T_p W$ and $\frac{\partial}{\partial y_i} \in T_{f(p)} B^2$ for all $p \in U$.

(d) Show that

$$< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >_p = < \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} >_{f(p)}$$

for all $p \in U$, and $i, j \in \{1, 2\}$. Using the previous part of the question, this demonstrates that f is an isometry.

3. Let $\mathbf{H}^2$ be the upper half-plane model of hyperbolic 2-space. We write $SL(2, \mathbf{R})$ for the 2×2 matrices with all real entries and determinant 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ and define the map

$$f_A : \mathbf{H}^2 \to \mathbf{H}^2, \; f_A(z) = \frac{az + b}{cz + d}.$$

It turns out that $f_A \circ f_B = f_{AB}$.

Show that the maps f_A are isometries of $\mathbf{H}^2$. You may find it useful to show first that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$