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Static solutions of a D-dimensional modified nonlinear Schrödinger equation

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Abstract
We study static solutions of a D-dimensional modified nonlinear Schrödinger equation (MNLSE) which was shown to describe, in two dimensions, the self-trapped (spontaneously localized) electron states in a discrete isotropic electron–phonon lattice [1, 2]. We show that this MNLSE, unlike the conventional nonlinear Schrödinger equation, possesses static localized solutions at any dimensionality when the effective nonlinearity parameter is larger than a certain critical value which depends on the dimensionality of the system under study. We investigate various properties of the equation analytically, using scaling transformations, within the variational scheme and numerically, and show that the results of these studies agree qualitatively and quantitatively. In particular, we prove that, for various values of D, when the coupling constant is larger than a certain critical value (which depends on D), this equation has two solutions, a stable (metastable) and an unstable one. We show that the solutions can be well approximated by a Gaussian ansatz and we also show that, in two dimensions, the equation possesses solutions with a nonzero angular momentum.

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1. Introduction

Nonlinear Schrödinger equations (NLSEs) (in various forms and for different spatial dimensionalities D) describe many interesting effects in condensed matter physics. Some of the examples are: the self-trapping of carriers (electrons, holes or excitons) in solid states (e.g. two-dimensional lattices have been studied in [1,2], for more general review see Rashba in [3]), nonlinear processes in plasma [4,5], the propagation of light pulses in nonlinear fibre optics [6], the self-focusing of light beams in nonlinear dispersive media [7] and others (more details can be found, e.g. in [8]). Note also that the NLSE arises as a result of the adiabatic approximation in describing a nonrelativistic particle interacting with a quantum Bose field [9] (in particular, a quasi-particle interacting with lattice vibrations in solids). Usually, the term NLSE is used for the equation with a local cubic nonlinearity.

The one-dimensional ($D = 1$) NLSE has been extensively studied as an integrable equation. It has stationary solutions which can be found by means of the inverse scattering method [10] and describe solitons that interact elastically. The equation describes the propagation of light pulses in fibre optics [6], the formation of envelope solitons in magnetized plasma [4,5], self-trapped excitations in quasi-one-dimensional molecular chains [11,12], etc.

The higher-dimensional NLSEs, on the other hand, are nonintegrable and can be solved only numerically. They have been studied intensively in various applications (in plasma physics, nonlinear optics, physics of solids, etc). For example, they describe self-trappings in two-dimensional ($D = 2$) and three-dimensional ($D = 3$) lattices in the continuum approximation. Calculation of the rate of a particle self-trapping in a three-dimensional lattice is connected with analogous equations in four dimensions ($D = 4$) [13]. For $D \geqslant 2$ stationary solutions of the NLSE are either unstable or do not exist, except for the very specific values of the coupling constant at $D = 2$ [14–16]. Instead of a soliton, the main NLSE solution structure for $D \geqslant 2$ is a collapsing cavity which leads to the formation of localized singularities of the wave amplitude within a finite time.

Note that the modelling of physical processes by NLSEs usually requires certain simplifications. Moreover, the various discrete versions of NLSEs (DNLSE) describe systems only within certain approximations. It is worth noting that some numerical studies and heuristic scaling arguments were presented in [17] to support the existence of spatially localized solutions of a particular form of a DNLSE for the dimensions $D = 1, 2, 3$. Namely, it has been shown that there is a lower bound on the energy of such a solution if the lattice dimension exceeds some critical value. These arguments were generalized in [18] where it was shown that there exists a critical size, called 'excitation threshold' by the authors, for the existence of a localized standing wave solution in an infinite system obeying a DNLS-type equation. Such a threshold depends on the dimensionality of the system and the degree of the nonlinearity, although in reality it is hard to imagine this latter as a varying parameter. In more realistic models, the additional terms due to the higher-order dispersion, nonlinearity saturation or nonlocal interactions should also be taken into account. Such models are usually studied within the continuum approximation. Various MNLSEs have also been studied [19,20]. These equations are, in general, not integrable but some of them do possess stationary localized solutions in two or higher dimensions (in what follows we will refer to these solutions as solitons even when they are not solitons of integrable equations). One or more extra terms are added to the equation and these extra terms sometimes stabilize the solitons in dimensions higher than or equal to two.

This is the case, for example, for the MNLSEs derived in [1, 2]. In these papers it was shown that the system of equations describing the self-trapped electron on a lattice can be reduced, in the continuum limit, to the following MNLSE:

$$\mathrm{i}\varphi_t + \Delta\varphi + 2(g|\varphi|^2 + G\Delta|\varphi|^2)\varphi = 0, \tag{1}$$

where g and G are the two coupling constants, and

$$\Delta = \sum_{\mu=1}^{D} \frac{\partial^2}{\partial x_\mu^2}, \tag{2}$$

where D is the number of dimensions. The extra term, that is proportional to G, arises due to the nonlocality of the nonlinear interaction. In the electron–phonon problems $G = \alpha g/12$, where α, typically, takes values in the range $1 \leqslant \alpha \leqslant 4$ [1, 2].

We would also like to add here that equation (1) is of the type that arises in many physical problems (see, e.g. [19–21]).

2. General characteristics of the MNLSE

In this paper, we study some properties of the MNLSE (1) numerically and using a variational scheme. We show that the equation admits stationary localized solutions at any dimensionality of the system in study, provided that the coupling constant g is larger than a certain critical value which depends on the number of dimensions D.

It is straightforward to show that the following values, which are determined as the functionals of the solutions of (1), are conserved: the norm functional

$$\mathcal{N} = \int \mathrm{d}x^D |\varphi|^2, \tag{3}$$

the Hamiltonian

$$H = \int \mathrm{d}x^D (|\vec{\partial}\varphi|^2 - g|\varphi|^4 + G(\vec{\partial}|\varphi|^2)^2), \tag{4}$$

the momentum

$$\vec{I} = \int \mathrm{d}x^D \vec{j}, \qquad j_\mu = -\frac{i}{2}\left(\varphi^* \frac{\partial\varphi}{\partial x_\mu} - \varphi \frac{\partial\varphi^*}{\partial x_\mu}\right) \tag{5}$$

and the angular momenta

$$L_{\mu\nu} = \int \mathrm{d}x^D (x_\mu j_\nu - x_\nu j_\mu), \tag{6}$$

i.e. the following holds:

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}\vec{I}}{\mathrm{d}t} = \frac{\mathrm{d}L_{\mu\nu}}{\mathrm{d}t} = 0. \tag{7}$$

Note that when φ is interpreted as a wavefunction it is normalized to one, i.e. $\mathcal{N} = 1$. The norm of the wavefunction can also correspond to a signal intensity, which is often called 'the number of particles', and that can take arbitrary values.

