Spherically symmetric solutions of the sixth order SU(N) Skyrme models

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Following the construction described by Ioannidou *et al.* [J. Math. Phys. **40**, 6353 (1999)], we use the rational map ansatz to construct analytically some topologically nontrivial solutions of the generalized SU(3) Skyrme model defined by adding a sixth order term to the usual Lagrangian. These solutions are radially symmetric and some of them can be interpreted as bound states of Skyrmions. The same ansatz is used to construct low-energy configuration of the SU(N) Skyrme model.
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I. INTRODUCTION

The Skyrme model[1] is widely accepted as an effective theory to describe the low-energy properties of nucleons. It was indeed shown[2–4] that in the large N_c limit, the Skyrme model is the low-energy limit of QCD. The classical static solutions of the model describe the bound states of nucleons and every configuration is characterized by a topological charge which, following Skyrme's idea, is interpreted as the baryon charge.

The Skyrme model can be used to predict the properties of the nucleons within 10% to 20%.[3,4] To improve these phenomenological predictions various extensions of the model have been proposed mostly by adding higher order terms[5–8] or extra fields[9] to the Lagrangian.

The study of the classical solutions of the Skyrme model has been done mostly using numerical methods, but recently Houghton *et al.*[10] showed that the classical solutions of the SU(2) model can be well approximated by using an ansatz that involves the harmonic maps from S^2 to S^2. The harmonic map describes the angular distribution of the solution while a profile function describes its radial distribution. This construction was later generalized[11] for the SU(N) model using harmonic maps from S^2 to CP^{N-1}. Moreover, it was shown that using a further generalization of this ansatz one can construct exact spherically symmetric solutions of the SU(N) Skyrme model.

The same method was also used in Ref. 12 to construct solutions of another SU(N) fourth order Skyrme model. In this article, we use the same generalized ansatz to construct solutions of the sixth order SU(3) Skyrme model and low-energy configurations of the SU(N) models defined in Ref. 13.

II. THE SIXTH ORDER SKYRME MODEL

The Skyrme model is described by an SU(N) valued field $U(\vec{x},t)$ which, to ensure finiteness of the energy, is required to satisfy the boundary condition $U \to I$ as $|\vec{x}| \to \infty$, where I is the unit matrix. This boundary condition implies that the three dimensional Euclidean space on which the model is defined can be compactified into S^3 and, as a result, the Skyrme field U corresponds to mappings from S^3 into SU(N). As $\pi_3(\mathrm{SU}(N)) = Z$ each configuration is characterized by its winding number, or topological charge, which can be obtained explicitly by evaluating the expression

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$$B=\frac{1}{24\pi^2}\int_{R^3}d\vec{x}^3\varepsilon^{ijk}\,Tr(R_i R_j R_k), \tag{1}$$

where $R_\mu=(\partial_\mu U)U^{-1}$ is the right chiral current. Skyrme's idea was to interpret the winding number associated with these topologically nontrivial mappings as the baryon charge.

The generalized sixth order Skyrme model is defined by the Lagrangian

$$E=-\frac{1}{12\pi^2}\int d\vec{x}^3\left(\frac{1}{2}TrR_i^2+\frac{1-\lambda}{16}Tr[R_i,R_j]^2+\frac{1}{96}\lambda Tr[R_i,R_j][R_j,R_k][R_k,R_i]\right), \tag{2}$$

where this parametrization of the model is chosen such that $\lambda\in[0,1]$ is a mixing parameter between the Skyrme term and the sixth order term: when $\lambda=0$ the model reduces to the usual pure Skyrme model while for $\lambda=1$ the Skyrme term vanishes and the model reduces to what we refer to in what follows as the pure Sk6 model.

The Euler–Lagrange equations derived from (2) for the static solutions are given by

$$\partial_i(R_i-\tfrac{1}{4}(1-\lambda)[R_j,[R_j,R_i]]-\tfrac{1}{16}\lambda[R_j,[R_j,R_k][R_k,R_i]])=0, \tag{3}$$

and the following inequality holds for every configuration:

$$E\geqslant\sqrt{1-\lambda}B. \tag{4}$$

The multi-Skyrmion solutions of the SU(2) Skyrme model have been studied in Ref. 13 where it was shown that they have the same symmetry as the pure Skyrme model. It was also shown that the harmonic map ansatz gives a good approximation to the solutions.

In the next section we describe the harmonic map ansatz. In the third section we prove that due to a constraint coming from the sixth order term, the multi-projector harmonic map ansatz provides exact solutions only for the SU(3) generalized model. We then show that one can nevertheless use the ansatz to construct low-energy configurations of the SU(N) models. In the fourth section we look at these configurations for the SU(4) model, while in the last section we look at some special ansatz for the SU(N) model.

III. HARMONIC MAP ANSATZ

The rational map ansatz, introduced by Houghton *et al.*,[10] is a generalization of the hedgehog ansatz found by Skyrme,[1] to approximate multi-Skyrmion solution of the SU(2) model. The ansatz was later generalized by Ioannidou *et al.*[14] to approximate solutions of the SU(N) Skyrme model using harmonic maps from S^2 into CP^{N-1}. This generalized ansatz is given by

$$U(r,\theta,\varphi)=e^{2if(r)(P(\theta,\varphi)-I/N)}=e^{-2if(r)/N}(I+(e^{2if(r)}-1)P(\theta,\varphi)), \tag{5}$$

where r,θ and φ are the usual polar coordinates. The profile function $f(r)$ must satisfy the boundary conditions $f(0)=\pi$ and $\lim_{r\to\infty}f(r)=0$ and $P(\theta,\varphi)$ is a projector in C^N which must be a harmonic map from S^2 into CP^{N-1} or equivalently a classical solution of the two dimensional CP^{N-1} σ model. These solutions are well known[15,16] and to construct them it is convenient to introduce the complex coordinate $\xi=\tan(\theta/2)e^{i\varphi}$ which corresponds to the stereographic projection of the unit sphere onto the complex plane.

In these coordinates, P must satisfy the equation

$$P\frac{\partial P}{\partial\xi}=0, \tag{6}$$

and the solutions of that equation are given by any projector of the form

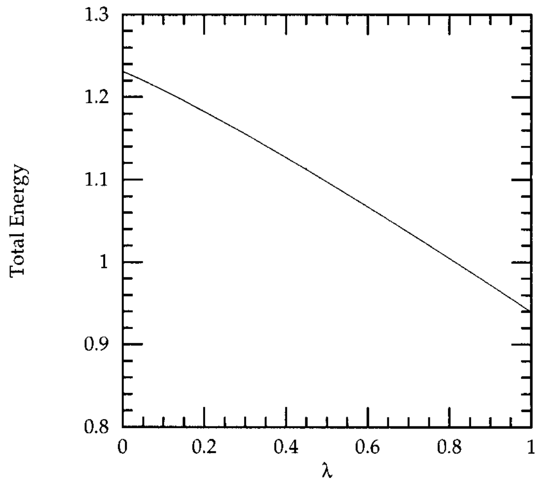

FIG. 1. Total energy of the 1 Skyrmion solution.

$$P(f)=\frac{h\otimes h^{\dagger}}{|h|^2}, \tag{7}$$

where $h\in C^N$ is holomorphic

$$\frac{\partial h}{\partial \bar{\xi}}=0. \tag{8}$$

The topological charge for the ansatz (5), with the prescribed boundary conditions for $f(r)$, is given by the winding number of the $S^2\rightarrow CP^{N-1}$. This winding number is itself given by the degree of the harmonic function h[15,16] which must then be a rational function of ξ.

