MATH4171:
Riemannian Geometry

Solution 2

Solutions

1. One purpose of a smooth atlas is to give us a way to define to smooth maps between spaces that are locally topologically the same as Euclidean space.

2. (*) Let (U_i, V_i, ϕ_i) $i \in I$ be the atlas for M, and (U_j, V_j, ψ_j) $j \in J$ be the atlas for N. The map F was said to be smooth at a point p if there exists U_i containing p and U_j containing $F(p)$ such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth at $\phi_i(p)$. The map F was said to be smooth everywhere if for any pair (i, j) the map $\psi_j \circ F \circ \phi_i^{-1}$ is smooth where defined.

   So clearly if a F is smooth everywhere it must be smooth at all points $p \in M$ (by the covering property of atlases).

   Now suppose F is smooth at every point p. Then as above we know there exists U_i containing p and U_j containing $F(p)$ such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth at $\phi_i(p)$. Suppose also $U_{i'}$ contains p and $V_{j'}$ contains $F(p)$. Then $\psi_{j'} \circ F \circ \phi_{i'}^{-1}$ is smooth at $\phi_{i'}(p)$ since we can express it as the composition of smooth maps:

   $$\psi_{j'} \circ F \circ \phi_{i'}^{-1} = (\psi_{j'} \circ \psi_j^{-1}) \circ (\psi_j \circ F \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_{i'}^{-1}).$$

   This implies that F is smooth everywhere since we have shown that for any pair (i', j') the map $\psi_{j'} \circ F \circ \phi_{i'}^{-1}$ is smooth at all points where it is defined.

3. Let I_1 be the open interval $(1/6, 5/6)$ and I_2 be the open interval formed as a quotient $((2/3, 1] \cup [0, 1/3))/(1 \sim 0)$. Then we have $S^1 = I_1 \cup I_2$ in the obvious way.

   Next we consider the spaces $M \times I_1$ and $M \times I_2$. The cartesian product of two smooth manifolds is certainly still a smooth manifold since if $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U_\beta, V_\beta, \psi_\beta)$ are atlases for two manifolds X and Y, then the maps

$$(\phi_\alpha, \psi_\beta) : U_\alpha \times U_\beta \to V_\alpha \times V_\beta$$

form a smooth atlas for $X \times Y$.

In the same way, if $(U_\alpha, V_\alpha, \phi_\alpha)$ for $\alpha \in A$ is an atlas for M, we see that the charts

$$\phi_\alpha \times id_i : U_\alpha \times I_i \to V_\alpha \times I_i$$

(for $id_i : I_i \to I_i$ the identity map) make each of $M \times I_1$ and $M \times I_2$ into smooth $(m+1)$-manifolds.

We now form the space $S(f)$ as a quotient space of the disjoint union of $M \times I_1$ and $M \times I_2$.

$$S(f) = [(M \times I_1) \cup (M \times I_2)]/\sim$$

where

$$(p, x_1) \sim (p, x_2) \text{ if } 1/6 < x_1 = x_2 < 1/3$$

and

$$(p, x_1) \sim (f(p), x_2)) \text{ if } 2/3 < x_1 = x_2 < 5/6.$$

The atlas on $(M \times I_1) \cup (M \times I_2)$ descends to an potential atlas on $S(f)$ whose charts are $\{\widetilde{\phi}_\alpha \times id_1, \widetilde{\phi}_\alpha \times id_2 \mid \alpha \in A\}$, but now there are more overlaps of the charts - and we need to make sure that changing coordinates on the new overlaps is a smooth operation.

So consider the charts $\widetilde{\phi}_\alpha \times id_1$ and $\widetilde{\phi}_\beta \times id_2$ (the other cases are easy). Then for $1/6 < t < 1/3$ we have

$$(\widetilde{\phi}_\beta \times id_2) \circ (\widetilde{\phi}_\alpha^{-1} \times id_1)(x^\alpha, t) = (\phi_\beta \circ \phi_\alpha^{-1}(x^\alpha), t)$$

and $\phi_\beta \circ \phi_\alpha^{-1}$ is certainly a smooth map by the definition of an atlas. And for $2/3 < t < 5/6$ we have

$$(\widetilde{\phi}_\beta \times id_2) \circ (\widetilde{\phi}_\alpha^{-1} \times id_1)(x^\alpha, t) = (\phi_\beta \circ f \circ \phi_\alpha^{-1}(x^\alpha), t)$$

and $\phi_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth by the definition of a smooth map of manifolds (f and f^{-1} were chosen to be smooth). The other direction $(\widetilde{\phi}_\alpha \times id_1) \circ (\widetilde{\phi}_\beta^{-1} \times id_2)$ is similar.

It remains to see that $S(f) = T(f)$, the mapping torus of f.

We define the multifunction

$$G_1 : M \times I_1 \to M \times [0,1]$$

by $G_1(p,x) = (p, x+1/4)$ for $x \leq 3/4$ and $G_1(p,x) = (f(p), x-3/4)$ for $x \geq 3/4$.

And

$$G_2 : M \times I_2 \to M \times [0,1]$$

by $G_2(p,x) = (f^{-1}(p), x+1/4)$ for $2/3 \leq x \leq 3/4$, $G_2(p.x) = (p, x-3/4)$ for $x \geq 3/4$, and $G_2(p,x) = (p, x+1/4)$ for $x \leq 1/3$.

Then you can check that G_1 and G_2 descend to a well-defined bijection

$$G : S(f) \to T(f).$$

It remains a delicate exercise for those more comfortable with topology to show that G and G^{-1} are both continuous so that $S(f)$ and $T(f)$ are homeomorphic as topological spaces.

4. (a) If we write $|w|^2 + |z|^2 = F(w,z) = F(a,b,c,d) = a^2+b^2+c^2+d^2$ then $S^3 = F^{-1}(1)$, the preimage of a regular value of F. Since F is constant along S^3, we have $DF|_p(v) = 0$ for any $p \in S^3$ and $v \in T_pS^3$. (You might like to try proving this by writing $v = c'(0)$ for some curve $c(t)$ contained in S^3 - heuristically it should be clear with a little thought - DF_p is the local change in F at p, and if we measure the local change in a direction along which F does not change (i.e. along a tangent $v \in T_pS^3$) we should get the answer 0).

   Now, $DF_{(1,0)} = (2a, 2b, 2c, 2d)|_{a=1,b=c=d=0} = (2,0,0,0)$, and this is zero on a 3-dimensional subspace which must coincide with the 3-dimensional space $T_{(1,0)}S^3$:

$$T_{(1,0)}S^3 = \langle \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d} \rangle.$$

   (b) Let's write the coordinates on $\mathbf{C}$ as $\alpha + i\beta$

   Let $s(t) = (1, it)$ be a path through $(1,0) \in \mathbf{C}^2$. Then $s'(0) = \partial/\partial d$. Now $D\pi_{(1,0)}(s'(0)) = (\pi \circ s)'(0)$ and $(\pi \circ s)(t) = it$ so we see that

$$D\pi_{(1,0)} \left(\frac{\partial}{\partial d} \right) = \frac{\partial}{\partial \beta} \in T_0\mathbf{C}$$

   Similarly we see that

$$D\pi_{(1,0)} \left(\frac{\partial}{\partial c} \right) = \frac{\partial}{\partial \alpha} \in T_0\mathbf{C}$$

   and

$$D\pi_{(1,0)}\left(\frac{\partial}{\partial b}\right) = 0 \in T_0\mathbf{C}$$

Hence we see that the kernel of $D\pi_{(1,0)}$ is just the 1-dimensional vector space spanned by $\frac{\partial}{\partial b}$.