# MATH4171:
## Riemannian Geometry

## Solution 6

## Solutions

1. (a) In class we showed using elementary methods that the distance between $z_1 = iy_1$ and $z_2 = iy_2$ is $d(z_1, z_2) = \log(y_1/y_2)$ (where we assume without loss of generality that $y_2 > y_1$).
   In this case, the LHS of the equation to be verified is

$$\begin{aligned}\sinh(\frac{1}{2}d(z_1, z_2)) &= \frac{e^{\frac{\log(y_1/y_2)}{2}} - e^{\frac{-\log(y_1/y_2)}{2}}}{2} \\ &= \frac{e^{\log(\sqrt{y_1/y_2})} - e^{\log(\sqrt{y_2/y_1})}}{2} \\ &= \frac{\sqrt{y_1/y_2} - \sqrt{y_2/y_1}}{2}.\end{aligned}$$

   And the RHS is

$$\begin{aligned}\frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} &= \frac{y_1 - y_2}{2\sqrt{y_1 y_2}} \\ &= \frac{\sqrt{y_1/y_2} - \sqrt{y_2/y_1}}{2},\end{aligned}$$

   and that's that.

   (b) The LHS is preserved since isometries preserve distances (if you like, as many of you did for homework, you can argue this from the definition of isometry given in terms of the Riemannian metric and the definition of distance given as an infimum of the values of certain integrals).
   The RHS requires some calculation. We recall the hint from the last problem sheet which told us that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

So let's play with the RHS:

$$\begin{aligned}\frac{|f_A(z_1)-f_A(z_2)|}{2\sqrt{\mathrm{Im}(f_A(z_1))\mathrm{Im}(f_A(z_2))}} &= \frac{|cz_1+d||cz_2+d|\left|\left(\frac{az_1+b}{cz_1+d}\right)-\left(\frac{az_2+b}{cz_2+d}\right)\right|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ &= \frac{|(az_1+b)(cz_2+d)-(az_2+b)(cz_1+d)|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ &= \frac{|z_1(ad-bc)-z_2(ad-bc)|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ &= \frac{|z_1-z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}},\end{aligned}$$

and that's that.

(c) You can achieve this by some basic Moebius transformations. If $x_1, x_2$ are real numbers $x_1 \neq x_2$, then

$$z \mapsto \frac{az - ax_2}{z - x_1}$$

takes $x_2$ to 0 and $x_1$ to $\infty$, where we choose $a \in \mathbf{R}$ so that the unimodular condition is satisfied.

Suppose now that $z_1$ and $z_2$ are in the upper half-plane and lie on the unique semicircle or half-line through $x_1$ and $x_2$ which meets the real axis at right angles. Then this transformation must take $z_1$ and $z_2$ to the upper imaginary axis, since Moebius transformations preserve circlines.

(d) Moebius transformations take circles and lines to circles and lines and also preserve angles. Since $a, b, c, d$ are all real, the class of Moebius transformations that we deal with all preserve the real axis. Since we know that vertical half-lines satisfy the shortest-distance property, we know that the only other lines which have a chance to are semicircles meeting the real axis at right angles. But (exercise!) given any two points in the upper half-plane, there is a unique semicircle or half-line through both points that meets the real axis at right angles: hence these are the geodesics.

2. Using the standard global coordinate chart on $\mathbf{H}^n$, the matrix $g_{i,j}$ is diagonal with all diagonal entries $g_{i,i} = 1/x_n^2$. We now use the formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{k,l} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}) \quad (*),$$

where

$$g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab} \text{ and } [g^{ij}] = [g_{ij}]^{-1}.$$

Since in our case the matrix $g_{ij}$ is diagonal so also is its inverse and we have

$$g^{ii} = x_n^2.$$

Looking at $(*)$, we see that we must have $k = l$ for any non-zero terms, and also at least one of $i, j, k$ has to be equal to $n$.

More precisely, there are four cases giving potentially non-zero answers: $i = j \neq n, k = n$; $i = k \neq n, j = n$; $j = k \neq n, i = n$; $i = j = k = n$. Of these, the second and the third are really the same since we have the identity $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Then it is a simple matter of differentiation and we see that

$$\Gamma^n_{nn} = \frac{-1}{x_n} = \Gamma^i_{in} = \Gamma^i_{ni}, \;\; \Gamma^n_{ii} = \frac{1}{x_n},$$

whenever $i \neq n$, with all other Christoffel symbols being 0.