MATH4171:
Riemannian Geometry

Solution 3

Solutions

1. To see that $c(t)$ lies on S^3, check that $||c(t)|| = 1$. We now compute $c'(\pi/4)(f)$.

$$\begin{aligned} c'(\pi/4)(f) &= \left.\frac{d}{dt}\right|_{t=\pi/4} f \circ c(t) \\ &= \left.\frac{d}{dt}\right|_{t=\pi/4} \mathrm{im}(\cos(t) + e^{it}\sin(t)) \\ &= \left.\frac{d}{dt}\right|_{t=\pi/4} \sin^2(t) \\ &= 2\cos(\pi/4)\sin(\pi/4) = 1 \end{aligned}$$

Suppose now that

$$c'(0) = \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial c} + \Delta\frac{\partial}{\partial d}.$$

Then we see, for example, that $c'(0)(b) = \beta$ where b is the function on $S^3 \subset \mathbf{R}^4$ given by projection onto the second coordinate. Similarly we can determine γ and Δ.

Computing we find that $\beta = 0$ and $\gamma = 1$ and $\Delta = 0$, for example (I apologise for c standing for both a circular curve and a coordinate function - hard to fix now),

$$\begin{aligned} \gamma &= c'(0)c = \left.\frac{d}{dt}\right|_{t=0} \cos(t)\sin(t) \\ &= \cos(0)^2 - \sin(0)^2 = 1 \end{aligned}$$

2. (*) Let $A(t)$ be a smooth path in $SO(n)$ with $A(0) = I$. Then we have

$$A^T(t)A(t) = I$$

for all t. Differentiating this identity with respect to t and evaluating at $t = 0$ we get

$$A'^T(0)A(0) + A^T(0)A'(0) = A'^T(0) + A'(0) = 0,$$

(you should convince yourself that differentiating commutes with taking transpose and that the Leibniz rule holds in matrix multiplication).

Hence we see that the tangent space $T_I SO(n)$ is contained in the skew-symmetric $n \times n$ matrices. Since both spaces are vector spaces of the same dimension, it follows that they are equal.

3. This is an important result and should be found in your lecture notes.