MATH4171:
Riemannian Geometry

Solution 4

Solutions

1. This is an exercise that gave no-one serious trouble.

2. (a) If (U_i, V_i, ϕ_i) $i \in I$ is an atlas for M and (U_j, V_j, ψ_j) for $j \in J$ is an atlas for N, then we get an atlas for $M \times N$ by taking products $(U_i \times U_j, V_i \times V_j, \phi_i \times \psi_j)$ for $(i,j) \in I \times J$. It follows that if $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are local coordinates at $p \in M$ and $q \in N$ that $x_1, \ldots, x_m, y_1, \ldots, y_n$ are local coordinates at $(p,q) \in M \times N$. Hence we see that

$$\begin{aligned} T_{(p,q)}(M \times N) &= < \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} > \\ &= < \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} > \oplus < \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} >= T_p M \oplus T_q N. \end{aligned}$$

   (b) Embedding S^3 as the unit sphere inside $\mathbf{R}^4$, we see that choosing the vector

$$(-y, x, -w, z) \in T_{(x,y,z,w)} S^3 \subset T_{(x,y,z,w)} \mathbf{R}^4$$

   describes a nowhere-vanishing vector field (you should check that this vector [*a priori* only a vector in $T_{(x,y,z,w)}\mathbf{R}^4$] actually lies in $T_{(x,y,z,w)}S^3$ by checking that it is normal to the normal direction to S^3 at this point).

   Then you should be able to find (up to an overall multiplication by ± 1) five further nowhere-vanishing vector fields by the same idea. Choosing carefully three of these, you should then check that the vectors at each point of S^3 are a basis for the tangent space. You can do this by checking that they are linearly independent.

   (c) Let me know if you have any ideas about this question or want me to confirm what you're thinking.

3. (a) You should get

$$[X,Y] = -z\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial x} + (2x+y)\frac{\partial}{\partial z}.$$

(b) The vector $X(x,y,z) \in T_{(x,y,z)}\mathbf{R}^3$ is inside the space $T_{(x,y,z)}S^2 \subset T_{(x,y,z)}\mathbf{R}^3$ if it is normal to the normal direction to S^2 at (x,y,z). The normal direction to S^2 at (x,y,z) is given by the vector $N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ (draw a picture to see this). Taking the dot product

$$N.X(x,y,z) = xz - 2yz + z(2y-x) = 0$$

as required. You can do something similar for Y, and for $[X,Y]$ in the final part of the question.