To approximate a solution, one plugs the ansatz (5) into the energy (2) and notices that if P satisfies (6), the integration over the polar angles and the radius decouple. One then has to minimize the integral over the polar angles of an expression which depends only on P. Taking for P the most general harmonic map of the desired degree, one then has to find the parameters of the general map which minimize that integral. Having done this, the profile function f is obtained by solving the Euler Lagrange equation derived from the effective energy.

A special case of this construction is the so-called hedgehog ansatz for the SU(2) model corresponding to one Skyrmion. In this case, we have $h=(1,\xi)^t$ and after inserting (7) into (2) the energy reduces to

$$E=\frac{1}{3\pi}\int dr\left(f_r^2 r^2+2\sin^2 f\,(1+(1-\lambda)f_r^2)+(1-\lambda)\frac{\sin^4 f}{r^2}+\lambda\frac{\sin^4 f}{r^2}f_r^2\right), \tag{9}$$

and the equation for f is given by

$$f_{rr}\left(1+2\,(1-\lambda)\frac{\sin^2 f}{r^2}+\lambda\frac{\sin^4 f}{r^4}\right)+\frac{2}{r}f_r\left(1-\lambda\frac{\sin^4 f}{r^4}\right)$$
$$+\frac{\sin 2g}{r^2}\left((1-\lambda)f_r^2-1+\frac{\sin^2 f}{r^2}(\lambda f_r^2-1+\lambda)\right)=0. \tag{10}$$

This actually corresponds to an exact solution of the model and it is radially symmetric. In Fig. 1 we present the λ dependence of the energy and in Fig. 2 we show the profile function f and the energy density for the pure Skyrme model, $\lambda=0$, and the pure Sk6 model, $\lambda=1$.

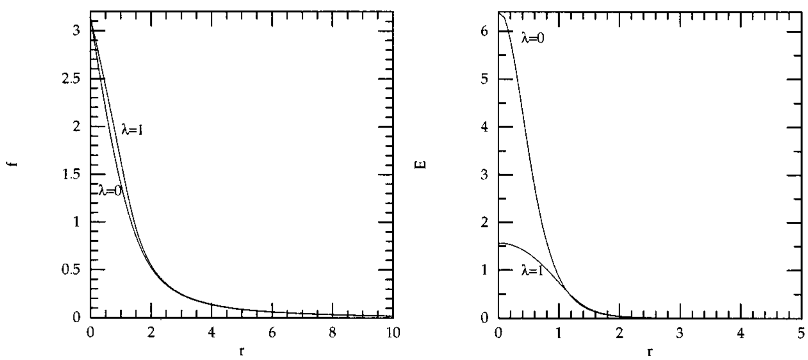

FIG. 2. Function profile f and energy density for the 1 Skyrmion solution of the pure Skyrme model, $\lambda=0$, and the pure Sk6 model, $\lambda=1$.

IV. SPHERICALLY SYMMETRIC SOLUTIONS FOR THE SU(N) MODEL

In this section we will follow the construction described in Ref. 14, to attempt to construct solutions of the extended SU(N) Skyrme model using a generalization of the harmonic map ansatz (5).

To build the new ansatz we need to introduce an operator P_+ which acts on any complex vector $u \in C^N$ and is defined as

$$P_+ u = \partial_\xi u - u \frac{u^\dagger \partial_\xi u}{|u|^2}. \tag{11}$$

Taking a holomorphic vector $h(\xi)$ we then define $P_0^+ h = h$ and by induction $V_k = P_+^k h = P_+(P_+^{k-1} h)$. These N vectors are mutually orthogonal[16] and so the corresponding projectors

$$P_k = P(P_+^k h) = \frac{P_+^k h (P_+^k h)^\dagger}{|P_+^k h|^2} \quad k=0,\ldots,N-1, \tag{12}$$

satisfy the orthogonality relations

$$\begin{aligned} P_k P_j &= \delta_{ij} P_k, \\ \sum_{k=0}^{N-1} P_k &= 1, \end{aligned} \tag{13}$$

as well as other properties discussed in detail in Ref. 14.

The generalized harmonic map ansatz is then defined as

$$\begin{aligned} U &= \exp\left\{ i g_0 \left(P_0 - \frac{I}{N} \right) + i g_1 \left(P_1 - \frac{I}{N} \right) - \ldots + i g_{N-2} \left(P_{N-2} - \frac{I}{N} \right) \right\} \\ &= e^{-i g_0/N}(I + A_0 P_0)\, e^{-i g_1/N}(I + A_1 P_1) \cdots e^{-i g_{N-2}/N}(I + A_{N-2} P_{N-2}), \end{aligned} \tag{14}$$

where $g_k(r)$ are $N-1$ profile functions and $A_k = e^{i g_k} - 1$. Moreover, for the ansatz to be well defined, the profile functions $g_k(r)$ must be a multiple of 2π at the origin and at infinity.

To proceed with our construction, it is convenient to rewrite the Euler–Lagrange equations of the model (3) using the usual spherical coordinates

$$
\begin{aligned}
&\partial_r\left\{ r^2 R_r + \frac{1-\lambda}{4}\left(A_{\theta r\theta} + \frac{1}{\sin^2\theta} A_{\varphi r\varphi}\right) + \frac{1}{16}\lambda\left[\frac{1}{\sin^2\theta}(B_{\theta\theta\varphi r\varphi}+B_{\varphi\varphi\theta r\theta})\right]\right\} \\
&\quad + \frac{1}{\sin\theta}\partial_\theta\left\{\sin\theta\left[R_\theta + \frac{1-\lambda}{4}\left(A_{r\theta r} + \frac{1}{r^2\sin^2\theta}A_{\varphi\theta\varphi}\right)\right] + \frac{\lambda}{16\,r^2\sin^2\theta}(B_{rr\varphi\theta\varphi}+B_{\varphi\varphi r\theta r})\right\} \\
&\quad + \frac{1}{\sin^2\theta}\partial_\varphi\left\{R_\varphi + \frac{1-\lambda}{4}\left(A_{r\varphi r} + \frac{1}{r^2}A_{\theta\varphi\theta}\right) + \frac{\lambda}{16\,r^2}(B_{rr\theta\varphi\theta}+B_{\theta\theta r\varphi r})\right\} = 0,
\end{aligned}
\tag{15}
$$

where $A_{jij} \equiv [R_j,[R_i,R_j]]$ and $B_{jjkik} \equiv [R_j,[R_j,R_k][R_i,R_k]]$. It is fairly easy to show that

$$
R_r = i\sum_{j=0}^{N-2}\dot{g}_j\left(P_j - \frac{I}{N}\right),
\tag{16}
$$

where $\dot{g}_j$ is the derivative of $g_j(r)$ with respect to r. Using the complex coordinates ξ and $\bar{\xi}$ introduced before we have

$$
R_\xi = \sum_{i=1}^{N-1}\left[e^{i(g_i-g_{i-1})}-1\right]\frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2},
\tag{17}
$$

and the derivatives with respect to θ and φ are given by

$$
\partial_\theta = \frac{1+|\xi|^2}{2\sqrt{|\xi|^2}}(\xi\,\partial_\xi + \bar{\xi}\,\partial_{\bar{\xi}}), \qquad \partial_\varphi = i(\xi\,\partial_\xi - \bar{\xi}\,\partial_{\bar{\xi}}).
\tag{18}
$$