Introducing

$$\Lambda = \int \mathrm{d}x^D \frac{i}{2}\left(\varphi^* \frac{\partial\varphi}{\partial t} - \varphi \frac{\partial\varphi^*}{\partial t}\right), \tag{8}$$

which we call the eigenenergy, the following useful relation can be easily obtained from equation (1):

$$\Lambda = \int \mathrm{d}x^D (|\vec{\partial}\varphi|^2 - 2g|\varphi|^4 + 2G(\vec{\partial}|\varphi|^2)^2). \tag{9}$$

Next, we define the average square radius of a field configuration as

$$\mathcal{R}^2 = \int \mathrm{d}x^D \sum_{\mu=1}^{D} x_\mu^2 |\varphi|^2 = \int \mathrm{d}x^D r^2 |\varphi|^2. \tag{10}$$

It is easy to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{R}^2 = 4\int \mathrm{d}x^D (2|\vec{\partial}\varphi|^2 - Dg|\varphi|^4 + (D+2)G(\vec{\partial}|\varphi|^2)^2), \tag{11}$$

when φ is a solution of (1). For localized solutions we interpret $\mathcal{R}$ as the radius of the soliton.

3. Stationary solutions

In this section, we consider stationary solutions of the MNLSE (1) of the form $\varphi = \phi e^{-i\lambda t}$. Then ϕ satisfies the equation

$$\Delta\phi + \lambda\phi + 2\left(g|\phi|^2 + \frac{\alpha g}{12}\Delta|\phi|^2\right)\phi = 0. \tag{12}$$

In this case one can easily prove that

$$\Lambda = \int \mathrm{d}x^D \left(|\vec{\partial}\phi|^2 - 2g|\phi|^4 + \frac{\alpha g}{6}(\vec{\partial}|\phi|^2)^2\right) = \lambda\mathcal{N} \tag{13}$$

and it is straightforward to show that the solutions of equation (12) correspond to the minimum of the functional

$$\mathcal{E} = H - \lambda N \tag{14}$$

together with the normalization condition

$$\mathcal{N} = \int \mathrm{d}x^D |\varphi|^2 = n. \tag{15}$$

In (14), λ is a Lagrange multiplier which has to be determined from the additional condition (15). Notice that $\mathcal{N}$ refers to the norm functional, while n refers to the actual value that this functional takes for a particular solution.

Notice incidentally that if $\phi(\vec{x})$ is a stationary solution of (12) then

$$\Psi(\vec{x}, t) = \phi(\vec{x} - \vec{v}t)e^{i((\vec{v}\cdot\vec{x})/2 - |\vec{v}|^2 t/4)} \tag{16}$$

satisfies (1) and so it corresponds to a soliton moving at the constant speed $\vec{v}$. This shows that one can easily 'boost' any static solution.

To study the existence of solutions of equation (12) we start by performing the following scaling, bearing in mind that λ is negative for bound state solutions:

$$\psi = \sqrt{\frac{g}{-\lambda}}\,\phi, \qquad \vec{\xi} = \sqrt{-\lambda}\,\vec{x}. \tag{17}$$

The scaled static MNLS equation (12) then becomes

$$\Delta\psi - \psi + 2\left(|\psi|^2 - \frac{\alpha\lambda}{12}\Delta|\psi|^2\right)\psi = 0. \tag{18}$$

Defining the quantities

$$\begin{aligned} N &= \int \mathrm{d}\xi^D |\psi|^2, & K &= \int \mathrm{d}\xi^D |\partial\psi|^2, \\ Y &= \int \mathrm{d}\xi^D |\psi|^4, & Z &= \int \mathrm{d}\xi^D (\vec{\partial}|\psi|^2)^2, \end{aligned} \tag{19}$$

the total energy H of the system, the eigenenergy Λ and the norm of the solution take the forms

$$H = \frac{(-\lambda)^{(4-D)/2}}{g}\left(K - Y - \frac{\lambda\alpha}{12}Z\right), \tag{20}$$

$$\Lambda = \frac{(-\lambda)^{(4-D)/2}}{g}\left(K - 2Y - \frac{\lambda\alpha}{6}Z\right) \tag{21}$$

and

$$N = -K + 2Y + \frac{\lambda\alpha}{6}Z, \tag{22}$$

respectively. Moreover, the coupling constant g is directly related to the norm of ψ through the relation

$$\gamma = ng = \frac{N}{(-\lambda)^{(D-2)/2}}. \tag{23}$$

Given that for stationary solutions $\mathrm{d}^2R^2/\mathrm{d}t^2 = 0$ the virial equation (11) becomes

$$2K - DY - (D+2)\frac{\lambda\alpha}{12}Z = 0 \tag{24}$$

and so we can eliminate K from (22) and (24) to get the relation

$$N = -Y\frac{D-4}{2} + Z\frac{\lambda\alpha}{12}\frac{D-2}{2} \tag{25}$$

allowing us to rewrite (20) as

$$H = \frac{(-\lambda)^{(4-D)/2}}{2g}\left((D-2)Y + \frac{\lambda\alpha}{12}DZ\right). \tag{26}$$

If ψ is a stationary solution of (18) the eigenvalue λ must be negative and we will see later that λ can be arbitrarily small. For a solution to be stable the Hamiltonian (26) must also be negative and this imposes the following condition on λ:

$$\alpha\lambda < 12\,\frac{(2-D)Y}{DZ}. \tag{27}$$

We notice immediately that for $D = 2$ and 1 the solutions are always stable and that when $D > 2$ one has to evaluate the integrals Y and Z for the solution and check whether the condition (27) holds.

When $\alpha = 0$ we see that (12) reduces to the NLSE and we see directly from (25) and (20) that the solutions exist when $D < 4$ and that they are stable ($E < 0$) for $D = 1$, unstable ($E > 0$) for $D = 3$ and marginally stable ($E = 0$) for $D = 2$.

In [23] it was proved that equation (18) admit radially symmetric solutions for all $\alpha\lambda < 0$. So, to compute the solutions of (12), one must first solve explicitly (18) for a given value of the product $\alpha\lambda$, evaluate N using (19) and then determine the corresponding value of γ, or g, using (23). Unfortunately, solving (18) explicitly is virtually impossible.

In the next section, we will determine the range of γ values for which (12) admits solutions by solving (18) numerically. This will show that solutions always exist when γ is larger than a certain critical value.

In section 5, we will tackle the problem analytically by approximating the solution of (12) by a radial ansatz and by using the variational principle to determine the parameters appearing in the ansatz. This will confirm the results obtained numerically in the previous section.

4. Numerical solutions

In this section, we consider the problem of finding some stationary solutions of equation (12) which, after the scaling introduced in the previous section, reduces to having to solve equation (18). Notice also that the effective coupling constant is now γ defined by (23) and that it is identical to g in the special case where $n = 1$.

For simplicity we first seek radially symmetric solutions of (18) and we look for solutions of the type $\psi = F(r)$ where r is the radius in the D-dimensional polar coordinates. From (18) we then see that the profile $F(r)$ must satisfy the equation

$$F_{rr} + \frac{D-1}{r}F_r - F + 2\left(F^2 - \frac{\lambda\alpha}{6}\left(F_r^2 + FF_{rr} + \frac{D-1}{r}FF_r\right)\right)F = 0, \tag{28}$$

which can be solved as an ordinary differential equation with the initial condition $F_0 = F(r = 0)$ and $dF/dr(r = 0) = 0$. One then has to find, for a given λ, the values F_0 for which $F(r)$ goes to 0 as r goes to infinity and compute the norm of ψ

$$N = V_{D-1} \int_0^\infty F^2 r^{D-1}\, dr, \tag{29}$$

where we have defined V_D as the area of the D-dimensional sphere

$$V_D = \int_0^{2\pi} d\theta_1 \prod_{i=2}^{D} \int_0^\pi \sin^{i-1}\theta_i\, d\theta_i. \tag{30}$$

Then we use (23) to determine the value of γ, for the given value of λ and the chosen value of n and g.

