MATH4171:

Riemannian Geometry

Solution 7

Solutions

1. There are numerous ways in which to approach this problem. The one that is most readily generalizable to other situations (such as that occurring in the other question on this sheet) calculates a system of ODEs to solve, the crucial ingredient being a knowledge of the Christoffel symbols, which in turn allows you to be very explicit about the action of the covariant derivative.

   In class we already computed the Christoffel symbols for this S^2. Recall that we gave an almost global coordinate chart

   $$\phi^{-1} : (\theta_1, \theta_2) \mapsto (\sin(\theta_1), \cos(\theta_1)\sin(\theta_2), \cos(\theta_1)\cos(\theta_2))$$

   where $(\theta_1, \theta_2) \in (-\pi/2, \pi/2) \times (0, 2\pi)$. We calculated that

   $$\Gamma^1_{22} = \cos(\theta_1)\sin(\theta_1), \ \ \Gamma^2_{12} = \Gamma^2_{21} = -\tan(\theta_1),$$

   with all other Christoffel symbols 0.

   Okay, awesome. Now we want to translate the 'parallel condition' into a system of ODEs. So let $X(t) \in T_{c(t)}S^2$ be a vector field along the curve c. We shall write

   $$X(t) = a(t)\frac{\partial}{\partial \theta_1} + b(t)\frac{\partial}{\partial \theta_2},$$

   for some real functions a and b. Now note that

   $$c(t) = \phi^{-1}(\pi/4, t),$$

   so that

   $$c'(t) = \frac{\partial}{\partial \theta_2}.$$

The parallel condition says that

$$\frac{D}{dt}X(t) = \frac{D}{dt}\left(a(t)\frac{\partial}{\partial\theta_1} + b(t)\frac{\partial}{\partial\theta_2}\right) = 0,$$

and using the properties of $\frac{D}{dt}$ this is the same as requiring

$$a(t)\left(\nabla_{\frac{\partial}{\partial\theta_2}}\frac{\partial}{\partial\theta_1}\right) + a'(t)\frac{\partial}{\partial\theta_1} + b(t)\left(\nabla_{\frac{\partial}{\partial\theta_2}}\frac{\partial}{\partial\theta_2}\right) + b'(t)\frac{\partial}{\partial\theta_2} = 0.$$

Here's where we need the Christoffel symbols. The Christoffel symbols tell us that

$$\nabla_{\frac{\partial}{\partial\theta_2}}\frac{\partial}{\partial\theta_1} = -\tan(\theta_1)\frac{\partial}{\partial\theta_2} \text{ and}$$

$$\nabla_{\frac{\partial}{\partial\theta_2}}\frac{\partial}{\partial\theta_2} = \cos(\theta_1)\sin(\theta_1)\frac{\partial}{\partial\theta_1}.$$

Furthermore since (on the curve c) θ_1 is constant $\theta_1 = \pi/4$, our parallel condition becomes

$$-a(t)\frac{\partial}{\partial\theta_2} + a'(t)\frac{\partial}{\partial\theta_1} + \frac{1}{2}b(t)\frac{\partial}{\partial\theta_1} + b'(t)\frac{\partial}{\partial\theta_2} = 0.$$

And $\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}$ is a basis for the tangent space at each point along c, so we must have

$$b'(t) - a(t) = 0, \;\; a'(t) + \frac{1}{2}b(t) = 0.$$

You should know how to solve this system of equations!

We get the solutions:

$$a(t) = A\cos(t/\sqrt{2}) + B\sin(t/\sqrt{2}), \;\; b(t) = A\sqrt{2}\sin(t/\sqrt{2}) - B\sqrt{2}\cos(t\sqrt{2}),$$

for arbitrary constants A and B. In our case we are told what $X(0)$ is, and that provides an initial condition so that we can solve for A and B. We have

$$X(0) = v = \sqrt{2}c'(0) = \sqrt{2}\frac{\partial}{\partial\theta_2},$$

so we see that $A = 0$ and $B = -1$.

Hence

$$X(t) = -\sin(t/\sqrt{2})\frac{\partial}{\partial\theta_1} + \sqrt{2}\cos(t/\sqrt{2})\frac{\partial}{\partial\theta_2}.$$

This would be a good place to stop, but the question asks for three coordinates, so we observe that in terms of ambient coordinates

$$\left.\frac{\partial}{\partial\theta_1}\right|_{c(t)} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\cos(t)\right) \quad \text{and}$$
$$\left.\frac{\partial}{\partial\theta_2}\right|_{c(t)} = \left(0, \frac{1}{\sqrt{2}}\cos(t), -\frac{1}{\sqrt{2}}\sin(t)\right),$$

and we can just substitute these into the expression that we already have.

2. This question follows much the same lines as the previous question, so we move a little quicker.

   Let

$$X(t) = a(t)\frac{\partial}{\partial x} + b(t)\frac{\partial}{\partial y}$$

be a parallel vector field along the curve c. Now $c'(t) = \frac{\partial}{\partial x}$ so that the parallel condition becomes

$$a'(t)\frac{\partial}{\partial x} + a(t)\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}\right) + b'(t)\frac{\partial}{\partial y} + b(t)\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\right) = 0.$$

On the previous problem sheet we figured out the Christoffel symbols for the hyperbolic plane, so we know that

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{1}{y}\frac{\partial}{\partial y} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \frac{-1}{y}\frac{\partial}{\partial x}.$$

Furthermore, along c the y-coordinate is fixed $y = 1$. So the parallel condition is equivalent to the following system of ODEs:

$$a'(t) - b(t) = 0, \ \ b'(t) + a(t) = 0,$$

which has the solutions

$$a(t) = A\cos(t) + B\sin(t), \ \ b(t) = -A\sin(t) + B\cos(t),$$

for arbitrary constants A, B. Now we know that $X(0) = \frac{\partial}{\partial x}$ so we can solve for the constants $A = 1, B = 0$. Thus

$$X(t) = \cos(t)\frac{\partial}{\partial x} - \sin(t)\frac{\partial}{\partial y}.$$