Averaging, slaving and balance dynamics in a simple atmospheric model

Djoko Wirosoetisno 1, Theodore G. Shepherd*

Department of Physics, University of Toronto, 60 St. George Street, Toronto, Ont., Canada M5S 1A7

Received 3 May 1999; received in revised form 7 January 2000; accepted 7 January 2000

Abstract

We report numerical results from a study of balance dynamics using a simple model of atmospheric motion that is designed to help address the question of why balance dynamics is so stable. The non-autonomous Hamiltonian model has a chaotic slow degree of freedom (representing vortical modes) coupled to one or two linear fast oscillators (representing inertia-gravity waves). The system is said to be balanced when the fast and slow degrees of freedom are separated. We find adiabatic invariants that drift slowly in time. This drift is consistent with a random-walk behaviour at a speed which qualitatively scales, even for modest time scale separations, as the upper bound given by Neishtadt’s and Nekhoroshev’s theorems. Moreover, a similar type of scaling is observed for solutions obtained using a singular perturbation (‘slaving’) technique in resonant cases where Nekhoroshev’s theorem does not apply. We present evidence that the smaller Lyapunov exponents of the system scale exponentially as well. The results suggest that the observed stability of nearly-slow motion is a consequence of the approximate adiabatic invariance of the fast motion. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 05.45.−a; 05.45.Pq; 45.20.Jj

Keywords: Balance dynamics; Adiabatic invariants; Nekhoroshev theorem

1. Introduction

The governing equations of dynamical meteorology, and approximate models such as the meteorological ‘primitive equations’ (widely used in numerical weather prediction and climate modeling), support both slow nonlinear vortical motions and fast inertia-gravity-wave oscillations. Although on statistical mechanical grounds one might expect the fast oscillations to be strongly excited [1], their amplitude in the real atmosphere is remarkably weak [2]. This has led to the central problem of balance dynamics: why is the observed motion so nearly devoid of fast oscillations?

* Corresponding author. Tel.: +1-416-978-6824; fax: +1-416-978-8905.
E-mail address: tgs@atmosp.physics.utoronto.ca (T. G. Shepherd)

1 Present address: Department of Mathematics and Statistics, University of Edinburgh, Edinburgh EH9 3JZ, UK.

0167-2789/00/$ – see front matter © 2000 Elsevier Science B.V. All rights reserved.
PII: S0167-2789(00)00022-1
Over the years, balance dynamics has been linked with the existence of the so-called slow manifold — a hypothetical manifold in phase space on which fast oscillations are absent [3,4]. Various perturbative methods have been developed to approximate this manifold [5,6], and it was soon discovered that these ‘initialization’ techniques do remove fast oscillations from the system, at least for a finite time, leading to their widespread use in operational weather forecast models [2].

It appears to be a settled question that an exact invariant slow manifold does not exist in general [7], and that the initialization techniques are at best asymptotic. This makes the near-balance observed in nature (and in approximate models) only more puzzling. In any case, previous studies have for the most part focused on the question of the existence of the slow manifold, and not on the stability of (nearly) slow motion. Clearly the latter question is the one that is required in order to address the central question of balance dynamics.

The purpose of this study is to address this question of why (nearly) slow motion is so stable in the atmosphere. In particular, we are interested in the restricted question of why it is that flows can stay nearly balanced over time scales much longer than the characteristic nonlinear interaction time scales. For this it is sufficient to focus on the nonlinear interactions, and avoid the issue of forcing and dissipation (which for large-scale atmospheric flow may be considered weak). Our hypothesis is that the stability of slow motion results from approximate adiabatic invariance of the fast motion, and is thus essentially linked to the Hamiltonian structure that characterizes ideal geophysical fluid dynamics (e.g., [8,9]).

It should be noted that there do exist stability estimates for more general dynamical systems which do not make use of the Hamiltonian structure. To be valid for long times, however, these results (e.g., [10]) rely on all the fast modes being dissipative, so they are not applicable here. Without dissipation, the best general estimate that one can get is asymptotic accuracy for a time scale comparable to the nonlinear interaction time scale (see discussion in Section 3 and the appendix).

In order to focus on the essential issues, most previous work on balance dynamics has been in the context of simple low-dimensional models, where results from dynamical systems often lead to new insights and understanding. One of the most well known of such models is a five-mode model introduced by Lorenz in 1986 [11]. The model consists of a slowly oscillating vorticity triad coupled to an inertia-gravity wave (having two components). It is thus an absolutely minimal model for the study of balance dynamics. Lorenz’s 1986 model is Hamiltonian [12,13] and can be reduced to canonical form. This feature, in some sense accidental, turns out to be crucial to its properties.

Using a Melnikov technique specially developed for the problem, Camassa [12] proved the existence of chaotic dynamics arising from the fast–slow coupling, thus ruling out the existence of an exactly invariant slow manifold and, consequently, of exact balance dynamics. However, as pointed out by Bokhove and Shepherd [13], due to the near-integrability of the model and its low-dimensionality, solutions are trapped in the gaps between KAM surfaces for all time. The widths of these gaps decrease exponentially as a function of the time scale separation, leading to an exponential increase in the accuracy, for all time, of balance dynamics in the model.