When α is nonzero we see from (28) that, without any loss of generality, we can restrict ourselves to the case $\alpha = 1$ as any other value changes γ, defined by (23), by the factor $\alpha^{(2-D)/2}$.

When computing a solution numerically we can also evaluate the eigenenergy (9)

$$\Lambda = -V_{D-1}\frac{\lambda n}{N}\int_0^\infty r^{D-1}\left(F_r^2 - \frac{2\alpha\lambda}{3}F^2F_r^2 - 2F^4\right)dr, \tag{31}$$

which, for a solution, should be equal to λn.

The eigenenergy Λ should not be confused with the conserved Hamiltonian energy H (20). In fact they differ by a factor of $\frac{1}{2}$ in the last two terms and one can write

$$\begin{aligned} H &= \frac{-\lambda n}{N}V_{D-1}\int_0^\infty r^{D-1}\left(F_r^2 - \frac{\alpha\lambda}{3}F^2F_r^2 - F^4\right)dr \\ &= \frac{1}{2}\Lambda - \frac{\lambda n}{2N}V_{D-1}\int_0^\infty r^{D-1}F_r^2\, dr. \end{aligned} \tag{32}$$

In figure 1 we exhibit various properties of the numerical solutions of (28). In particular, figure 1(*a*) shows the eigenenergy $-\Lambda = -\lambda$ as a function of γ. In particular, it demonstrates that a solution exists only when the parameter γ is larger than some critical values, which are given in table 1. Other data shown in table 1 correspond to values derived later using an ansatz approximation. We observe also that for $D = 2$ there is only one branch of solutions while for $D > 2$ there are two solutions for each value of $\gamma > \gamma_{\rm cr}$. According to the Vakhitov–Kolokolov criteria [24], the upper branch corresponds to the stable (metastable) solution, and the other (lower) one corresponds to an unstable solution. Figure 1(*b*) presents the plot of the ratio of the energy (32) to the eigenvalue λ as a function of γ. As shown above, the energy is always negative for $D = 2$ and, hence, the corresponding solution is stable. For $D > 2$ the upper branch is metastable at $\gamma_{\rm cr} < \gamma < \gamma_{\rm c}$, and is stable, i.e. has negative energy, only at $\gamma > \gamma_{\rm c}$. In table 1 we give the critical values of $\gamma_{\rm c}$ corresponding to the zero energy, and, respectively, the value of $\lambda_{\rm c}$. When λ is large, we see from (32) that the ratio $-H/\lambda$ goes asymptotically to $-\frac{1}{2}$.

The size of the solutions for the rescaled function is given by the radius

$$R = \left(\frac{1}{N}\int_0^\infty F^2 r^{D+1} dr\right)^{1/2}, \tag{33}$$

where N is defined by (29) as a function of γ. The actual size of the soliton, $\mathcal{R}$, is related to R defined by (33) as follows:

$$\mathcal{R} = \frac{R}{\sqrt{-\lambda}}. \tag{34}$$

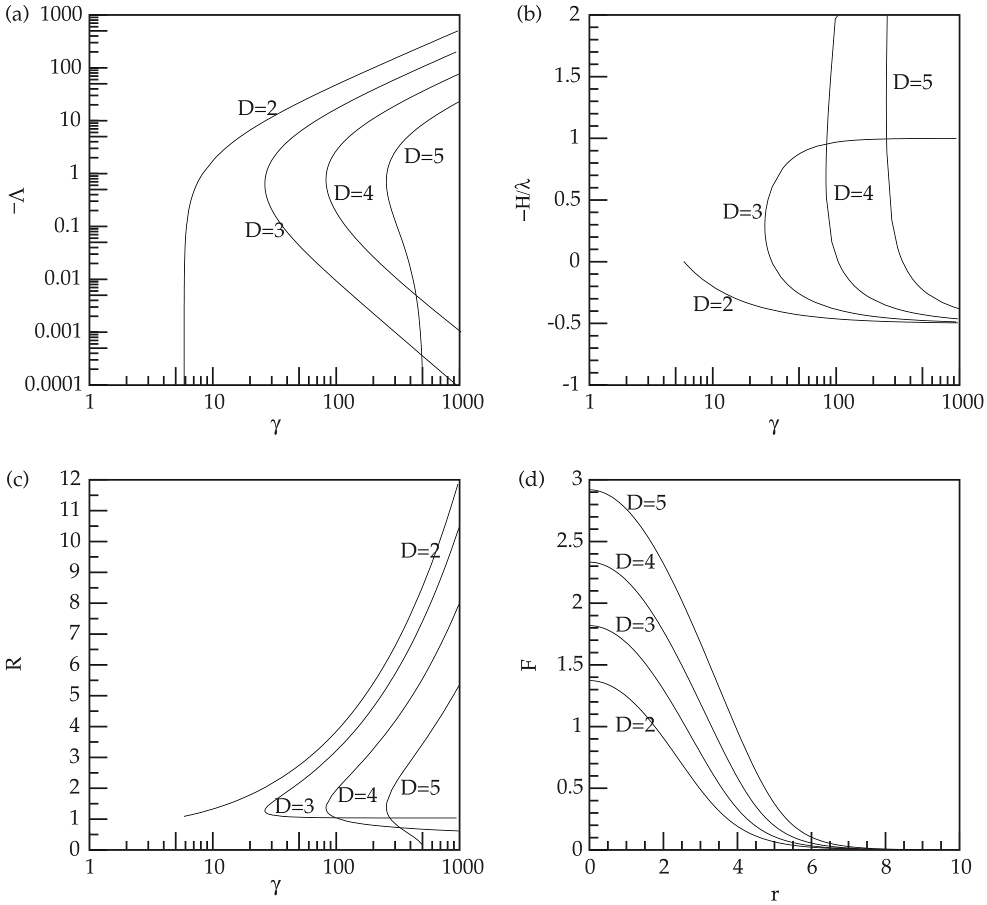

Figure 1. Numerical solutions (28) for $D = 2, 3, 4$ and 5: (a) eigenvalues $-\Lambda = -n\lambda$ as functions of γ; (b) energy (20), eigenvalue ratios $-H/\lambda$; (c) Radii R; (d) profiles for $\lambda = -10$ and $\gamma = 26.7231$ ($D = 2$), $\gamma = 68.5808$ ($D = 3$), $\gamma = 188.576$ ($D = 4$) and $\gamma = 552.273$ ($D = 5$).

Table 1. Critical values of γ and λ, determined numerically and using equations (57) and (58), respectively.