Substituting the above into Eqs. (15) we get

$$
\begin{aligned}
&\partial_r\left[r^2 R_r + (1-\lambda)\frac{(1+|\xi|^2)^2}{8}(A_{\bar{\xi} r \xi} + A_{\xi r \bar{\xi}})\right] + \frac{(1+|\xi|^2)^2}{2}((R_{\bar{\xi}})_\xi + (R_\xi)_{\bar{\xi}}) \\
&\quad + (1-\lambda)\frac{(1+|\xi|^2)^3}{8r^2}(\xi A_{\xi\xi\bar{\xi}} - \bar{\xi}A_{\bar{\xi}\xi\bar{\xi}}) + (1-\lambda)\frac{(1+|\xi|^2)^4}{16r^2}([A_{\xi\xi\bar{\xi}}]_{\bar{\xi}} - [A_{\bar{\xi}\xi\bar{\xi}}]_\xi) \\
&\quad + (1-\lambda)\frac{(1+|\xi|^2)^2}{8}([A_{r\bar{\xi}r}]_\xi + [A_{r\xi r}]_{\bar{\xi}}) + \frac{\lambda}{16}\left\{\partial_r\left[\frac{(1+|\xi|^2)^4}{4}(B_{\bar{\xi}\xi\bar{\xi}r\bar{\xi}} - B_{\xi\xi\bar{\xi}r\bar{\xi}})\right]\right. \\
&\quad + \frac{(1+|\xi|^2)^2}{4r^2}(\partial_{\bar{\xi}}[(1+|\xi|^2)^2 B_{rr\xi\xi\bar{\xi}}] - \partial_\xi[(1+|\xi|^2)^2 B_{rr\bar{\xi}\xi\bar{\xi}}]) \\
&\quad + \frac{(1+|\xi|^2)^2}{2|\xi|^2 r^2}\left(\xi\partial_\xi\left[\frac{(1+|\xi|^2)^2}{4|\xi|^2}(-\xi\xi\xi B_{\xi\xi r\xi r})\right] + \bar{\xi}\partial_{\bar{\xi}}\left[\frac{(1+|\xi|^2)^2}{4|\xi|^2}(-\overline{\xi\xi\xi}B_{\overline{\xi\xi}r\bar{\xi}r})\right]\right) \\
&\quad + \frac{(1+|\xi|^2)^2}{8r^2}(\partial_\xi[(1+|\xi|^2)^2(B_{\xi\bar{\xi}r\bar{\xi}r} + B_{\bar{\xi}\xi r\bar{\xi}r} - B_{\overline{\xi\xi}r\xi r})] + \partial_{\bar{\xi}}[(1+|\xi|^2)^2 \\
&\quad \left.\times(-B_{\xi\xi r\bar{\xi}r} + B_{\xi\bar{\xi}r\xi r} + B_{\bar{\xi}\xi r\xi r})])\right\} = 0.
\end{aligned}
\tag{19}
$$

In Ref. 14 it is shown that if one takes the special holomorphic vector

$$
V_0 = h = (h_0, h_1, \ldots, h_{N-1})^t,
\tag{20}
$$

where

$$h_k = \xi^k \sqrt{C_k^{N-1}} \tag{21}$$

and C_k^{N-1} denotes the binomial coefficients, then the terms in (19) coming from the usual Skyrme model, i.e., all the terms except the ones proportional to $\lambda/16$, are all proportional to $P_i - P_{i-1}$ and $P_i - I/N$. Using (13) one can get rid of the projector P_{N-1} and (19) will then be the sum of the $N-1$ terms $P_i - I/N$ for $i=0,\ldots,N-2$, with coefficients that depend only on r. This implies that the equations for the Skyrme model reduce to $N-1$ ordinary differential equations for the profile functions g_i and their solutions, if they exist, will provide us with exact solutions of the SU(N) Skyrme model.

In what follows we will show that the angular dependence of the terms proportional to λ in (19), i.e., the terms coming from the sixth order term, is also coming exclusively from the projectors $P_i - I/N$ or $P_i - P_{i-1}$, but that we have to impose an extra constraint on the profile functions g_i.

We start by noting that

$$[R_\xi, R_{\bar\xi}] = -\sum_{i=1}^{N-1} a_i^2 \frac{|V_i|^2}{|V_{i-1}|^2}\left(\frac{V_i V_i^\dagger}{|V_i|^2} - \frac{V_{i-1}V_{i-1}^\dagger}{|V_{i-1}|^2}\right), \tag{22}$$

$$[R_r, R_\xi] = i\sum_{i=1}^{N-1} (\dot g_i a_i - \dot g_{i-1} a_i)\frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2} = \sum_{i=1}^{N-1} K_i \frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2}, \tag{23}$$

$$[R_r, R_{\bar\xi}] = i\sum_{i=1}^{N-1} (\dot g_i a_i - \dot g_{i-1} a_i)\frac{V_{i-1} V_i^\dagger}{|V_{i-1}|^2} = \sum_{i=1}^{N-1} K_i \frac{V_{i-1} V_i^\dagger}{|V_{i-1}|^2}, \tag{24}$$

where $a_i = e^{i(g_i - g_{i-1})} - 1$. It is then straightforward to check that

$$B_{\bar\xi\xi\bar\xi r\xi} - B_{\xi\xi\bar\xi r\bar\xi} = \sum_{i=1}^{N-1}\left(b_i \frac{|V_{i-1}|^2}{|V_{i-2}|^2}\frac{|V_i|^2}{|V_{i-1}|^2} + c_i \frac{|V_i|^4}{|V_{i-1}|^4} + d_i \frac{|V_{i+1}|^2}{|V_i|^2}\frac{|V_i|^2}{|V_{i-1}|^2}\right)(P_i - P_{i-1}), \tag{25}$$

where b_i, c_i and d_i are functions of g_k only. However, as shown in Ref. 14, if V_0 is given by (20) and (21), then $|V_i|^2/|V_{i-1}|^2 \propto (1+|\xi|^2)^{-2}$ and thus

$$\frac{(1+|\xi|^2)^4}{4}(B_{\bar\xi\xi\bar\xi r\xi} - B_{\xi\xi\bar\xi r\bar\xi}) \propto (P_i - P_{i-1}). \tag{26}$$

Furthermore, we have

$$B_{rr\xi\xi\bar\xi} = i\sum_{i=1}^{N-1}\left(e_i \frac{|V_i|^2}{|V_{i-1}|^2} + s_i \frac{|V_{i-1}|^2}{|V_{i-2}|^2}\right)\frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2} \tag{27}$$

with $e_i = e(g_i)$ and $s_i = s(g_i)$. But in Eq. (19) this term appears as

$$\partial_{\bar\xi}[(1+|\xi|^2)^2 B_{rr\xi\xi\bar\xi}] = 2\xi(1+|\xi|^2)B_{rr\xi\xi\bar\xi} + (1+|\xi|^2)^2\,\partial_{\bar\xi}(B_{rr\xi\xi\bar\xi}). \tag{28}$$

Since $\partial_{\bar\xi}|V_i|^2/|V_{i-1}|^2 \propto -2\xi(1+|\xi|^2)^{-3}$ the only parts of (28) that are nonzero are the ones that involve the derivatives of $V_i V_{i-1}^\dagger/|V_{i-1}|^2$ with respect to $\bar\xi$. Since it can be shown that the latter are proportional to $\Sigma_{i-1}^{N-1} C_i (1+|\xi|^2)^{-2}(P_i - P_{i-1})$ where $C_i = C(g_i)$, then one sees that the term that involves $B_{rr\xi\xi\bar\xi}$ in (19) is proportional to $(P_i - P_{i-1})$.