In Lorenz’s 1986 model, the most important factor contributing to the balance behaviour is the near-integrability of the Hamiltonian system. Indeed, Lorenz [11] devised an algorithm for finding balanced solutions based on temporal periodicity, which has nothing to do with a fast–slow time scale separation. In the near-integrable case, the frequency spectrum of the unperturbed slow dynamics consists of a countable set of harmonics which, in general, does not overlap with the fast spectrum — whether or not there is a time scale separation. It is this fact that allows a separation of the fast and slow degrees of freedom (on the KAM surfaces).

However, the near-integrability of Lorenz’s 1986 model is a highly unphysical feature. The generic behaviour of atmospheric models is chaotic slow dynamics coupled to linear fast dynamics [14]. In such a case, the frequency spectrum of the unperturbed slow dynamics is full, and must overlap with the fast spectrum. It follows that the mechanism for balance dynamics in Lorenz’s model is irrelevant for more realistic models. Moreover, it masks what is thought to be the true mechanism for balance: the time scale separation between the vortical and gravity-wave modes [5–7].
Therefore, in order to address the question of the stability of balance dynamics, it is necessary to consider a more complex model. At a minimum, the unperturbed slow dynamics must be chaotic. Furthermore, it is interesting to allow the possibility of resonance between fast modes, because this allows a mechanism of fast–slow coupling through fast–fast–slow triad interactions which is suppressed in Lorenz’s 1986 model. On the other hand, our hypothesis is that the special properties of fluid dynamics that arise from its Hamiltonian structure are important in the maintenance of balance dynamics, and we therefore seek a Hamiltonian model. (Note that the stratified compressible Euler equations are Hamiltonian, as are the meteorological primitive equations.)

We seek the simplest possible model that contains all these ingredients. However, this presents us with a dilemma. The Lorenz 1986 model is obtained by a truncation of the shallow-water equations (SWE) — widely considered as the prototype PDE to study balance dynamics — so an obvious way to make the slow dynamics chaotic would be to include more vorticity triads. Unfortunately, there are no known consistent spectral truncations of the SWE that include more than one vorticity triad and preserve the Hamiltonian structure of the parent model; in this respect, too, the Lorenz model is very special. (It is a curious fact that while the SWE are Hamiltonian and the Lorenz 1986 model is Hamiltonian, the latter is derived from the former via a nine-component truncation that appears not to be Hamiltonian!)

Thus, in order to address the hypothesis of this study, we must construct a model that is somewhat ‘ad hoc’ but which nevertheless contains the essential physical ingredients in question. In order to make the unperturbed slow dynamics chaotic, we add a prescribed time dependence to the canonical form of the Lorenz model. The resulting system is a $2^1/2$-d.o.f. Hamiltonian system which is equivalent to a forced pendulum coupled to a linear oscillator. The chaotic pendulum can be thought of as representing the random effect of other vorticity triads on a given vorticity triad; the linear oscillator, as before, represents a fast gravity wave. The second ingredient we add to the model is another fast gravity wave. This enables us to study the effect of fast–fast–slow interactions, which provide a means for energy transfer between the fast and slow motions, and thus for the rapid destruction of balance dynamics. In this paper we report results of numerical experiments on this ‘extended Lorenz model’ (2.4) and connect them to the available analytical predictions.

An important tool in the previous investigation of the Lorenz model [13] is the Poincaré section. This is not practical for our larger model, so we shall use the fact that the analogue of the invariant tori of the Lorenz model is the level surfaces of some adiabatic invariants $I_i$ and we compute (the variation of) these quantities in place of the geometric picture. When the frequencies of the gravity waves are not resonant, this can be done using the canonical averaging procedure (cf. e.g., [15]).

As we shall see below, it takes a trajectory an exponentially long time to traverse these $I_i$ surfaces, providing an explanation to the high degree of balance observed. This is in accordance with Neishtadt’s [16] and Nekhoroshev’s [17,18] theorems, which state that $dI_i/dt$ is bounded by a quantity that is exponentially small in the time scale separation parameter $\varepsilon$; our numerical study here shows that this scaling holds for physically realistic values of $\varepsilon$, which are orders of magnitude larger than what is required in the hypothesis of the analytical results. We shall also establish the equivalence of the slaving procedure [see below] with finding the $\tilde{I}_i = 0$ surface in the non-resonant case. As expected, the drift of a solution from the slaving manifold is again exponentially slow, but, somewhat surprisingly, this is also true for resonant frequencies. This points to a potential extension of the averaging-type results to resonant cases, for certain special (i.e., nearly balanced) initial conditions. To obtain a better picture of the behaviour of the system, we compute the Lyapunov exponents of the system, and an interesting behaviour is observed: the largest exponent, corresponding to the slow chaotic dynamics, is independent of the time scale separation parameter $\varepsilon$, but the smaller exponents appear to depend sensitively on $\varepsilon$. This confirms the fact that in Hamiltonian systems, just as an exact invariant implies a pair of zero exponents, each independent adiabatic invariant implies a pair of exponentially small Lyapunov exponents.
The rest of this paper is arranged as follows. In Section 2 we introduce the model; a quick overview of the averaging and slaving procedures is given in Section 3. The numerical results obtained using these procedures are presented in Sections 4 and 5. The behaviour of the Lyapunov exponents in the system is discussed in Section 6. A summary and discussion section concludes the paper.

