HOROSPHERICAL GEOMETRY OF SUBMANIFOLDS IN HYPERBOLIC SPACE

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Recently we discovered a new geometry on submanifolds in hyperbolic space[2, 5, 6, 7, 8, 9] which is called *horospherical geometry*. In this talk I explain the outline of this geometry. I start to explain the elementary horospherical (horocyclic) geometry on the Poincaré disk model of hyperbolic plane. On the Poincaré disk, if we adopt geodesics as lines, we have the model of non-Euclidean geometry of Gauss-Bolyai-Lobachefskii. It is well-known that the axiom 5 of the Euclidean geometry is not satisfied. However, if we adopt horocycles as lines, the axiom 5 of the Euclidean geometry is satisfied. But the axiom 1 is not satisfied. We call such a geometry horocyclic geometry. In this case, what is the angle between lines? Under a suitable definition, we can show that the total sums of interior angles of a triangle is π. This suggests us a kind of the Gauss-Bonnet type theorem holds if we define a suitable curvature of a surface in hyperbolic space. In the previous theory of surfaces in hyperbolic space, there appeared two kinds of curvatures. One is called the *extrinsic Gauss curvature* K_e and another is the *intrinsic Gauss curvature* K_I. The intrinsic Gauss curvature is nothing but the Gauss curvature defined by the induced Riemannian metric on the surface. The relation between these curvatures is known that $K_e = K_I + 1$. Of course the Gauss-Bonnet type theorem holds for intrinsic Gauss curvature by Chern-Weil theory. In [5] we defined a new curvature K_h called a *hyperbolic curvature* of the surface by using the modified notion of the hyperbolic Gauss map[1, 3]. This curvature is an extrinsic hyperbolic invariant because we have the relation $K_h = 2 - 2H + K_I$, where H is the mean curvature of the surface. In [7] we have modified the hyperbolic Gauss curvature into the *horospherical Gauss curvature* $\widetilde{K}_h$ and shown that the Gauss-Bonnet type theorem holds. Moreover, Chern-Lashof type inequality for absolute Gauss curvature holds. This curvature is not a hyperbolic invariant but it is invariant under the canonical action of $SO(3)$. We can show that the value of $\widetilde{K}_h(p)$ is a hyperbolic invariant if and only if $\widetilde{K}_h(p) = 0$. Moreover, $\widetilde{K}_h(p) = 0$ if and only if $K_h(p) = 0$. Therefore the horospherical flatness is a hyperbolic invariant. Totally umbilical and horospherical flat surfaces are horospheres. We call the geometry related to this curvature *horospherical geometry*. Moreover, there is an important class of surfaces called *linear Weingarten surfaces* which satisfy the relation $aK_I + b(2H - 2) = 0$. In [4], the Weierstrass-Bryant type representation formula for such surfaces with $a + b \neq 0$ (called, the *Bryant type*) has been shown. This class of surfaces contains flat surfaces (i.e., $a \neq 0, b = 0$) and CMC-1 surfaces ($a = 0, b \neq 0$). However, the horospherical flat surface is the exceptional case (the *non-Bryant type* : $a + b = 0$). Therefore the horospherical flat surfaces are also importnat subjects in hyperbolic geometry. I will describe the horospherical flat surfaces and singularities in another talk at geometry seminar.

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