MATH4171:

Riemannian Geometry

Solution 1

Solutions

1. A smooth manifold is a set M equipped with an atlas, such that the induced topology on M is Hausdorff. See lecture notes for relevant definitions.

2. (*) We shall make an atlas for M with the required property. M is a smooth n-manifold so we know that we already have a smooth atlas (U_j, V_j, ψ_j) for $j \in J$ where J is some indexing set. We shall use this as our starting point.

   Let $x \in M$, and consider $j \in J$ such that $x \in U_j$ (since the U_j cover M we know that there is at least one such j). Now by the definition of an atlas we know that V_j is an open set in $\mathbf{R}^n$ with $\psi_j(x) \in V_j$. Hence there is a small ball $B(\psi_j(x); \epsilon_{x,j})$ that is centred at $\psi_j(x)$ that lies entirely inside V_j, $B(\psi_j(x); \epsilon_{x,j}) \subseteq V_j$. By rescaling and shifting we give a diffeomorphism

   $$\eta_{(x,j)} : B(\psi_j(x); \epsilon_{x,j}) \to B(0; 1).$$

   We now define an atlas with indexing set $I = \{(x, j) \mid x \in U_j\}$ such that $V_i = B(0; 1)$ for all $i \in I$. The sets $U_{(x,j)}$ are defined by

   $$U_{(x,j)} = \psi_j^{-1}(B(\psi_j(x); \epsilon_{x,j})),$$

   and the coordinate charts $\phi_(x, j) : U_{(x,j)} \to V_{(x,j)} = B(0; 1)$ are defined by

   $$\phi_{(x,j)} = \eta_{(x,j)} \circ \psi_j|_{U_{(x,j)}}.$$

   We leave it for the reader to check that the coordinate change maps are smooth (use the fact that composition of smooth maps is smooth) and that the topology induced by this atlas is the same as the original topology.

3. Observe that Γ looks like a Figure 8. Clearly there is a special point p on Γ which doesn't have a neighbourhood topologically equivalent to an open ball in $\mathbf{R}$ (in other words, to an open interval).

   To make this intuition a little more precise, we can see that any connected open neighbourhood of p is either an 8, an R, or an X. One can remove a point from both an R and from an 8 to leave something connected, whereas removing the special point p from X means X falls into four connected pieces. But removing *any* point from an open interval makes the open interval fall into *two* connected pieces. So this argument shows that no open neighbourhood of p is topologically equivalent to an open interval.

   (See the internet for the definition of connectedness [although it should be clear intuitively what we mean] - in this case we can use the easier notion of "path-connectedness").

4. We use the implicit function theorem. The space Γ_a is realized as the zeroes of the function $F_a(x, y, z) = xyz - a$. The differential of F_a is computed as

$$dF_a = (yz, xz, xy),$$

   which is never zero for $a \neq 0$ (otherwise one of x, y, z would be 0 forcing $F_a(x, y, z) = -a \neq 0$.

   Hence when $a \neq 0$, Γ_a is a smooth 2-manifold.

   When $a = 0$, Γ_a is the union of the (x, y)-, (y, z)-, and (z, x)-planes. So we see that Γ_0 fails to be locally like $\mathbf{R}^2$ along the x-, y-, and z-axes. (You could try and give, if you like, some further justification as in the answer to the previous question).