2. The extended Lorenz model

We begin with Lorenz’s 1986 [11] model, with the scaling of [13]:

\[
\begin{align*}
\frac{du}{dt} & = -uv + b_1u z_1, & \frac{dv}{dt} & = uw - b_1u z_1, & \frac{dw}{dt} & = -uv, \\
\frac{dx_1}{dt} & = -\frac{1}{\varepsilon}z_1, & \frac{dz_1}{dt} & = \frac{1}{\varepsilon}x_1 + b_1uv.
\end{align*}
\]

(2.1)

The system is derived by spectral truncation of the shallow-water equations; \(u, v, w\) corresponds to a vorticity triad, and \((x_1, z_1)\) to an inertia-gravity wave. The time scale separation between the two kinds of motion is \(\varepsilon = \text{Ro} b_1/(1 + b_1^2)^{1/2}\), where \(\text{Ro}\) is the Rossby number and \(b_1\) the rotational Froude number at the wavelength of the gravity wave. We are interested in the case \(\varepsilon \ll 1\), for which the gravity-wave dynamics is “fast” and the vortical dynamics is “slow”. The time evolution is scaled on the slow time scale, since that is the time scale of physical interest for balance dynamics (i.e., we want to know the behaviour of the system on the slow time scale in the limit \(\varepsilon \to 0\)).

Noticing that \(C = u^2 + v^2\) is a constant (corresponding to the enstrophy), the system can be reduced to four variables by writing \(u =: \sqrt{C} \cos \varphi, v =: \sqrt{C} \sin \varphi\), and then defining \(\varphi := \varphi' - \varepsilon b_1x_1\), to give

\[
\begin{align*}
\frac{d\varphi}{dr} & = w, & \frac{dw}{dr} & = -\frac{C}{2} \sin(2\varphi + 2\varepsilon b_1x_1), \\
\frac{dx_1}{dr} & = -\frac{1}{\varepsilon}z_1, & \frac{dz_1}{dr} & = \frac{1}{\varepsilon}x_1 + \frac{b_1 C}{2} \sin(2\varphi + 2\varepsilon b_1x_1).
\end{align*}
\]

(2.2)

If we now define the action–angle variables \((\theta_1, I_1)\) by \(z_1 =: \sqrt{2I_1} \sin \theta_1, x_1 =: \sqrt{2I_1} \cos \theta_1\), the system (2.2) can be written in the Hamiltonian form

\[
H(\varphi, w, \theta_1, I_1) = \frac{1}{2}[C \sin^2(\varphi + \varepsilon \sqrt{2I_1} b_1 \cos \theta_1) + w^2] + I_1,
\]

\[\{\varphi, w\} = -\{w, \varphi\} = 1, \quad \{\theta_1, I_1\} = -\{I_1, \theta_1\} = \varepsilon^{-1},\]

(2.3)

with the rest of the Poisson brackets being zero.

As mentioned in the introduction, this system is near-integrable, although chaotic motion has been proved to exist between the invariant KAM tori. It is therefore not a very good model to study the balance behaviour of the atmosphere, so we shall modify it by making the Casimir \(C\) time-dependent: \(C(t)\). For this study, we shall take \(C(t) = k_0 + k_1 \cos \gamma t\), with \(k_0 = 1, k_1 = 0.8\) and \(\gamma = 0.93\). The constants have been chosen to give a broad, almost uniformly chaotic region with few large islands in the \(\varphi-w\) plane when \(b_1 = 0\). For \(b_1 \neq 0\), this picture is undoubtedly modified, but the near-constancy of the largest Lyapunov exponent found below [Section 6] suggests that the general features persist in this case. With this we attempt to simulate the presence of several vortical modes (whose behaviour is chaotic in general), but without including the complex resonant interactions that may exist among them (and between them and the gravity waves). This is the system we shall study in Section 4.
To study the basic effects of fast–fast–slow interactions [Section 5], we shall attach another gravity-wave mode \((\theta_2, I_2)\) with frequency \(\Omega/\varepsilon\) in the same manner as the first one. (The possibility that the results might depend on the precise nature of this coupling is discussed in Section 5.) Thus we arrive at our most general system

\[
H(\varphi, w, \theta, I) = \frac{1}{2} \left[ C(t) \sin^2 \left( \varphi + \varepsilon \sum_i \sqrt{2} b_i \cos \theta \right) + w^2 \right] I_1 + \Omega I_2,
\]

\[
\{\varphi, w\} = \{-w, \varphi\} = 1, \quad \{\theta_i, I_i\} = -\{I_i, \theta_i\} = \varepsilon^{-1},
\]

(2.4)

where \(i = 1, 2\) and all the other brackets are zero. For \(\varepsilon \ll 1\), the system corresponds to a “slow” oscillator \((\varphi, w)\) with frequency of \(O(1)\), coupled to two “fast” oscillators \((\theta_1, I_1)\) and \((\theta_2, I_2)\) with frequencies \(\varepsilon^{-1}\) and \(\Omega \varepsilon^{-1}\). Here \(b_i\) is the rotational Froude number at the wavelength of each gravity wave. For scaling consistency, we need to have \(\Omega^2 = (1 + b_2^{-2})/(1 + b_1^{-2})\). With no loss of generality, we shall take \(\Omega \geq 1\).

### 3. Averaging and slaving

In both the limits \(b_1 \to 0\) and \(\varepsilon \to 0\), the fast actions \(I_i\) become invariants of motion. For finite but sufficiently small values of either parameter, these constants of motion are replaced by adiabatic invariants — quantities which vary only slowly in time. These adiabatic invariants can be computed by performing a change of variables \(u \mapsto \tilde{u}\) such that the coupling term in the Hamiltonian (2.4), which initially appears at \(O(\varepsilon)\) or \(O(b_i)\), is pushed to higher orders.