Using similar arguments, it is easy to check that the terms involving $B_{rr\bar\xi\xi\bar\xi}$, $B_{\xi\bar\xi r\bar\xi r}$, $B_{\bar\xi\xi r\bar\xi r}$, $B_{\bar\xi\bar\xi r\xi r}$, $B_{\xi\xi r\bar\xi r}$, $B_{\xi\bar\xi r\xi r}$ and $B_{\bar\xi\xi r\xi r}$ factorize in the same way.

There are a few terms in (19) which we still have to consider. They involve the expressions

$$B_{\xi\,\xi\,r\,\xi\,r}=\sum_{i=3}^{N-1}\,(a_iK_{i-1}K_{i-2}-a_{i-2}K_iK_{i-1})\,\frac{V_iV_{i-3}^{\dagger}}{|V_{i-3}|^2},\tag{29}$$

$$B_{\overline{\xi}\,\overline{\xi}\,r\,\overline{\xi}\,r}=\sum_{i=3}^{N-1}\,(a_iK_{i-1}K_{i-2}-a_{i-2}K_iK_{i-1}).\,\frac{V_{i-3}V_i^{\dagger}}{|V_{i-3}|^2}\,,\tag{30}$$

where $K_i=i(\dot{g}_ia_i-\dot{g}_{i-1}a_i)$. It is clear that these terms will always give a $\xi,\overline{\xi}$ dependence besides the projectors P_i and, hence, if we want (19) to reduce to $N-1$ equations that involve only the profile functions g_i, then we have to make sure that (29) and (30) vanish i.e., we must impose the conditions

$$a_iK_{i-1}K_{i-2}-a_{i-2}K_iK_{i-1}=0\ \Leftrightarrow\ \dot{g}_{i-2}-\dot{g}_{i-3}=\dot{g}_i-\dot{g}_{i-2}.\tag{31}$$

This last constraint, which is a result of the addition of the sixth order term, implies that we can only consider two profile functions g_0 and g_1 and that we should thus have only two equations. Unfortunately we have $N-1$ equations which are not compatible with each other. From this we see that the ansatz (5) will provide exact solutions of the generalized Skyrme model for the SU(2) and the SU(3) model only. For larger values of N, the ansatz will nevertheless give some low-energy radially symmetric configurations. The SU(2) case is nothing but the usual hedgehog ansatz and we will focus on the solutions of the SU(3) model in the next section.

In order to derive the equations for the profile functions, it is convenient to write the energy density of the model in terms of $(\xi,\overline{\xi})$:

$$\begin{aligned}E=-\frac{i}{12\pi^2}\int r^2dr\,d\xi d\overline{\xi}\,Tr\Bigg(&\frac{1}{(1+|\xi|^2)^2}R_r^2+\frac{1}{r^2}|R_\xi|^2+\frac{1-\lambda}{4r^2}[R_r,R_\xi][R_r,R_{\overline{\xi}}]\\&-(1-\lambda)\frac{(1+|\xi|^2)^2}{16r^4}[R_{\overline{\xi}},R_\xi]^2+\lambda\frac{(1+|\xi|^2)^2}{64r^4}[[R_r,R_{\overline{\xi}}],[R_r,R_\xi]][R_\xi,R_{\overline{\xi}}]\Bigg).\end{aligned}\tag{32}$$

Defining

$$\begin{aligned}F_i&=g_i-g_{i+1}\quad\text{for}\quad i=0,\ldots,N-3,\\F_{N-2}&=g_{N-2},\end{aligned}\tag{33}$$

as well as $W_i=(|V_i|^2/|V_{i-1}|^2)\,(1-\cos(F))$ and $W_{N-1}=(|V_{N-1}|^2/|V_{N-2}|^2)\,(1-\cos(g))$, the terms in the above expression can be rewritten as

$$Tr\,R_r^2=\frac{1}{N}\left(\sum_{i=0}^{N-2}\,\dot{g}_i\right)^2-\sum_{i=0}^{N-2}\,\dot{g}_i^2,\tag{34}$$

$$Tr|R_\xi|^2=-2\sum_{i=1}^{N-1}\,W_i,\tag{35}$$

$$Tr[R_r,R_\xi][R_r,R_{\overline{\xi}}]=-2\sum_{k=1}^{N-1}\,W_k\dot{F}_{k-1}^2,\tag{36}$$

$$Tr[R_{\overline{\xi}},R_\xi]^2=4\left(W_1^2+\sum_{i=1}^{N-2}\,(W_i-W_{i+1})^2+W_{N-1}^2\right),\tag{37}$$

$$Tr[[R_r,R_{\bar{\xi}}],[R_r,R_\xi][R_\xi,R_{\bar{\xi}}]]=4\left(\dot{F}_0^2W_1^2+\sum_{i=1}^{N-2}(\dot{F}_{i-1}W_i-\dot{F}_iW_{i+1})^2+\dot{F}_{N-2}^2W_{N-1}^2\right). \quad (38)$$

In Ref. 14 it was shown that

$$\frac{|V_k|^2}{|V_{k-1}|^2}=k(N-k)(1+|\xi|^2)^{-2}, \quad (39)$$

and from this we see that all the terms in (32) are proportional to $(1+|\xi|^2)^{-2}$ and that after integrating out the angular dependence the energy reduces to

$$\begin{aligned}E=\frac{1}{6\pi}\int r^2dr\Bigg\{&-\frac{1}{N}\left(\sum_{i=0}^{N-2}\dot{g}_i\right)^2+\sum_{i=0}^{N-2}\dot{g}_i^2+\frac{2}{r^2}\sum_{k=1}^{N-1}Z_k+\frac{(1-\lambda)}{2r^2}\sum_{k=1}^{N-1}(\dot{g}_k-\dot{g}_{k-1})^2Z_k\\&+\frac{(1-\lambda)}{4r^4}\left(Z_1^2+\sum_{k=1}^{N-2}(Z_k-Z_{k+1})^2+Z_{N-1}^2\right)\\&+\frac{\lambda}{16r^4}\left(\dot{F}_0^2Z_1^2+\sum_{k=1}^{N-2}(\dot{F}_{k-1}Z_k-\dot{F}_kZ_{k+1})^2+\dot{F}_{N-2}^2Z_{N-1}^2\right)\Bigg\},\end{aligned} \quad (40)$$

where $Z_k=k(N-k)(1-\cos(F_{k-1}))$.

