MATH4171:

Riemannian Geometry

Homework 6

Assigned on 22nd November 2011

Starred problems due on Tuesday December 6th

Please submit solutions on or before the due date to Andrew Lobb's pigeonhole in the Mathematics Coffee Room on the 1st floor of the Mathematics Dept.

Problems:

1. We work in the upper-half plane model of hyperbolic 2-space $\mathbf{H}^2$. We will show that for $z_1, z_2 \in \mathbf{H}^2$ the distance function is given by the formula

$$\sinh(\frac{1}{2}d(z_1, z_2)) = \frac{|z_1 - z_2|}{2\sqrt{\text{Im}(z_1)\text{Im}(z_2)}}.$$

   (a) Let $z_1 = iy_1$ and $z_2 = iy_2$ for $y_1, y_2 \in \mathbf{R}$, and verify that the formula holds in this case. (We derived the distance between two such points in class, you may use this result).

   (b) (*) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ and let $f_A(z) = \frac{az+b}{cz+d}$ be the isometry of $\mathbf{H}^2$ considered on the last problem sheet.
   Show that both sides of the formula are invariant under f_A (you may use the hint about $\text{Im}(f_A(z))$ given on the last sheet).

   (c) Finally, given two points $z_1, z_2 \in \mathbf{H}^2$, find an $A \in SL(2, \mathbf{R})$ such that both $f_A(z_1)$ and $f_A(z_2)$ lie on the imaginary axis.

   (d) Given what you know about Moebius transformations of $\mathbf{C}$, explain how you would draw the shortest path connecting two points $z_1, z_2 \in \mathbf{H}^2$.

2. Let $\mathbf{H}^n$ be the upper-half plane model of hyperbolic n-space,

$$\mathbf{H}^n = \{x \in \mathbf{R}^n : x_n > 0\}, \ g(v, w) = \frac{v.w}{x_n^2},$$

where we write g for the metric on $\mathbf{H}^n$ and we we identify each tangent space canonically with $\mathbf{R}^n$.

Calculate all Christoffel symbols Γ_{ij}^k for the global coordinate chart just given by the identity map $\phi : \mathbf{H}^n \to \mathbf{R}^n$, $\phi(x) = x$.