With \(\varepsilon\) as the small parameter, this can be done easily using Lie series, which is essentially a perturbative way to compute the required change of variable. This singular limit is in some sense easier to handle as we do not need to solve for the behaviour of the slow variables, which is chaotic in general. There are several variants of the method [15,19,20], although they must obviously give the same answer; here we shall follow the approach of [21] with minor modifications to account for the \(\varepsilon^{-1}\) factor in our Poisson bracket (see [22] for more details).

Introducing the notations \(C = C(t)\), \(C'(t) := dC/dt\), and writing the gravity waves in terms of \((x_i, z_i)\) being defined as with \((x_1, z_1)\), the adiabatic invariants are given to \(O(\varepsilon^2)\) by

\[
\tilde{I}_1 = I_1 + \varepsilon C b_1 x_1 \sin 2\varphi + \varepsilon^2 \left[ \frac{C^2 b_1^2}{16} (1 - \cos 2\varphi) - \frac{C'}{2} b_1 z_1 \sin 2\varphi \right.
\]
\[
+ C \cos 2\varphi \left( \frac{b_1^2}{4}(x_1^2 - z_1^2) - b_1 w z_1 + b_1 b_2 \frac{x_1 x_2 + \Omega z_1 z_2}{1 - \Omega^2} \right),
\]
\[
\tilde{I}_2 = I_2 + \varepsilon C 2\Omega b_2 x_2 \sin 2\varphi + \varepsilon^2 \left[ \frac{C^2 b_2^2}{16\Omega^2} (1 - \cos 4\varphi) - \frac{C'}{2\Omega^2} b_2 z_2 \sin 2\varphi \right.
\]
\[
+ C \cos 2\varphi \left( \frac{b_2^2}{4\Omega^2}(x_2^2 - z_2^2) - b_2 \Omega^{-2} w z_2 - b_1 b_2 \frac{z_1 z_2 + \Omega x_1 x_2}{1 - \Omega^2} \right). \quad (3.1)
\]

For the one-frequency case [Section 4], one can simply take \(b_2 = 0\) and consider \(\tilde{I}_1\). The resonant denominator \(1 - \Omega^2\), which arises from the product \(x_1 x_2\) at \(O(\varepsilon^2)\) of the Taylor expansion of \(H\), causes the approximation to break down at second order when \(\Omega = 1\). At higher orders, other resonant denominators exclude all rational values of \(\Omega\), so this expansion is sensitive to the resonant properties of the frequency. It is well known that this effect is real and physical [15,20], at least for arbitrary values of \(I_i\). In the rest of this paper, we will often use the term (fast) actions to mean the adiabatic invariants \(\tilde{I}_i\) (computed to some finite order).
This change-of-variable technique is the basis for most proofs of the Kolmogorov–Arnol’d–Moser and Nekhoro-
shchev theorems (see [15,23] for instructive proofs). In the first case, it is shown that a convergent approximation
can be obtained for a given invariant torus provided that certain conditions are met. In the second, it is shown that
one can carry out the perturbation theory to order $n \sim 1/\varepsilon$ under less restrictive conditions, which results in the
exponential accuracy of the adiabatic invariants.

More precisely, Nekhoroshev’s theorem gives us the following estimate [15]. For non-resonant frequencies $\{\Omega_i\}$,
the Hamiltonian system

$$H(\theta, I, p, q) = \sum_i \Omega_i I_i + \varepsilon H_1(I, \theta, p, q; \varepsilon)$$

(3.2)

with a canonical Poisson bracket $\{q_i, p_i\} = \{\theta_i, I_i\} = 1$, admits a change of variable $I \mapsto \tilde{I}$, etc., such that the
time variation of $\tilde{I}$ is exponentially small,

$$\left| \frac{d\tilde{I}_i}{dt} \right| \leq C_i \exp \left( -\frac{\kappa_1}{\varepsilon^d} \right)$$

(3.3)

for $0 \leq t \leq T$ and $\varepsilon \leq \varepsilon_0$. Here $T$, $\varepsilon_0$, $\kappa_1$, $C_i$ and $d$ are constants, with $T$ becoming longer (typically exponentially)
as $\varepsilon \to 0$. Our system (2.4) has a non-canonical Poisson bracket, but it is easily checked that this result still holds
in that case.

This version of the result differs from what is perhaps the more common statement of the theorem in two respects:
First, instead of steepness (or convexity, cf. [15]) of the leading-order Hamiltonian, here we have terms that are
linear in the actions $I_i$; the proof of the theorem is in fact simpler in this case than in the steep case. Second, there are
additional variables $(p, q)$ in the perturbation Hamiltonian $H_1$; this fact, which was pointed out in Nekhoroshev’s
original paper [18], means that unlike the KAM theorem this result applies to systems that are not near-integrable.

An essentially unrelated result which applies to the one-frequency case, $b_2 = 0$, has been obtained by Neishtadt [16]. For our purposes here, the theorem states that when $b_2 = 0$, a bound of the form (3.3) with $d = 1$
Applies to $\tilde{I}_1$ in the system (2.3), giving an improvement over Nekhoroshev’s theorem for $\varepsilon \to 0$ in that it gives
a larger $d$ in (3.3): a straightforward (but possibly non-optimal) re-scaling of (2.3) to conform with the form
(3.2) makes $\sqrt{\varepsilon}$ the perturbation parameter, and a direct application of Nekhoroshev’s theorem then gives $d = \frac{1}{4}$
in (3.3).