In Ref. 14 the fields F_i defined by (33) were used, and very special solutions were obtained by taking $F_0=F_1=\cdots=F_{N-2}$. It was observed that when $F_i(0)=2\pi$ and $F_i(\infty)=0$ this solution of the SU(N) pure Skyrme model has a topological charge $B=(N/6)(N^2-1)$ and has an energy equal exactly to $(N/6)(N^2-1)$ times the energy of the single Skyrmion solutions. It is easy to show that, if one uses the same ansatz for the sixth order Skyrme model, the profile $f=F_0/2$ satisfies the hedgehog profile equation (10) and the energy of the configuration is given by $E(\lambda)=4E_0(\lambda)$ where $E_0(\lambda)$ is the energy of the hedgehog solution for the generalized model. These configurations are not exact solutions, except for the SU(3) model.

To consider the most general ansatz, one can derive from (40) the following equations for the profile functions F_l, $l=0,\ldots,(N-2)$:

$$\begin{aligned}&-\frac{2(l+1)}{N}\sum_{i=0}^{N-2}(i+1)\ddot{F}_i+2\sum_{k=0}^{l}\sum_{i=k}^{N-2}\ddot{F}_i+\frac{(1-\lambda)}{r^2}\ddot{F}_l(l+1)(N-l-1)(1-\cos F_l)\\&+\frac{2}{r}\left(-\frac{2(l+1)}{N}\sum_{i=0}^{N-2}(i+1)\dot{F}_i+2\sum_{k=0}^{l}\left(\sum_{i=k}^{N-2}\dot{F}_i\right)\right)+\frac{(1-\lambda)}{2r^2}\dot{F}_l^2(l+1)(N-l-1)\sin F_l\\&-\frac{2}{r^2}(l+1)(N-l-1)\sin F_l-\frac{(1-\lambda)}{r^4}(l+1)^2(N-l-1)^2(1-\cos F_l)\sin F_l\\&+\frac{(1-\lambda)}{2r^4}(l+1)(N-l-1)\sin F_l[l(N-l)(1-\cos F_{l-1})+(l+2)(N-l-2)(1-\cos F_{l+1})]\\&+\frac{\lambda}{8r^4}\{2\ddot{F}_l(l+1)^2(N-l-1)^2(1-\cos F_l)^2-\ddot{F}_{l-1}l(l+1)(N-l)(N-l-1)(1-\cos F_{l-1})\\&\times(1-\cos F_l)-\ddot{F}_{l+1}(l+1)(l+2)(N-l-1)(N-l-2)(1-\cos F_l)(1-\cos F_{l+1})\}\\&+\frac{-\lambda}{4r^5}\{2\dot{F}_l(l+1)^2(N-l-1)^2(1-\cos F_l)^2-\dot{F}_{l-1}l(l+1)(N-l)(N-l-1)(1-\cos F_{l-1})\\&\times(1-\cos F_l)-\dot{F}_{l+1}(l+1)(l+2)(N-l-1)(N-l-2)(1-\cos F_l)(1-\cos F_{l+1})\}\\&+\frac{\lambda}{8r^4}\{2\dot{F}_l^2(l+1)^2(N-l-1)^2(1-\cos F_l)\sin F_l-\dot{F}_{l-1}^2l(l+1)(N-l)(N-l-1)\sin F_{l-1}\\&\times(1-\cos F_l)-\dot{F}_{l+1}^2(l+1)(l+2)(N-l-1)(N-l-2)(1-\cos F_l)\sin F_{l+1}\}=0.\end{aligned} \quad (41)$$

When $N=3$, the solution of the two equations lead to exact solutions of the model, while for larger values of N, the ansatz (14) corresponds to low-energy configurations.

We would like to point out at this stage that as proved in Ref. 14, the topological charge for the configuration (14) is given by

$$B=\sum_{i=0}^{N-2}\mathcal{D}_k(F_i-\sin F_i)_{r=0}^{r=\infty}, \tag{42}$$

where

$$\mathcal{D}_k=-i\frac{1}{4\pi^2}\int\frac{|P_+^{k+1}h|^2}{|P_+^k h|^2}d\xi d\overline{\xi} \tag{43}$$

takes integer values given by the degree in ξ of the wedge product[16] of h and its derivatives

$$\mathcal{D}_k=\frac{1}{2\pi}\deg(h^{(k)}),\quad h^{(k)}=h\wedge\partial_+h\wedge\cdots\wedge\partial_+^k h,\quad k=0,\ldots,N-1. \tag{44}$$

Each configuration is thus characterized by the boundary conditions for the profile function F_i and we can without loss of generality impose the condition $\lim_{r\to\infty}F_i(r)=0$. For the configuration to be well-defined at the origin we must also impose a condition of the type

$$F_i(0)=n_i 2\pi, \tag{45}$$

where $n_i\in N$.

V. RADIALLY SYMMETRIC SU(3) SOLUTIONS

To describe the solution of the SU(3) model, we use the profile $F=F_0$ and $g=F_1$ and the energy (40) simplifies to

$$\begin{aligned}E=\frac{1}{6\pi}\int r^2dr\Bigg\{&\frac{2}{3}(\dot g^2+\dot F^2+\dot g\dot F)+\frac{1}{r^2}((1-\cos F)((1-\lambda)\dot F^2+4)+(1-\cos g)\\&\times((1-\lambda)\dot g^2+4))+(1-\lambda)\frac{2}{r^4}((1-\cos F)^2-(1-\cos F)(1-\cos g)+(1-\cos g)^2)\\&+\frac{\lambda}{2r^4}(\dot F^2(1-\cos F)^2+\dot g^2(1-\cos g)^2-(1-\cos F)(1-\cos g)\dot g\dot F)\Bigg\}.\end{aligned} \tag{46}$$

The equations for the profile function F and g are then given by

$$\begin{aligned}&g_{rr}+\frac{1}{2}F_{rr}+\frac{F_r}{r}+2\frac{g_r}{r}+\frac{3}{2r^2}\left((1-\lambda)(1-\cos g)g_{rr}+\frac{1}{2}\sin g((1-\lambda)g_r^2-4)\right)\\&+\frac{1}{2}\sin g((1-\lambda)g_r^2-4)+(1-\lambda)\frac{3}{2r^4}((1-\cos F)-2(1-\cos g))\sin(g)+\frac{3\lambda}{8r^4}(1-\cos g)\\&\times\left(2\left(\sin g g_r^2+(1-\cos g)\left(g_{rr}-2\frac{g_r}{r}\right)\right)-\sin F F_r^2-(1-\cos F)\left(F_{rr}-2\frac{Fr}{r}\right)\right)=0,\end{aligned} \tag{47}$$

TABLE I. Topological charge and energy of the hedgehog SU(2) solution.

SU(2)		Energy	
n_g	B	$E(0)$	$E(1)$
1	1	1.2315	0.9395

$$F_{rr}+\frac{1}{2}g_{rr}+2\frac{F_r}{r}+\frac{g_r}{r}+\frac{3}{4r^2}(\sin F((1-\lambda)F_r^2-4)+2(1-\lambda)(1-\cos F)F_{rr})$$
$$-(1-\lambda)\frac{3}{2r^4}(2(1-\cos F)-(1-\cos g))\sin F+\frac{3\lambda}{8r^4}(1-\cos F)$$
$$\times\left(12\left(\sin F\,F_r^2+(1-\cos F)\left(F_{rr}-2\frac{F_r}{r}\right)\right)-\sin g\,g_r^2-(1-\cos g)\left(g_{rr}-2\frac{g_r}{r}\right)\right)=0. \tag{48}$$

The topological charge of the solution now reads

$$B=\frac{1}{\pi}((F-\sin F)|_{r=0}^{r=\infty}+(g-\sin(g))|_{r=0}^{r=\infty}) \tag{49}$$

and, if we take the boundary conditions

$$F(0)=n_F 2\pi,$$
$$g(0)=n_g 2\pi, \tag{50}$$

where n_F and n_g are integers, we have $B=2(n_f+n_g)$. When n_F and n_g are of opposite signs, we can interpret the solutions as a mixture of Skyrmions and anti-Skyrmions.

