MATH4171:
Riemannian Geometry

Solution 5

Solutions

1. We shall use polar coordinates ρ, θ. The hyperbolic metric on B^2 is given by

$$\langle v, w\rangle = \frac{4v.w}{(1-\rho^2)^2},$$

where we write '.' for the usual dot product.

Now $x = \rho\cos\theta$ and $y = \rho\sin\theta$ so

$$\frac{\partial}{\partial\rho} = \frac{\partial y}{\partial\rho}\frac{\partial}{\partial y} + \frac{\partial x}{\partial\rho}\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},$$

and similarly

$$\frac{\partial}{\partial\theta} = -\rho\sin\theta\frac{\partial}{\partial x} + \rho\cos\theta\frac{\partial}{\partial y}.$$

Hence

$$\langle\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\rangle = \frac{1}{(1-\rho^2)^2}, \langle\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\theta}\rangle = 0, \langle\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\rangle = \frac{\rho^2}{(1-\rho^2)^2}.$$

Hence the matrix g_{ij} (considered with respect to ρ, θ coordinates) is diagonal with entries $\frac{1}{(1-\rho^2)^2}$ and $\frac{\rho^2}{(1-\rho^2)^2}$.

So we see that the area $A_{R,r}$ is given by

$$\begin{aligned} A_{R,r} &= \int_0^{2\pi}\int_r^R \frac{\rho}{(1-\rho^2)^2}d\rho d\theta \\ &= \pi\left(\frac{1}{1-R^2} - \frac{1}{1-r^2}\right). \end{aligned}$$

2. (a) Write we $f(X,Y,Z) = (x,y)$. Then since $(x,y,0)$ lies on the line containing (X,Y,Z) and $(0,0,-1)$, we have $x/X = y/Y = 1/Z+1$ so that

$$x = \frac{X}{Z+1}, y = \frac{Y}{Z+1}.$$

Likewise one can show that

$$f^{-1}(x,y) = \left(\frac{2x}{1-x^2-y^2}, \frac{2y}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right).$$

(b) Since $\psi = \phi \circ f^{-1}$, we have $\psi^{-1} = f \circ \phi^{-1}$, and we know both f and ϕ^{-1} explicitly. So we see that

$$\psi^{-1}(y_1,y_2) = \left(\frac{\sinh y_2 \cos y_1}{1+\cosh y_2}, \frac{\sinh y_2 \sin y_1}{1+\cosh y_2}, 0\right).$$

(c) By the definition of ψ, the map between the two charts $\psi \circ f \circ \phi^{-1}$ is just the identity map. Hence $y_i \circ f = x_i$ so $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$.

(d) We compute:

$$\frac{\partial}{\partial x_1} = (-\sin(x_1)\sinh(x_2), \cos(x_1)\sinh(x_2), 0),$$

$$\frac{\partial}{\partial x_2} = (\cos(x_1)\cosh(x_2), \sin(x_1)\cosh(x_2), \sinh(x_2)).$$

And using the metric defined via the form q on the hyperboloid, we see that

$$\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle = \sinh^2(x_2),$$

$$\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle = 0,$$

$$\left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} \right\rangle = \cosh^2(x_2) - \sinh^2(x_2) = 1.$$

We also compute:

$$\frac{\partial}{\partial y_1} = \left(\frac{-\sinh y_2 \sin y_1}{1+\cosh y_2}, \frac{\sinh y_2 \cos y_1}{1+\cosh y_2}\right),$$

$$\frac{\partial}{\partial y_2} = \left(\frac{\cos(y_1)}{1+\cosh(y_2)}. \frac{\sin(y_1)}{1+\cosh(y_2)}\right).$$

And using the metric on the Poincare unit ball model of hyperbolic space, we see that

$$\begin{aligned}
\left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1} \right\rangle &= \frac{4}{\left(1 - \left(\frac{\sinh^2(y_2)}{(1+\cosh^2(y_2))^2}\right)\right)^2} \frac{\sinh^2(y_2)}{(1+\cosh(y_2))^2} \\
&= \frac{4}{\left(\frac{2}{1+\cosh(y_2)}\right)^2} \frac{\sinh^2(y_2)}{(1+\cosh(y_2))^2} \\
&= \sinh^2(y_2),
\end{aligned}$$

and similarly

$$\left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right\rangle = 0,$$

$$\left\langle \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_2} \right\rangle = 1.$$

3. We first follow the hint!

$$\begin{aligned}
\operatorname{Im}(f_A(z)) &= \operatorname{Im}\left(\frac{az+b}{cz+d}\right) \\
&= \operatorname{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) \\
&= \operatorname{Im}\left(\frac{(ac|z|^2+bd+adz+bc\bar{z})}{|cz+d|^2}\right) \\
&= \operatorname{Im}\left(\frac{i(ad-bc)\operatorname{Im}(z)}{|cz+d|^2}\right) \\
&= \frac{\operatorname{Im}(z)}{|cz+d|^2}.
\end{aligned}$$

Now we want to show that f_A is an isometry of $\mathbf{H}^2$ - in other words, that it preserves the Riemannian metric. In fact, as we saw in class, it will be enough to show that it preserves the Riemannian norm $||*||^2 = \langle *, * \rangle$.

First we need to calculate the differential of f_A. Let $z(t)$ be a curve in $\mathbf{H}^2 \subset \mathbf{C}$, $z : \mathbf{R} \to \mathbf{H}^2$, then

$$\begin{aligned}
Df_A(z'(0)) &= \left.\frac{d}{dt}\right|_{t=0} \frac{az(t)+b}{cz(t)+d} \\
&= \frac{(ad-bc)z'(0)}{(cz(0)+d)^2} \\
&= \frac{z'(0)}{(cz(0)+d)^2}.
\end{aligned}$$

Then we see that

$$\begin{aligned}\langle Df_A(z'(0)), Df_A(z'(0))\rangle &= \frac{1}{[\mathrm{Im} f_A(z(0))]^2}\frac{|z'(0)|^2}{|(cz(0)+d)|^4} \\ &= \frac{|z'(0)|^2}{[\mathrm{Im} z(0)]^2} \\ &= \langle z'(0), z'(0)\rangle.\end{aligned}$$

So f_A preserves the Riemannian norm and hence is an isometry.