In the resonant case, $\Omega = p/q$, this latter result can be used to explain another well-known phenomenon: the
stability of the resonance. Here one uses partial averaging [24] to compute an adiabatic invariant $\tilde{I}_{pa}$ (in our system
this will be the only exponentially slow invariant) and apply Neishtadt’s result to this quantity. As far as our purpose
of studying long-term balance dynamics is concerned, however, direct application of this procedure does not yield
a substantial improvement. Taking $\Omega = \frac{1}{3}$ as a concrete example, following the procedure in [24] we find that
the leading-order slow action is $I_{pa} = 2I_1 + 3I_2$, giving rise to the exponential invariant $\tilde{I}_{pa}$. We then find that
$\tilde{I}_{pa} = 2\tilde{I}_1 + 3\tilde{I}_2 + O(\varepsilon)$, which is not useful to bound the individual variations of $\tilde{I}_1$ and $\tilde{I}_2$, since standard averaging
tells us that the latter two quantities are $O(\varepsilon^{1/2})$-invariant.

It should be noted that the above slowness estimates on the drift are valid for asymptotically much longer times than
the more well-known classical estimates — generally known as the Bogolyubov–Krylov–Mitropol’skii averaging
procedure — which do not make use of the Hamiltonian structure [25,26]. Using these estimates, one typically
obtains asymptotic accuracy over time scales of order one, viz., if $u(t)$ denotes the original variables and $\tilde{u}(t)$
denotes their averaged counterpart, the method gives

$$\frac{d\tilde{u}}{dt} = W(\tilde{u}; \varepsilon),$$

(3.4)
where

\[ |u(t) - R(\tilde{u}(t), t; \varepsilon)| \leq \varepsilon^n C_1 \]  

(3.5)

for \( t \leq C_2 \); here \( W \) and \( R \) are determined from the original equations of motion, and \( C_1 \) and \( C_2 \) are constants. Under certain conditions this can be improved to give asymptotic accuracy for slightly longer time scales, \( t \leq C_3 \log \varepsilon \).

Apart from the difference in time scales, this is a qualitatively different result from the Hamiltonian case: It gives an accuracy estimate for all the variables, as opposed to just the fast action in our canonical averaging, and the existence of an action-like quantity is not assumed — all the averaged quantities \( \tilde{u} \) may have \( O(1) \) variations over the slow time scale.

When there is dissipation in the system, the accuracy time scale may be extended for all time. However, this typically requires the damping of all the fast variables, or in other words, that the slow manifold be contained in an attracting centre manifold. For our physical problem, this case is not (directly) relevant, as discussed earlier.

The estimate (3.3) also implies the stability of the slow solution obtained using the slaving method [14], where the fast degrees of freedom are completely eliminated. This is done by writing the equations of motion in the \( x_i \) and \( z_i \) variables above, and making an ansatz that these fast variables, denoted collectively by \( f \), are slaved to the slow variables \( s := (\varphi, w) \):

\[ f = U(s; \varepsilon). \]  

(3.6)

Substituting this relation into the equation of motion and expanding \( U(s; \varepsilon) \) in \( \varepsilon \), we can solve order-by-order for a modified equation of motion in \( s \) alone and the slaving relation \( U \) (see the appendix for more details). To \( O(\varepsilon^2) \), the slaving relations for the system (2.4) are given by

\[
\begin{align*}
U_{z1} &= \varepsilon^2 [Cb_1 w \cos 2\varphi + \frac{1}{2} C' b_1 \sin 2\varphi] + O(\varepsilon^3), \\
U_{x1} &= -\frac{1}{2} \varepsilon Cb_1 \sin 2\varphi + O(\varepsilon^3), \\
U_{z2} &= \varepsilon^2 [C \Omega^{-2} b_2 w \cos 2\varphi + \frac{1}{2} C' \Omega^{-1} b_2 \sin 2\varphi] + O(\varepsilon^3), \\
U_{x2} &= -\frac{1}{2} \varepsilon C \Omega^{-1} b_2 \sin 2\varphi + O(\varepsilon^3). \end{align*}
\]  

(3.7)

As with any asymptotic method that does not rely on the existence of an attractor or Hamiltonian structure, the best estimate one can obtain for the stability of the slaving solution is an asymptotically accurate estimate for time scales of order unity (or of order \( \log \varepsilon \), which is practically \( O(1) \) for physical applications). The demonstration, included in the appendix, is essentially the same as that for the accuracy of BKM-averaged solutions.

With some algebra, it can be verified that these slaving relations agree with the surface \( \tilde{I}_i = 0 \) given by (3.1) to the order shown in the non-resonant cases. If both methods are asymptotically correct, this will be true for all orders, since the functional forms of the adiabatic invariants and of the slaving solutions are independent of \( \varepsilon \) and \( \Omega \). We shall assume without proof that this is true for the slaving solution (that the adiabatic invariants computed here are asymptotically correct follows from Nekhoroshev’s theorem), and conclude that the two methods yield the same solution for the \( \tilde{I}_i = 0 \) surface, to the order of computation, in the absence of resonances.