In Table I, we give the energy of the hedgehog solution $(B=1)$ for the SU(2) model. This solution is an embedded solution of any SU(N) model and it is the solution with the lowest energy. We thus use it as the reference energy for all the other solutions.

In Table II we present the properties of the different solutions for the SU(3) models. The first two columns specify the boundary condition of the solution, and the third column gives the topological charge of that solution. In columns 4 and 5 we give the energy of the solutions for the pure Skyrme model and the pure Sk6 model while columns 6 and 7 give the corresponding relative energy per Skyrmion, that is, the energy divided by the energy of the single Skyrmion and the total number of Skyrmions. For the solutions corresponding to the superposition of Skyrmions and anti-Skyrmion, we define the total number of Skyrmions as the total number of Skyrmions and anti-Skyrmions. Notice that the cases $n_g=0, n_F=1$ and $n_g=1, n_F=0$ correspond to the same solution modulo an internal rotation.

In Fig. 3, we present the energy of the three different types of solution as a function of λ.

TABLE II. Topological charge and energy of some SU(3) solutions.

SU(3)			Total energy		Relative energy	
n_F	n_g	B	$E(0)$	$E(1)$	$E_B(0)/(\lvert B\rvert E_1(0))$	$E_B(1)/(\lvert B\rvert E_1(1))$
1	1	4	4.928	3.758	1	1
1	0	2	2.377	1.819	0.965	0.968
0	1	2	2.377	1.819	0.965	0.968
1	−1	2-2	3.862	3.191	0.784	0.849

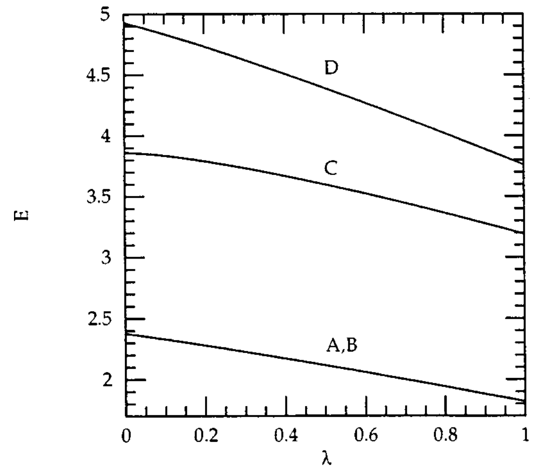

FIG. 3. Energy of the SU(3) solution for the boundary conditions (a) $n_F=0,n_g=1$, (b) $n_F=1,n_g=0$, (c) $n_F=1,n_g=-1$, (d) $n_F=1,n_g=1$.

VI. LOW-ENERGY SU(4) CONFIGURATIONS

As was shown in the last two sections, the ansatz (14) provides an exact solution of the sixth order model only for the SU(3) model, or when $\lambda=0$, that is for the usual Skyrme model. For the SU(N) model with $N\geqslant 4$, the ansatz still produces low-energy configurations. In particular, when λ is small, we can expect the ansatz to be very close to an exact solution. In this section we look at some configurations of the SU(4) model. For this model, we have three profile functions F_0, F_1 and F_2 and the energy for the general ansatz (14) is explicitly given by

$$\begin{aligned}E=\frac{1}{6\pi}\int r^2dr\Bigg\{&\frac{1}{4}(3\dot F_0^2+4\dot F_1^2+3\dot F_2^2+4\dot F_0\dot F_1+4\dot F_1\dot F_2+2\dot F_0\dot F_2)+\frac{2}{r^2}[3(1-\cos F_0)\\&+4(1-\cos F_1)+3(1-\cos F_2)]+(1-\lambda)\Bigg\{\frac{1}{2r^2}[3\dot F_0^2(1-\cos F_0)+4\dot F_1^2(1-\cos F_1)\\&+3\dot F_2^2(1-\cos F_2)]+\frac{1}{2r^4}\{9\,(1-\cos F_0)^2+16\,(1-\cos F_1)^2+9\,(1-\cos F_2)^2\\&-12(1-\cos F_0)(1-\cos F_1)-12(1-\cos F_1)(1-\cos F_2)\}\Bigg\}+\frac{\lambda}{8r^4}\{9\dot F_0^2(1-\cos F_0)^2\\&+16\dot F_1^2(1-\cos F_1)^2+9\dot F_2^2(1-\cos F_2)^2-12F_0F_1(1-\cos F_0)(1-\cos F_1)\\&-12F_1F_2(1-\cos F_1)(1-\cos F_2)\}\Bigg\}\end{aligned}\tag{51}$$

from which we can derive the following equations:

$$\begin{aligned}&\left(\frac{3\lambda(1-\cos F_0)^2}{2r^4}+\frac{2(1-\lambda)(1-\cos F_0)}{r^2}+1\right)\ddot F_0+\left(\frac{2}{3}-\frac{\lambda(1-\cos F_0)(1-\cos F_1)}{r^4}\right)\ddot F_1+\frac{1}{3}\ddot F_2\\&\quad-\frac{4\sin F_0}{r^2}+\frac{6\,\dot F_0+4\dot F_1+2\dot F_2}{3r}+\frac{(1-\lambda)\dot F_0^2\sin F_0}{r^2}+(1-\lambda)\frac{\sin F_0}{r^4}(4(1-\cos F_1)\\&\quad-6(1-\cos F_0))+\lambda\frac{(1-\cos F_0)}{r^4}\left(\frac{3}{2}\dot F_0^2\sin F_0-\dot F_1^2\sin F_1\right)-\lambda\frac{(1-\cos F_0)}{r^5}(3\dot F_0(1-\cos F_0)\\&\quad-2\,\dot F_1(1-\cos F_1))=0,\end{aligned}\tag{52}$$

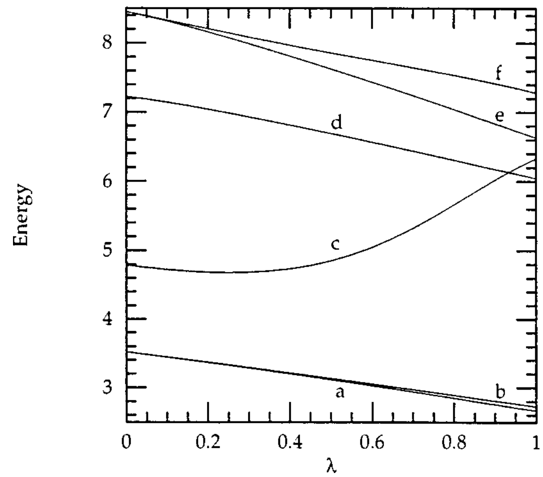

FIG. 4. Energy density of the Su(4) multi-projector ansatz (a) $n_0=0,\ n_1=0,\ n_2=1$; (b) $n_0=1,\ n_1=0,\ n_2=0$; (c) $n_0=0,\ n_1=1,\ n_2=0$; (d) $n_0=1,\ n_1=0,\ n_2=1$; (e) $n_0=1,\ n_1=1,\ n_2=0$; and (f) $n_0=0,\ n_1=1,\ n_2=1$.