D	γ_c	λ_c	γ_c^{ans}	λ_c^{ans}
2	—	0	—	0
3	30.0982	−1.79	$3(3\pi/2)^{3/2} \approx 30.7$	−2
4	103.91	−3.035	$32\pi^2/3 \approx 105.28$	−3
5	345.466	−3.95	$5^5(\pi/2)^{5/2}/3^3 \approx 357.9$	−3.6

In figure 1(c) we present the plot of the radius R. The top and the bottom branches of figure 1(c) correspond, respectively, to the top and the bottom branches of the curves in figure 1(a). The stable solutions are thus the ones with the bigger value of R. In figure 1(d) we present the profiles $F(r)$ of one solution corresponding to the eigenvalue $\lambda = -10$, for various dimensionalities. The corresponding values of γ are given in the figure caption.

5. Radial ansatz

In this section, we approximate the solutions by an analytical ansatz. First we note that equation (28) can be obtained by minimizing the density

$$S = \int_0^\infty r^{D-1}\,\mathrm{d}x \left(F_r^2 - F^4 + F^2 - \frac{\lambda\alpha}{3} F^2 F_r^2 \right). \qquad (35)$$

Note that, in this section, we will allow α to take any value. We now use the fact that the MNLSE admits radially symmetric solutions [22, 23, 1, 2].

To estimate the values of g for which equation (1) admits solutions normalized to n, we consider the Gaussian ansatz

$$F = A\mathrm{e}^{-kr^2}, \qquad (36)$$

compute N given by equation (29) and use the fact that $N = \gamma(-\lambda)^{(D-2)/2}$ to express A as the following function of γ and k:

$$A^2 = \gamma(-\lambda)^{(D-2)/2} \frac{2^{(D+2)/2} k^{D/2}}{\Gamma(D/2) V_{D-1}}. \qquad (37)$$

Next we evaluate (35) for the above ansatz and obtain

$$S = \gamma \left(kD(-\lambda)^{(D-2)/2} + (-\lambda)^{(D-2)/2} - \frac{2\gamma(-\lambda)^{D-2} k^{D/2}}{\Gamma(D/2) V_{D-1}} + \frac{\alpha\gamma D(-\lambda)^{D-1} k^{(D+2)/2}}{3\Gamma(D/2) V_{D-1}} \right). \qquad (38)$$

Minimizing it with respect to k we get the equation

$$\Gamma\left(\frac{D}{2}\right) V_{D-1} - \gamma\kappa^{(D-2)/2} + \frac{D+2}{6}\gamma\alpha\kappa^{D/2} = 0, \qquad (39)$$

where we have introduced $\kappa = -\lambda k$. Once we have determined the value of κ which minimizes S, we find the eigenenergy (21):

$$\lambda = \kappa D - \frac{4\gamma\kappa^{D/2}}{\Gamma(D/2) V_{D-1}} + \frac{2\alpha\gamma D\kappa^{(D+2)/2}}{3\Gamma(D/2) V_{D-1}} \qquad (40)$$

and obtain an estimate of the eigenvalue λ from the ansatz. Notice that if we define $\tilde{\kappa} = \kappa\alpha$ and $\tilde{\gamma} = \gamma a^{(2-D)/2}$, α disappears from (39) and (40). Thus, without loss of generality, we take $\alpha = 1$ from now on.

The simplest cases are summarized in table 2. We easily solve the algebraic equations when $D = 1, 2$ and 4 and the results are given in table 3. The case $D = 3$ and 5 can be easily solved using an algebraic manipulation package like Maple.

Table 2. Equation for κ and eigenenergy λ for the ansatz (36) for $D = 2$–5.

D	V_{D-1}	$\Gamma(D/2)$	Equation	Λ
1	2	$\sqrt{\pi}$	$2\pi^{1/2}\kappa^{1/2} - \gamma + \gamma\frac{1}{2}\kappa = 0$	$\kappa - \frac{2\gamma}{\pi^{1/2}}\kappa^{1/2} + \frac{\gamma}{3\pi^{1/2}}\kappa^{3/2}$
2	2π	1	$2\pi - \gamma + \gamma\frac{4}{6}\kappa = 0$	$2\kappa - \frac{2\gamma}{\pi}\kappa + \frac{2\gamma}{3\pi}\kappa^2$
3	4π	$\frac{1}{2}\sqrt{\pi}$	$2\pi^{3/2} - \gamma\kappa^{1/2} + \gamma\frac{5}{6}\kappa^{3/2} = 0$	$3\kappa - \frac{2\gamma}{\pi^{3/2}}\kappa^{3/2} + \frac{\gamma}{\pi^{3/2}}\kappa^{5/2}$
4	$2\pi^2$	1	$2\pi^2 - \gamma\kappa + \gamma\kappa^2 = 0$	$4\kappa - \frac{2\gamma}{\pi^2}\kappa^2 + \frac{4\gamma}{3\pi^2}\kappa^3$
5	$\frac{8}{3}\pi^2$	$\frac{3}{4}\sqrt{\pi}$	$2\pi^{5/2} - \gamma\kappa^{3/2} + \gamma\frac{7}{6}\kappa^{5/2} = 0$	$5\kappa - \frac{2\gamma}{\pi^{5/2}}\kappa^{5/2} + \frac{5\gamma}{3\pi^{5/2}}\kappa^{7/2}$

Table 3. Values of κ for $D = 1$, 2 and 4.

D	κ	
1	$\left(\frac{\sqrt{4\pi + 2\gamma^2} - 2\sqrt{\pi}}{\gamma}\right)^2$,	$\gamma > 0$
2	$(\gamma - 2\pi)\frac{3}{2\gamma}$,	$\gamma > 2\pi$
4	$\frac{1}{2}\left(1 \pm \sqrt{1 - \frac{8\pi^2}{\gamma}}\right)$,	$\gamma > 8\pi^2$