For resonant frequencies, the slaving solution is then identical to that obtained using averaging up to the order \( n \) at which the latter breaks down. In this case, the error for both solutions remains small for time scales \( T \sim O(\varepsilon^{-n}) \). The authors are not aware of any result that guarantees accuracy beyond this time scale (except for the stability of the resonance itself).

4. Numerical results: one gravity wave

We integrate the model (2.3) numerically and investigate the time evolution of the adiabatic invariants and of the slaving solutions. A fourth-order symplectic integrator [27] is used because of its stability and efficiency (although
Fig. 1. (a) Top panel: Time evolution of $I_1$ computed to $O(1)$ [bottom], $O(\varepsilon)$, $O(\varepsilon^2)$, and $O(\varepsilon^3)$ [top]. The parameters were $b_1 = 0.71$, $b_2 = 0$, and $\varepsilon^{-1} = 8.0$. The initial conditions were $\varphi = 0.23$, $w = 0.54$, $I_1 = 0.1$. Except for the bottom curve, the curves have been shifted up to improve visibility. (b) Bottom panel: Blow-up of the $O(\varepsilon^2)$ (shifted down by 0.005) and $O(\varepsilon^3)$ approximations in (a).

our results obviously do not depend on the integration scheme). In this section the system is restricted to one gravity wave ($b_2 = 0$); the adiabatic invariants are computed to $O(\varepsilon^3)$ while the slaving solutions are computed to $O(\varepsilon^4)$.

As can be seen in Fig. 1a, there is a considerable amount of ‘noise’, consisting of oscillations at about the fast frequency, in $I_1^{(0)}$ (the bracketed superscript denotes the order to which the quantity is computed; if none is specified, $O(\varepsilon^3)$ in this section and $O(\varepsilon^2)$ in the next is implied). The amplitude of these oscillations decreases as higher-order approximations are used, presumably until an asymptotic limit (which depends on $\varepsilon$) is reached. Superimposed on these fast oscillations, however, there is a slower secular drift that cannot be removed by the small corrections of the higher-order approximations [Fig. 1b].

The drift over long time scales is better seen in Fig. 2, which shows time series of $I_1$, computed to $O(\varepsilon^3)$, for three different values of $\varepsilon$. The three curves appear to be qualitatively very similar in the time scale of their variations. The time axes of the three curves, however, have been scaled by $\exp(2.0)$, $\exp(0.0)$, and $\exp(-2.0)$ (the origin of the numerical factor will become clear shortly), so the actual time scale of the top curve is $\exp(-2.0) \times 10^5$, and that of the bottom one is $\exp(2.0) \times 10^5$. In fact, the variation in $I_1$ reaches $O(1)$ over a time scale ten times longer than what is shown in Fig. 2. Therefore, as far as our purpose of studying the slow drift is concerned, we only need to compute $I_1$ to the order at which this slow drift is sufficiently resolved; this is satisfied for all the numerical results reported here.

In Fig. 3, we show the variations $\Delta I_1^{(n)}$ over a fixed length of time for different $n$ and $\varepsilon$. The ‘envelope’ seen for $\varepsilon > 0.1$ is evidence of the asymptotic nature of the perturbation theory: for $\varepsilon > 0.1$, higher-order approximations will not contribute to any further reduction of the noise, while for smaller $\varepsilon$, better approximations to $I_1$ are possible.
Fig. 2. Time series of $\tilde{I}_1$ computed to $O(\varepsilon^3)$. The parameters were $b_1 = 0.71$, $b_2 = 0$, and $\varepsilon^{-1} = 8.0$ (top curve), 9.0 (middle), 10.0 (bottom). The time axis has been scaled by $\exp(2)$, 1, and $\exp(-2)$, respectively, for the three curves [see text]. The initial conditions were $\varphi = 0.23$, $w = 0.55$, $I_1 = 0.1$ in all cases; the top and bottom curves have been shifted for visual clarity.

Fig. 3. The variation in action $\Delta I^0_n$ over $0 \leq t \leq 500$ as a function of $\varepsilon$ and order of approximation $n$ [top curve: $O(1)$, bottom curve: $O(\varepsilon^3)$] in the $b_2 = 0$ case. The initial conditions were $\varphi = 0.12$, $w = 0.88$, $I_1 = 0.2$.

To better quantify the drift rate, we compute the variance $\langle (\Delta \tilde{I}_1(t))^2 \rangle$, where $\Delta \tilde{I}_1(t) := \tilde{I}_1(t) - \tilde{I}_1(0)$, over an ensemble of $N_{\text{ens}}$ trajectories with nearby initial conditions. As shown in Fig. 4, when we take $N_{\text{ens}}$ large enough, we find that $\langle (\Delta \tilde{I}_1(t))^2 \rangle$ grows linearly for some time. This behaviour is consistent with what one would expect if $\tilde{I}_1(t)$ undergoes a random walk and the evolution of its probability distribution function can be modelled as a diffusive process.

Fig. 4. The growth of $\langle (\Delta \tilde{I}_1)^2 \rangle$, computed using 400 realizations with nearby initial conditions. The parameters and initial conditions are: $\varepsilon = 0.125$, $b_1 = 0.41$, $b_2 = 0$, $\varphi = 0.23$, $w = 0.54$, $I_1 = 0.1$. 
process using a Fokker–Planck equation (cf. Section 5.4 in [20]). The straightness of the line improves when a larger ensemble is taken, and degrades for smaller \( N_{\text{ens}} \); due to the computational costs, here we use the smallest \( N_{\text{ens}} \) that allows the growth rate to be determined. This linear growth ceases when factors such as the finite size of accessible phase space become important, but as Fig. 2 makes clear, the sensitive dependence on \( \epsilon \) continues to hold over much longer time scales.