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{3\lambda(1-\cos F_0)(1-\cos F_1)}{4r^4}\right)\ddot{F}_0+\left(1+\frac{2\lambda(1-\cos F_1)^2}{r^4}+\frac{2(1-\lambda)(1-\cos F_1)}{r^2}\right)\ddot{F}_1\\
&\quad+\left(\frac{1}{2}-\frac{3\lambda(1-\cos F_1)(1-\cos F_2)}{4r^4}\right)\ddot{F}_2+\frac{(1-\lambda)\dot{F}_1^2\sin F_1}{r^2}+\frac{\dot{F}_0+2\,\dot{F}_1+\dot{F}_2}{r}-4\,\frac{\sin F_1}{r^2}\\
&\quad+(1-\lambda)\frac{\sin F_1}{r^4}(3(1-\cos F_0)+3(1-\cos F_2)-8(1-\cos F_1))-\frac{\lambda}{r^5}(1-\cos F_1)\\
&\quad\times\left(4\,\dot{F}_1(1-\cos F_1)-\frac{3}{2}\dot{F}_0(1-\cos F_0)-\frac{3}{2}\dot{F}_2(1-\cos F_2)\right)+\frac{\lambda}{r^4}(1-\cos F_1)\\
&\quad\times\left(2\,\dot{F}_1^2\sin F_1-\frac{3}{4}\dot{F}_0^2\sin F_0-\frac{3}{4}\dot{F}_2^2\sin F_2\right)=0,
\end{aligned}\tag{53}$$

and

$$\begin{aligned}
&\left(\frac{2}{3}-\frac{\lambda(1-\cos F_1)(1-\cos F_2)}{r^4}\right)\ddot{F}_1+\left(\frac{3\lambda(1-\cos F_2)^2}{2r^4}+\frac{2(1-\lambda)(1-\cos F_2)}{r^2}+1\right)\ddot{F}_2\\
&\quad+\frac{1}{3}\ddot{F}_0+\frac{2\dot{F}_0+4\dot{F}_1+6\,\dot{F}_2}{3r}-4\,\frac{\sin F_2}{r^2}+\frac{(1-\lambda)\dot{F}_2^2\sin F_2}{r^2}+(1-\lambda)\frac{\sin F_2}{r^4}(4(1-F_1)\\
&\quad-6(1-F_2))-\lambda\,\frac{(1-\cos F_2)}{r^5}(3\,\dot{F}_2(1-\cos F_2)-2\,\dot{F}_1(1-\cos F_1))\\
&\quad+\lambda\,\frac{(1-\cos F_2)}{r^4}\left(\frac{3}{2}\dot{F}_2^2\sin F_2-\dot{F}_1^2\sin F_1\right)=0.
\end{aligned}\tag{54}$$

Describing the boundary condition for the profile functions as before, $F_i(0)=n_i 2\pi$, the topological charge is given by

$$B=3n_0+4n_1+3n_2.\tag{55}$$

TABLE III. Topological charge and energy of some SU(4) configurations.

SU(4)				Total energy		Relative energy	
n_0	n_1	n_2	B	$E(0)$	$E(1)$	$E_B(0)/(\|B\|E_1(0))$	$E_B(1)/(\|B\|E_1(1))$
0	0	1	3	3.517 39	2.666 53	0.952 10	0.945 98
1	0	0	3	3.517 39	2.729 15	0.952 10	0.968 19
0	1	0	4	4.788 07	6.333 22	0.972 04	1.685 07
1	0	1	6	7.224 64	6.046 04	0.977 80	1.072 44
1	1	0	7	8.452 19	6.629 98	0.980 52	1.008 02
0	1	1	7	8.452 19	7.280 58	0.980 52	1.106 94
1	1	1	10	12.311	9.396 05	1	1

In Table III we present the energy values of various types of configurations when $\lambda=0$ and $\lambda=1$. We notice that when $\lambda=0$, the solutions are symmetric under the exchange $f_0 \leftrightarrow f_2$, but that the sixth order term breaks the symmetry. This results in a difference of energy between the configuration with $n_0=0, n_1=0, n_2=1$ and $n_0=1, n_1=0, n_2=0$ as well as between the configurations with $n_0=1, n_1=1, n_2=0$ and $n_0=0, n_1=1, n_2=1$. In Fig. 4, we present the curve for the energy of the configurations as a function of λ.

VII. SU(N) LOW-ENERGY CONFIGURATION

After inserting the ansatz (5) in the full equation for the SU(N) model, we found that we had only two independent profile functions g_0 and g_1 and that the ansatz would only provide solutions for the SU(3) model. One can nevertheless use the SU(N) ansatz to compute low-energy configurations. For example, if we consider the reduced ansatz defined by (5) together with the constraint $\dot{g}_{i-2}-\dot{g}_{i-3}=\dot{g}_i-\dot{g}_{i-2}$ and define the profiles $F=g_0-g_1$ and $g=g_{N-2}$ we can minimize the energy (40) and solve the equations for F and g for various boundary conditions. We found that to get configurations corresponding to a bound state, i.e., a configuration with an energy per Skyrmion smaller than the energy of the hedgehog solution, we must take $n_F=0$ and $n_g=1$. The energies that we found are given in Table IV.

In Figs. 5 and 6 we present the profile and the energy density for different values of N and for $\lambda=0.5$. It shows that the energy density has the shape of a hollow sphere of radius $r=0.7\sqrt{N}$. The profile g has the same shape for all values of N but is shifted to the right as N increases. The profile F, on the other hand, is also shifted as the shell radius increases, but its amplitude decreases like $1/N^2$. Note that in Fig. 6, the profiles for $N=100$ and $N=200$ have been multiplied by 100 to make them visible. For other values of λ the graphics look very much the same except that the shell radius and width are slightly different, but the conclusions remain the same.

Figure 6(b) suggests to simplify the ansatz further for large N by taking $F(r)=0$. This implies that $g_i=g\ \forall i$ and the multi-projector ansatz (5) becomes

$$U=\exp(-ig(P_{N-1}-I/N)), \tag{56}$$

where P_{N-1} can also be written as

TABLE IV. Topological charge and energy for the reduced ansatz with $n_F=0$ and $n_g=1$.