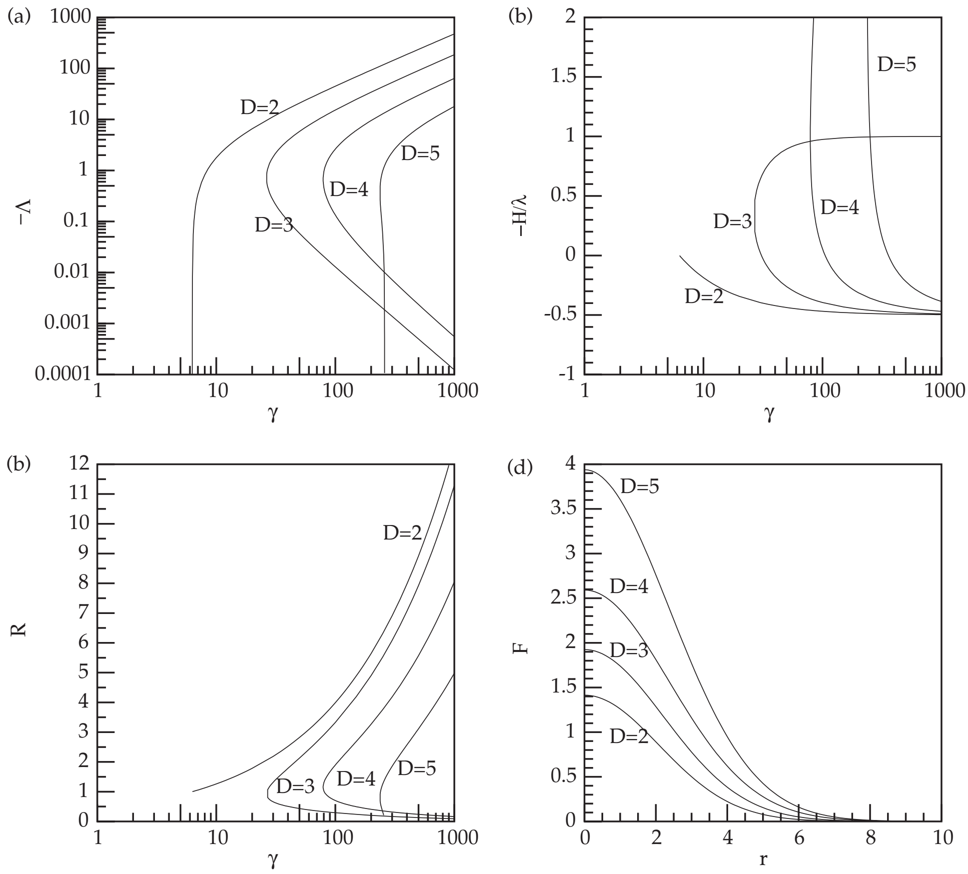

Figure 2. Radial ansatz (36) for $D = 2, 3, 4$ and 5: (a) eigenenergies $-\Lambda$ (40) as functions of γ; (b) energy (20) and eigenenergy (40) ratios $-H/\Lambda$; (c) radii $\mathcal{R}$; (d) profiles for $\lambda = -10$ and ($D = 2$): $\gamma = 27.2271$, $A = 1.4142$, $k = 0.115\,38$; ($D = 3$): $\gamma = 72.098$, $A = 1.9252$, $k = 0.100\,80$; ($D = 4$): $\gamma = 207.52$, $A = 2.5914$, $k = 0.089\,35$; and ($D = 5$): $\gamma = 643.59$, $A = 3.9398$, $k = 0.088\,93$.

In figure 2 we present some properties of the ansatz (36) when $D = 2, 3, 4$ and 5. Figures 2(a)–(c) present, respectively, the eigenenergy $-\lambda$, the ratio of the energy (20) to the eigenenergy (40), and the radius R, as functions of γ. From table 4 we conclude that the predicted value of $\gamma_{\rm cr}$ is very close to the one calculated numerically.

Table 4. Critical values of γ determined numerically and for the ansatz (36).

D	Numerical $\gamma_{\rm cr}$	Ansatz $\gamma_{\rm cr}$
2	5.85	$2\pi \approx 6.2832$
3	26.4094	$3\pi(5\pi/2)^{1/2} \approx 26.4129$
4	82.6714	$8\pi^2 \approx 78.957$
5	254.964	$5(\pi)^{5/2}(35/18)^{3/2} \approx 237.16$

Figure 2(d) presents the profile functions of the configurations corresponding to $\Lambda = -10$. The values of γ, k and A, given in the caption, were computed as follows: first we used figure 2(a) to determine the value of γ corresponding to the solution $\Lambda = 10$. Then we calculated K by solving equation (39) and then, taking $k = -\kappa/\lambda$, we determined A using (37). Comparing figures 1 and 2, we see that the ansatz (36) gives a very good approximation to the solutions of (28). In fact, the approximation is so good that figures 1 and 2, at first sight, may appear to be almost the same (especially for $D < 5$).

The relations (39) and (40) determine the dependence of λ on the constants γ and α in parametric form via the parameter κ. To obtain this dependence explicitly, we rewrite (39) as follows:

$$\gamma = V_{D-1}\Gamma\left(\frac{D}{2}\right)\left(\frac{1}{\kappa} - \frac{D+2}{6}\alpha\right)^{-1}\left(\frac{1}{\kappa}\right)^{D/2}. \tag{41}$$

Rewriting (40) as

$$\lambda = D\kappa - 4\kappa\frac{1/\kappa - (D/6)\alpha}{1/\kappa - (D+2)/6\alpha} \tag{42}$$

and solving this algebraic expression we can compute $\kappa^{D/2}$ as a function of λ and substitute it into (40) to get

$$\frac{1}{\kappa} = \frac{\alpha}{12}\frac{Q + (4-D)/2}{\mu}, \tag{43}$$

where

$$Q = (D+2)\mu + \sqrt{\left(\frac{4-D}{2}\right)^2 + (D^2 - 2D + 8)\mu + (D+2)^2\mu^2} \tag{44}$$

and

$$\mu = -\frac{\lambda\alpha}{12} > 0. \tag{45}$$

Substituting (43) into (41), we find

$$\gamma = \frac{V_{D-1}}{16}\Gamma\left(\frac{D}{2}\right)\left[\frac{\alpha}{12}\right]^{(D-2)/2} F(\mu), \tag{46}$$

where $F(\mu)$ is given by

$$F(\mu) = \left(Q - \frac{4-D}{2}\right)\left[\frac{Q + (4-D)/2}{\mu}\right]^{D/2}. \tag{47}$$

Using this result, we can analyse the conditions of the existence of solutions for arbitrary values of D.

The function $F(\mu)$ satisfies the following asymptotic conditions: $F(\mu) \to \infty$ as $\mu \to \infty$ for any value of D and when $\mu \to 0$ we have

$$F(\mu \to 0) \to \begin{cases} \infty & \text{if } D \leqslant 4, \\ F_0 & \text{if } D > 4. \end{cases} \tag{48}$$

Moreover, $F(\mu)$ has a minimum at the finite value

$$\mu_0 = \frac{(D-2)(6-D)}{2(D+2)^2} \tag{49}$$

when $2 < D \leqslant 6$. For $D > 6$, $F(\mu)$ is a monotonically increasing function of μ thus attaining its minimum value at $\mu = 0$.

Thus equation (46) has solutions only when

$$\gamma \geqslant \frac{V_{D-1}}{16}\Gamma\left(\frac{D}{2}\right)\left[\frac{\alpha}{12}\right]^{(D-2)/2} F_{\min}(\mu), \tag{50}$$

where $F_{\min}(\mu)$ is the corresponding minimum value of F:

$$F_{\min}(\mu) = \begin{cases} F(\mu_0) & \text{if } D \leqslant 6, \\ F(0) & \text{if } D > 6. \end{cases} \tag{51}$$