Estimating \( d(h_1 Q_1)/dt \) for \( 0 < t < 10^3 \) and plotting it as a function of \( \epsilon \), we obtain the data points connected by the solid curve in Fig. 5. Over the range of \( \epsilon \) shown, especially for \( \epsilon < 7 \), the scaling of the drift rate is approximated very well by \( \exp(-2.0/\epsilon) \). As mentioned above, higher-order approximations to \( \tilde{I}_1 \) would be needed to determine the drift rate for smaller \( \epsilon \), however, the trend of the figure [see also the discussion below] strongly suggests that the scaling holds as \( \epsilon \to 0 \). This \( \epsilon \)-dependence of the drift rate implies that individual trajectories with different \( \epsilon \) should look qualitatively alike provided that the time axis is scaled by \( \exp(-2.0/\epsilon) \); as alluded to earlier, this is confirmed in Fig. 2. More importantly, it suggests that the excitation of the fast modes occurs over exponentially long time scales as \( \epsilon \to 0 \), and thus that balance is exponentially accurate (for finite times).

We also find that the drift rate decreases when we start with smaller \( \tilde{I}_1 \), scaling roughly linearly as \( \tilde{I}_1 \) ranges from 0.3 to 0.01 (here the drift rate is computed before \( \tilde{I}_1 \) can vary much), although presumably there is a lower bound on the drift rate that depends on the parameters (and not on the initial conditions). This lower bound, which corresponds to that of the slaving solution, implies that the “slowest invariant manifold” (SIM) of Lorenz [11] is unstable in this model, since trajectories starting on our slaving manifold will eventually drift away from it, with the drift rate increasing as the distance from the manifold grows.

We measure this distance using the norm

\[
\text{Imb}_1 := (x_1 - U_{x_1})^2 + (z_1 - U_{z_1})^2
\]

(\note{\text{Imb}_1 \simeq 2\tilde{I}_1 for small values of \( \tilde{I}_1 \).} As with \( \langle \Delta \tilde{I}_1^2 \rangle \), we find that \( \langle \text{Imb}_1 \rangle \) grows linearly in time when slaving solutions are used as initial conditions. This growth rate is plotted as the solid curve in Fig. 6. Again, we find that the drift rate depends sensitively on \( \epsilon \) and is well approximated by \( \exp(-2.2/\epsilon) \), although the uncertainties here are somewhat larger than those in the case of the adiabatic invariants.

To our knowledge, there is no analytical result predicting this dependence of the drift rate on \( \tilde{I}_1 \) that applies to our system, although for near-integrable systems such a result has been proved [28].
Fig. 6. The drift rates \( \frac{dK}{dt} \) for \( 0 < t < 10^3 \), plotted as a function of \( \varepsilon \). Solid line: \( K = 1b_1 \) in the one-frequency case [Section 4]. Dashed lines: the \( \Omega = \sqrt{2} \) (two-frequency) case [Section 5], with \( K = 1b_1 \) (top curve), and the corresponding \( K = 1b_2 \) (bottom curve). The remaining parameters are as in Fig. 7.

5. Two gravity waves

The situation is more complicated with two fast modes, Eq. (2.4) with \( b_2 \neq 0 \). First, we note that since \( \Omega > 1 \), we would expect \( \tilde{I}_2 \) to vary more slowly than \( \tilde{I}_1 \) (the time scale separation is greater). This is indeed what we observe numerically, although in some cases [see below] we find that \( \Delta \tilde{I}_2 \simeq -\Delta \tilde{I}_1 \). In the limit of large \( \Omega \) (in practice we only need to take \( \Omega \geq 2 \) for \( \varepsilon \leq 0.1 \)), the drift rate of \( \tilde{I}_2 \) is orders of magnitude smaller than that of \( \tilde{I}_1 \), which in turn is comparable to that in the one-frequency case, as expected.

A ‘typical’ situation is shown in Fig. 7. [In this section, we use \( \mathcal{O}(\varepsilon^2) \) approximations to \( \tilde{I}_i \).] Here it is seen that sometimes the time variations of \( \tilde{I}_1 \) and \( \tilde{I}_2 \) are negatively correlated, with their average \( \frac{1}{2}(\tilde{I}_1 + \tilde{I}_2) \) having a much weaker variation. This ‘exchange of action’ behaviour — which we should stress does not always happen — is most often seen for sufficiently small values of \( \Omega \) and \( \varepsilon \). In a few cases, notably for large \( \Omega \) and small \( \varepsilon \), we do find that a linear combination \( (\tilde{I}_1 + \alpha' \tilde{I}_2) \), with \( \alpha' > 1 \), varies much more slowly than either fast action.