		Total energy		Relative energy	
Model	B	$E(0)$	$E(1)$	$E_B(0)/(\|B\|E_1(0))$	$E_B(1)/(\|B\|E_1(1))$
SU(3)	2	2.377	1.819	0.965	0.968
SU(4)	3	3.624	2.759	0.981	0.979
SU(5)	4	4.811	3.632	0.977	0.966
SU(6)	5	6.015	4.518	0.977	0.962

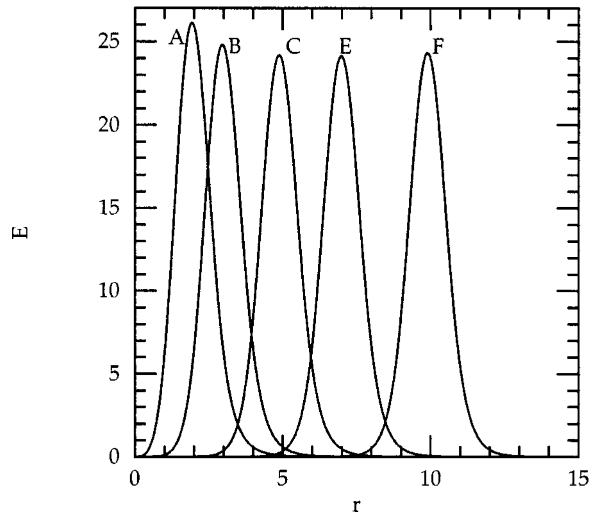

FIG. 5. Energy density of the multi-projector solution with $n_F=0$, $n_g=1$, $\lambda=0.5$. (a) $N=10$, (b) $N=20$, (c) $N=50$, (d) $N=100$, and (e) $n=200$.

$$P_{N-1}=\frac{\tilde{h}\tilde{h}^{\dagger}}{|\tilde{h}|^{2}}, \tag{57}$$

where $\tilde{h}$ is equal, up to a unitary rotation, to the complex conjugate of the holomorphic vector V_0 defined in (20) and (21): $\tilde{h}=A\overline{V_0}$ for some $A\in \mathrm{SU}(N)$ with $\partial_{\xi}A=\partial_{\bar{\xi}}A=0$. This is shown by using the fact that P_{N-1} is an antiholomorphic projector[16] and that solving (39) recursively we have

$$|V_k|^2=\frac{k!(N-1)!}{(N-1-k)!}|1+|\xi|^2|^{N-1-2k} \tag{58}$$

and so $|V_{N-1}|^2=(N-1)!^2|1+|\xi|^2|^{1-N}$. Knowing that up to an overall coefficient $|V_{N-1}|^2$ is a polynomial in $\bar{\xi}$ of degree $N-1$, we can conclude that up to a unitary iso-rotation, V_{N-1} is equal to the complex conjugate of V_0.

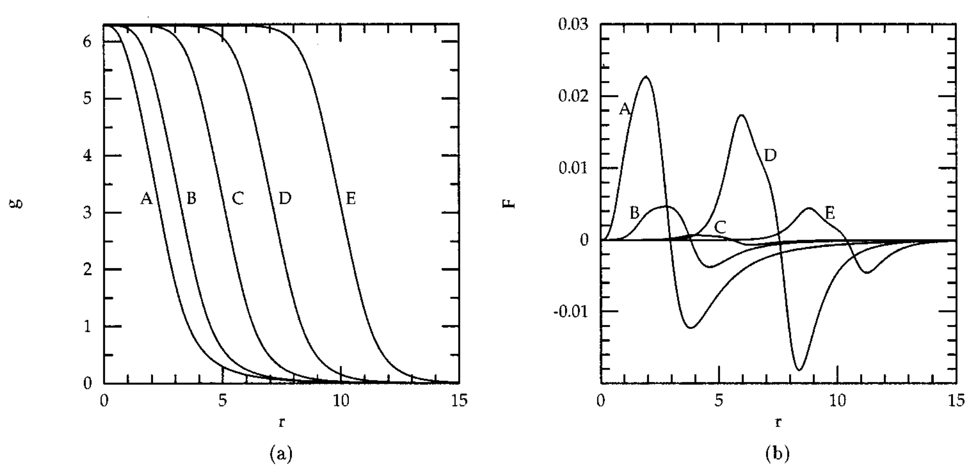

FIG. 6. Profile (a) g and (b) F of the multi-projector solution with $n_F=0$, $n_g=1$, $\lambda=0.5$. (a) F for $N=10$, (b) F for $N=20$, (c) F for $N=50$, (d) $100\times F$ for $N=100$, and (e) $100\times F$ for $N=200$.

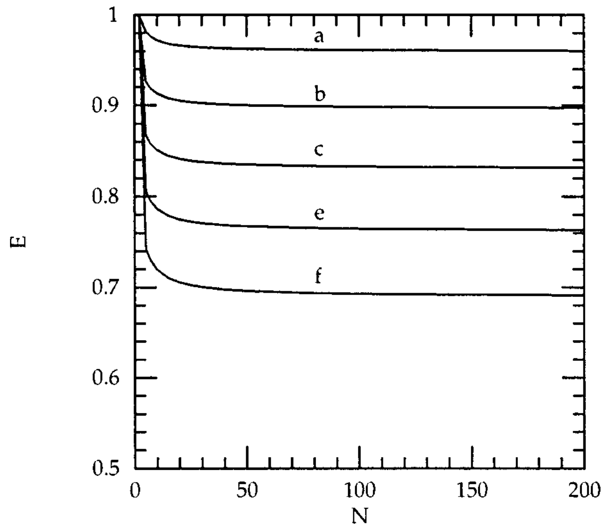

FIG. 7. Energy $E/(E_{B=1}(N-1))$ for the SU(N), configuration (56) for (a) $\lambda=0$, (b) $\lambda=0.25$, (c) $\lambda=0.5$, (d) $\lambda=0.75$, and (e) $\lambda=1$.

The topological charge of the antiholomorphic projector P_{N-1} is equal to $1-N$ and as the profile function is $-g$, the baryon number for this configuration is $N-1$. The ansatz (56) is not a solution, but its energy

$$E=\frac{1}{6\pi}\int r^2dr\left\{\frac{N-1}{N}\dot{g}^2+\frac{1}{2r^2}+(N-1)(1-\cos g)((1-\lambda)\dot{g}^2+4)\right.$$
$$\left.+\frac{1}{2r^4}(N-1)^2(1-\cos g)^2\left((1-\lambda)+\frac{\lambda}{4r^4}\dot{g}^2\right)\right\}, \tag{59}$$

can easily be computed by solving the equation

$$2g_{rr}+4\frac{g_r}{r}+\frac{N}{r^2}\left((1-\lambda)(1-\cos g)g_{rr}+\frac{1}{2}\sin g((1-\lambda)g_r^2-4)\right)+\frac{\lambda}{4r^4}N(N-1)(1-\cos g)$$
$$\times\left(\sin g g_r^2+(1-\cos g)\left(g_{rr}-2\frac{g_r}{r}\right)\right)=0. \tag{60}$$

In Fig. 7, we present the relative energy, $E(\lambda)/(E_{B=1}(\lambda)(N-1))$, of this configuration as a function of N for different values of λ. We see that this configuration corresponds to a bound state of Skyrmions and that the energy per Skyrmion decreases with N. The energy of this configuration corresponds to an upper bound for the energy of the $B=N-1$ radially symmetric solution of the SU(N) model and these configurations correspond to bound states of Skyrmions for all values of N and all values of λ. As every SU(p) solution can be trivially embedded in an SU(q) solution when $p\leqslant q$ we can claim that for every $B<N$ the SU(N) model has a radially symmetric solution of charge B corresponding to a bound state. With the exception of the hedgehog solutions, these solutions are expected to be unstable when the radial symmetry is broken as their energies are larger than the known SU(2) solutions.[13]

VIII. CONCLUSIONS

In this article we have shown how to construct some radially symmetric solutions of the SU(3) sixth order Skyrme model. The construction is similar to the one used for the pure Skyrme model in Ref. 14 except that, because of an extra constraint, the construction only works for the SU(3) model. The same ansatz can nevertheless be used to compute low-energy configurations of the SU(N) model. In particular we showed that for every N there is a radially symmetric solution of charge $B<N$ which corresponds to a bound state of Skyrmions.

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