Here

$$F(\mu_0) = 8D\left(2D\frac{D+2}{D-2}\right)^{(D-2)/2}, \tag{52}$$

$$F(0) = (D-4)\left(2D\frac{D-2}{D-4}\right)^{D/2}. \tag{53}$$

When $D \geqslant 6$, according to the argument above, equation (46) has thus only one solution when $\gamma > \gamma_{\rm cr}$. For $D < 5$ and $\gamma > \gamma_{\rm cr}$ this equation has two solutions, which correspond to the stable and unstable solutions, respectively. When $D = 5$ and $\gamma_{\rm cr} < \gamma < \gamma_0$ the equation has two solutions (one stable and one unstable), and when $\gamma > \gamma_0$ it has only one solution, which corresponds to a stable soliton. Here $\gamma_{\rm cr}$ is proportional to $F(\mu_0)$, and γ_0 is proportional to $F(0) = F_0$. For $D > 6$ the value of $\gamma_{\rm cr}$ determined from equation (50) is given by the expression:

$$\gamma_{\rm cr} = V_{D-1}\frac{D-2}{4}\Gamma\left(\frac{D}{2}+1\right)\left[\frac{\alpha D(D-2)}{6(D-4)}\right]^{(D-2)/2}. \tag{54}$$

The solution of the equation corresponds to the absolute minimum of the Hamiltonian only when the total energy is negative. The latter is given by the expression

$$E = \frac{(-\lambda)^{(4-D)/2}}{8g}A^2 b^{(D-2)/2}\Gamma\left(\frac{D}{2}\right)\left(\frac{3D-4}{2} - Q\right). \tag{55}$$

Thus, the energy is negative when $\mu > \mu_1 = (D-2)/2D$. At $\mu = \mu_1$ the function F takes the value

$$F(\mu_1) = 8D^2\left(\frac{2D^2}{D-2}\right)^{(D-2)/2}. \tag{56}$$

Therefore, the soliton solution corresponds to the absolute minimum of the energy functional for $\gamma > \gamma_{\rm c}$, where $\gamma_{\rm c}$ is given by

$$\gamma_{\rm c} = V_{D-1}\frac{D}{2}\Gamma\left(\frac{D}{2}+1\right)\left[\frac{\alpha D^2}{6(D-2)}\right]^{(D-2)/2}. \tag{57}$$

The corresponding eigenvalue $\lambda_{\rm c}$ is

$$\lambda_{\rm c} = -\frac{6(D-2)}{\alpha D}. \tag{58}$$

The values of $\gamma_{\rm c}$ and $\lambda_{\rm c}$ are given in the last two columns of table 1 and compare well with the values obtained by the numerical calculations. However, although the Gaussian ansatz gives a good approximation to what we have seen in our numerical simulations, the ansatz is not exact. In fact, for $D = 6$ and 7 the numerical integration of equation (28) shows the existence of two solutions (like for $D = 5$) except that the critical values of the coupling constants are very large (e.g. the critical values of γ_c (as $\lambda \to 0$) are given by $\sim$808 and $\sim$10637 for $d = 6$ and 7, respectively). So we see that this approximation is very good, though not perfect.

A similar analysis of equation (46) can be carried out straightforwardly for $D/2 \gg 1$ using the asymptotic expression for the Γ-function and the relation

$$V_{D-1} \approx \sqrt{2e}\left(\frac{2\pi e}{D}\right)^{(D-1)/2}, \qquad \frac{D}{2} \gg 1. \tag{59}$$

Using (44) we can expand (47) in power series of $1/D$ to find

$$F(\mu) \approx D(2D)^{D/2}\left(1+\frac{2}{D}\right)^{D/2}\left[1-\frac{4}{D}+2\left(1+\frac{2}{D}\right)\mu\right]. \tag{60}$$

Substituting these results into equation (46), and solving it with respect to the eigenenergy, using (45), we get

$$\lambda = -\frac{6}{\alpha}\frac{\gamma-\gamma_{cr}}{\gamma_{cr}}, \tag{61}$$

where γ_{cr} can be approximated at $D/2 \gg 1$ by its asymptotic value:

$$\gamma_{cr} \to \frac{3De}{2\alpha}\left(\frac{\alpha\pi D}{6}\right)^{D/2}, \qquad \frac{D}{2} \gg 1. \tag{62}$$

Note that the asymptotic value of γ_{cr} can also be obtained from (54) and that the dependence of λ on γ at large dimensionalities is similar to the case $D = 2$ (see below).

The dependence of the total energy (20) on γ is given by the expression

$$E = \frac{3n}{\alpha}\frac{2\gamma_{cr}-\gamma}{\gamma_{cr}}. \tag{63}$$

As was mentioned above, the solutions are metastable when $E > 0$, and are absolutely stable when $E < 0$. For $D/2 \gg 1$ these regimes, according to (63), occur when $\gamma_{cr} < \gamma < \gamma_c = 2\gamma_{cr}$, and $\gamma > \gamma_c$, respectively. We also find from (61) that $\lambda = \lambda_c = -6/\alpha$ corresponds to the given value of $\gamma = \gamma_c$, which is in agreement with the general analysis of (46) for an arbitrary value of D (see (57) and (58)).

The energy ratio, according to (61) and (63), takes the form

$$\frac{E}{-\Lambda} = \frac{2\gamma_{cr}-\gamma}{2(\gamma-\gamma_{cr})}, \qquad \Lambda = n\lambda. \tag{64}$$

The radius (10) of the configuration corresponding to the anstaz (36) is given by

$$R^2 = \frac{D}{4k}. \tag{65}$$

At large D it takes the asymptotic value

$$R = \frac{D}{2}\sqrt{-\lambda\frac{\alpha}{6}} \tag{66}$$

and, thus, the asymptotic value of the actual soliton size (34) is

$$\mathcal{R} = \frac{D}{2}\sqrt{\frac{\alpha}{6}}. \tag{67}$$

Therefore, the soliton radius depends only on the nonlinearity coefficients α and is independent of γ for large D.

It is worth mentioning here that the two branches of solutions, stable (metastable) and unstable, exist when $D > 2$. The stability of the corresponding solutions can be analysed using the Vakhitov–Kolokolov [24] criteria, which state that a soliton is stable if $\mathrm{d}(-\lambda)/\mathrm{d}n > 0$.

6. Special solutions for $D = 2$

The two-dimensional case of the MNLSE is very special. First, in equation (23) the value of γ is equal to N for all values of $-\lambda$. Moreover, (22) and (24) imply that $N = Y$ and so we have

$$K - Y = 2\frac{\alpha\lambda}{12}Z < 0. \tag{68}$$

Hence

$$\gamma = ng = N = Y > K \tag{69}$$

for all values of λ and, for the radial function $\psi = F(r)$, we get

$$K = 2\pi \int_0^\infty \mathrm{d}r\, r F_r^2. \tag{70}$$

In particular, as was shown in [1, 15], when $\alpha = 0$, $K = 5.85$ and this value is thus the critical value for γ below which there is no solution when $D = 2$.

When $D = 2$ equation (12) also has solutions of the form $\psi = F(r)\mathrm{e}^{\mathrm{i}l\theta}$ where the radial profile $F(r)$ now satisfies the equation

$$-F + F_{rr} + \frac{1}{r}F_r - \frac{l^2}{r^2}F + 2\left(F^2 - \frac{\lambda\alpha}{6}\left(F_r^2 + FF_{rr} + \frac{1}{r}FF_r\right)\right)F = 0. \tag{71}$$

Notice that the angular momentum (6) for these solutions is simply given by $L_{10} = ln$. When $\alpha = 0$, we see from (24) and (22) that $K = Y = N$ for any value of l. Solving (71) numerically for different values of l, we find the values of K given in table 5. Note that they do not depend on λ.

Notice also that as $N = \gamma$, for $D = 2$, we can compute a solution numerically by first choosing a value of λ, then solving (71) with the appropriate boundary conditions and then computing N which gives the value γ for which this is a solution.

In figure 3, we exhibit various properties of the numerical solutions of (71) for $l = 0, 1$ and 2. Figure 3(a) shows the eigenenergy $-\Lambda = -n\lambda$ as a function of γ and it confirms the fact that solutions exist only when the parameter γ is larger than the critical values listed in the table. Figures 3(b) and (c) show, respectively, the ratio of the total energy to the eigenvalue $-E/\Lambda$, and the radius (33) as functions of γ. In figure 3(d) we present the plots of the profiles $F(r)$ of the solutions for $l = 0, 1$ and 2 when $\gamma = 55$. The corresponding values of λ are given in the figure caption.

Note also that (71) can be obtained by minimizing the density

$$S = \int_0^\infty \mathrm{d}r\, r\left(F_r^2 - F^4 + F^2 + \frac{l^2}{r^2}F^2 - \frac{\lambda\alpha}{3}F^2F_r^2\right). \tag{72}$$

It is straightforward to show that the asymptotic behaviour of F near the origin is given by $F = Br^l$, for some constant B, while the finiteness of N imposes the requirement that F goes to 0 at infinity. Therefore we approximate the solutions by evaluating S for the ansatz

$$F = Ar^l\mathrm{e}^{-kr^2}. \tag{73}$$

Table 5. Critical values of γ predicted from (69), $D = 2$.

l	$\gamma_{\mathrm{cr},l} = K$
0	5.85
1	24.1492
2	44.876
3	66.2117

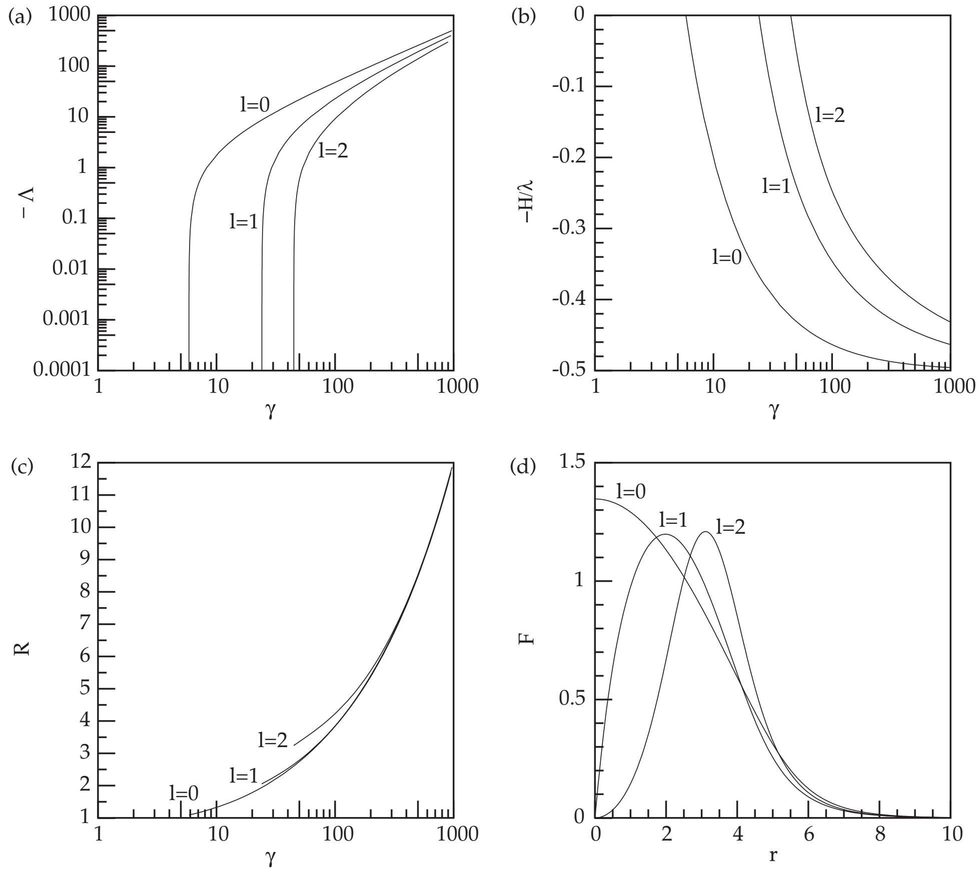

Figure 3. Numerical solutions (28) for $D = 2$ and $l = 0, 1, 2$: (*a*) eigenvalues $-\Lambda = -n\lambda$ as functions of γ; (*b*) energy (20) and eigenenergy (40) ratios $-E/\Lambda$; (*c*) radii R; (*d*) solution profiles $F(r)$ when $\gamma = 55$ for $l = 0$ ($\lambda = -24.4781$) $l = 1$ ($\lambda = -7.7915$) and $l = 2$ ($\lambda = -1.2255$).

First, we compute N using equation (29) and as equation (23) for $D = 2$ implies $N = -\lambda\gamma$, we can express A as a function of γ and k, finding

$$A^2 = \frac{\gamma(2k)^{l+1}}{\Gamma(l+1)\pi}. \tag{74}$$

Substituting (73) and (74) into (72) and using the fact that, as shown in the previous section, α can be scaled away and so that we can take $\alpha = 1$ without any loss of generality, we get

$$S = \frac{\gamma}{\pi}\left(\frac{1}{2} + k(1+l)\right) + \frac{\gamma^2\Gamma(2l+1)}{2^{2l+1}\pi^2\Gamma^2(l+1)}\left(\frac{-\lambda k^2}{3} - k\right). \tag{75}$$

Minimizing S with respect to k and defining, as before, $\kappa = -k\lambda$, we find the solution

$$\kappa_l = \frac{3}{2\gamma}(\gamma - \gamma_{\mathrm{cr},l}), \tag{76}$$

where

$$\gamma_{\mathrm{cr},l} = \gamma_{\mathrm{cr},0}\frac{l!(l+1)!}{(2l)!}2^{2l}, \qquad \gamma_{\mathrm{cr},0} = 2\pi. \tag{77}$$

Table 6. Critical value for γ determined numerically and using (77) for the ansatz (73).

l	Numerical $\gamma_{\mathrm{cr},l}$	Ansatz $\gamma_{\mathrm{cr},l}$
0	5.85	6.28
1	24.14	25.13
2	44.87	50.27

Expression (76), taken together with (74), gives us the minimizing parameters for the ansatz (73). Note also from (76) that κ_l is positive only if γ is larger than the critical values $\gamma_{\mathrm{cr},l}$ (77), which depend on l and which are given in table 6.

The eigenenergy (9), evaluated for the new ansatz, is now

$$\lambda(\kappa) = 2\kappa(1+l) + \frac{\gamma\Gamma(2l+1)\kappa}{2^{2l-1}\pi\Gamma^2(l+1)}\left(\frac{\kappa}{3}-1\right). \tag{78}$$

Substituting (76) into (78), we get

$$-\lambda_l = \kappa^2 = \frac{3(l+1)}{\alpha\gamma_{\mathrm{cr},l}}(\gamma-\gamma_{\mathrm{cr},l}). \tag{79}$$

The corresponding energy takes the value

$$E_l = -\frac{3(l+1)}{2\alpha g\gamma_{\mathrm{cr},l}}(\gamma-\gamma_{\mathrm{cr},l})^2, \tag{80}$$

and, thus,

$$\frac{E_l}{-\Lambda_l} = -\frac{\gamma-\gamma_{\mathrm{cr},l}}{2\gamma}, \tag{81}$$

where $\Lambda_l = n\lambda_l$.

The radius of localization $R = \sqrt{(l+1)/2k} = \sqrt{-\lambda(l+1)/2\kappa}$ as a function of γ is given by the expression

$$R_l = (l+1)\sqrt{\frac{\gamma}{\gamma_{\mathrm{cr},l}}}, \tag{82}$$

from which it follows, that $R_l = l+1$ at $\gamma = \gamma_{\mathrm{cr},l}$. The actual size of the localization, $\mathcal{R}$, which was defined in (10) is, according to (34) and (82), given by the expression

$$\mathcal{R}_l = \sqrt{\frac{\alpha\gamma(l+1)}{3(\gamma-\gamma_{\mathrm{cr},l})}} \tag{83}$$

(at $l = 0$ this value had been calculated in [1,2]).

In figure 4, we exhibit various properties of the ansatz (73) for $l = 0, 1$ and 2. Figures 4(a)–(c) present, respectively, the eigenenergy $-\Lambda$, the ratio of the energy (20) to the eigenenergy (40), and the radius (83). Notice that the radius curves of $l = 0$ and 1 overlap exactly for the ansatz. Comparing figure 4 with 3, we see that the ansatz (73) provides a very good approximation to the solutions of (71). In figure 4(d) we present the profiles $F(r)$ of one solution of each $l = 0$, 1 and 2 for $\gamma = 55$. The corresponding values of Λ, obtained from figure 4(a), are given in the figure caption.

7. Conclusion

We have shown that the MNLSE (1) admits solutions at all dimensionalities, D, when the nonlinearity coupling constant g is larger than a certain critical value which depends on the

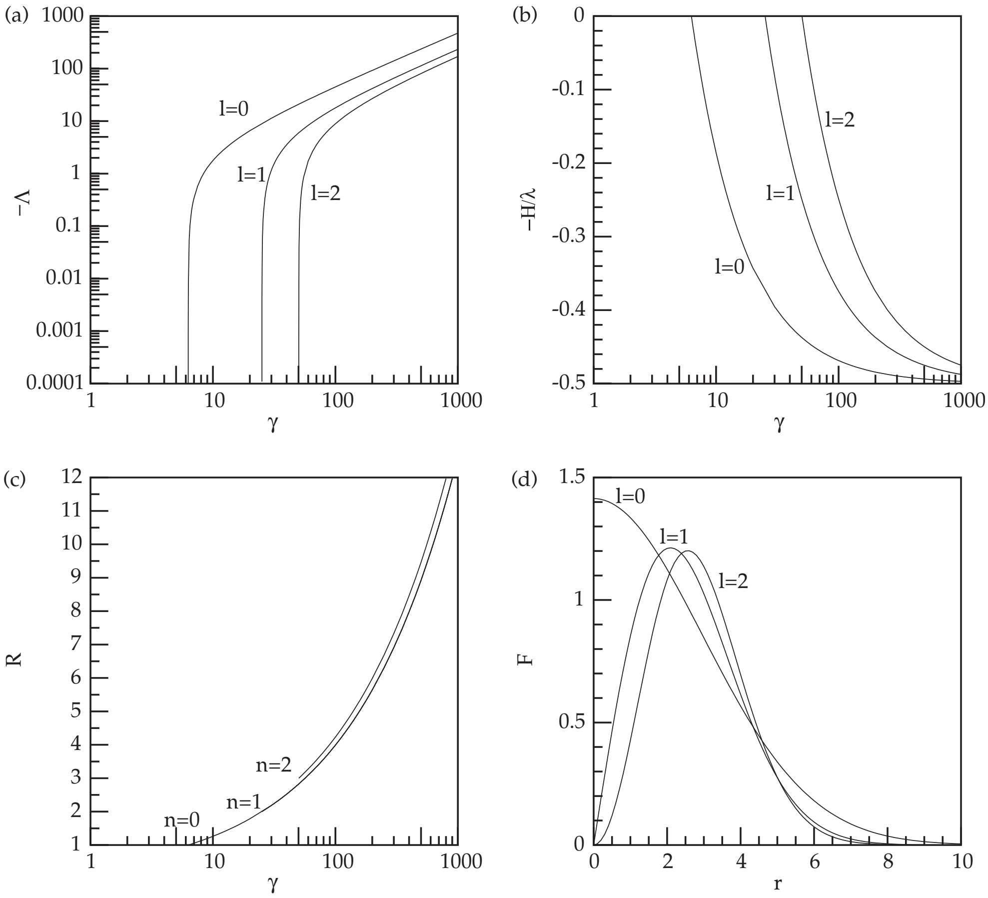

Figure 4. Numerical solutions (28) for $D = 2$ and $l = 0, 1, 2$: (*a*) eigenvalues $-\lambda$ as functions of γ; (*b*) energy (20) and eigenenergy (40) ratios $-H/\Lambda$; (*c*) radii R; (*d*) solution profiles $F(r)$ when $\gamma = 55$ for $l = 0$ ($\lambda = -23.2606$, $A = 1.4142$, $k = 0.057\,12$), $l = 1$ ($\lambda = -7.130\,28$, $A = 0.9560$, $k = 0.114\,24$) and $l = 2$ ($\lambda = -0.847\,712$, $A = 0.4975$, $k = 0.152\,32$).

dimensionality and the norm n of the solution. We have also shown that, as the value of g increases, the solutions are more bound, their energies are smaller and they correspond to solitons which are narrower. Obviously, the values of energy and $\gamma_{\rm cr}$ depend on the second nonlinearity parameter of the equation, G, which characterizes the dispersion of the nonlinearity. We have also shown that all solutions can be well approximated by an appropriate Gaussian ansatz. This approximation is exceptionally good although it tends to overestimate the values of $\gamma_{\rm c}$. At $D \geqslant 6$ it predicts only one solution while, in fact, there are two.

We have shown that all two-dimensional solutions are stable as their total energy is negative, while in higher dimensions only solutions corresponding to sufficiently large values of the coupling constant g are stable, since at $D \gg 2$ we have $\gamma_{\rm c} = 2\gamma_{\rm cr}$, where $\gamma = ng$.

In two dimensions, equation (1) also has stationary solutions with a nonzero angular momentum. These solutions are also all stable but they exist only when g is larger than a critical value which grows with the value of the angular momentum. These solutions can also be successfully approximated by a modified Gaussian ansatz.

In a future work, we plan to present our investigations of the scattering properties of the solutions we derived in this paper.

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