To obtain a more quantitative measure of the drift, we again turn to ensemble averages, computing \( \langle \Delta \tilde{I}_1^2 \rangle \) and the correlation \( \langle \Delta \tilde{I}_1 \Delta \tilde{I}_2 \rangle \). As before, we find that the variance \( \langle \Delta \tilde{I}_1^2 \rangle \) grows linearly in time. The dependence of \( \frac{d\langle \Delta \tilde{I}_1^2 \rangle}{dt} \) on \( \varepsilon \) is plotted as the top dashed curve in Fig. 5. It is clear that for smaller \( \varepsilon \) the drift rate

Fig. 7. Time series of (from the top) \( \tilde{I}_1 \), \( \frac{1}{2}(\tilde{I}_1 + \tilde{I}_2) \), and \( \tilde{I}_2 \). The parameters and initial conditions were: \( \varepsilon^{-1} = 7.5, b_1 = 0.41, \Omega = \sqrt{2}; \varphi = 0.23, w = 0.55, I_1 = 0.1, I_2 = 0.0618.\)
also drops rapidly, although not as steeply as in the one-frequency case. However, as \( \varepsilon \to 0 \), the drift rate becomes less sensitive to \( \varepsilon \) as the variations of the actions take on opposite signs, \( \Delta I_2 \simeq -\Delta I_1 \), more and more often.

We introduce the following quantities. Let \( \alpha^2 := \langle \Delta I_1^2 \rangle / \langle \Delta I_2^2 \rangle \), and let \( J := (\Delta I_1 + \alpha \Delta I_2) / \sqrt{2} \). If the fast actions are not correlated, \( \langle \Delta I_1 \Delta I_2 \rangle = 0 \), then the drift rate of \( J \) will simply be equal to the drift rate of \( \bar{I}_1 \), so (the inverse of) the ratio of the two drift rates is a measure of how well the actions are negatively correlated. Obviously this definition does not cover the cases where the actions become correlated after drifting independently for some time, nor where the proportionality constant \( \alpha \) above varies within the ensemble, but we still find it a useful measure even if it underestimates the coupling between the two fast actions.

In the numerical experiments, we indeed find that \( \langle \Delta I_1 \Delta I_2 \rangle \leq 0 \) to within statistical fluctuations. As \( \Omega \) grows or as \( \varepsilon \) decreases, the inequality becomes a strict one and \( \langle \Delta I_1 \Delta I_2 \rangle \) becomes correlated with \( \langle \Delta I_1^2 \rangle \). We plot \( \text{d}(J^2)/\text{d}t \) as the bottom dashed curve of Fig. 5. It is seen that the drift rate of \( J \) is slightly but consistently smaller than that of \( \bar{I}_1 \) in the corresponding one-frequency case; this is true for all values of \( \Omega \) we have used, although as mentioned above, the difference becomes small quickly as \( \Omega \) increases. We also find that the drift rate of \( J \) is smaller than that of \( \bar{I}_1 \) (top dashed curve), by a small factor for larger \( \varepsilon \), but by a large factor for smaller \( \varepsilon \).

Note that the functional form \( \Delta I_1 + \alpha \Delta I_2 \) does not lead (at least in a straightforward way) to the conclusion that \( J \) is the partially-averaged action discussed in Section 3, since the latter has an \( \mathcal{O}(\varepsilon) \) cross-term that can vary on a time scale of order unity.

The value of \( \alpha \) varies from 3.6 (for \( \varepsilon^{-1} = 4.5 \)), through 1.2 (\( \varepsilon^{-1} = 7.5 \)), to 1.0 (\( \varepsilon^{-1} = 10.5 \)). It should be clear that this number is to be taken as (minus) the ratio \( \Delta I_1 / \Delta I_2 \) only when \( J \) drifts much more slowly than \( \bar{I}_1 \), i.e., when the actions are well correlated. In this case, we find that invariably \( \alpha \) is very close to 1.

We find that the resonance property of \( \Omega \) has relatively weak effects in the study reported here. When the effect of the magnitude of \( \Omega \) is factored out, there is little difference between the drift rates of resonant frequencies such as \( \Omega = \frac{3}{2} \) or \( \frac{4}{3} \), say, and non-resonant ones such as \( \sqrt{2} \) or \( \frac{1}{5} (1 + \sqrt{5}) \). This may be caused by the high order at which the frequencies become resonant — \( \frac{3}{2} \) is resonant at \( \mathcal{O}(\varepsilon^4) \) and \( \frac{4}{3} \) at \( \mathcal{O}(\varepsilon^7) \). The only exception seems to be \( \Omega = 2 \), which is resonant at \( \mathcal{O}(\varepsilon^3) \); in this case we find that the ratio \( \alpha \) has a sharp peak at \( \Omega = 2 \) and the actions appear to lose correlations. Interestingly, we do not find any sign of unusual behaviour near \( \Omega = 1 \) (using the \( \mathcal{O}(\varepsilon) \) approximation to the actions).

Computing the drift of the solution from the slaving manifold as before, we find that the drift rate of \( \text{Im} b_1 \) does not depend on \( \Omega \) — in Fig. 6 the curves for different \( \Omega \) would overlap each other to within the error bars — and it is practically identical to that in the \( b_2 = 0 \) case (cf. top dashed curve in Fig. 6). As expected, \( \text{Im} b_2 \) drifts at a slower rate than \( \text{Im} b_1 \) (bottom dashed curve in the figure), and its drift rate does depend sensitively on \( \Omega \) and \( \varepsilon \). Unlike with the adiabatic invariants, no singularity in the drift rates is found at or near \( \Omega = 1 \) or 2, indicating that the destabilizing resonance effect is absent or strongly suppressed near the slaving manifold.

It is important to determine whether the exchange-of-action behaviour discussed above is an artefact of the way in which the second fast wave is included in our model. To this end, we considered two other systems which are Taylor expansions of (2.4) to either \( \mathcal{O}(\varepsilon) \) or \( \mathcal{O}(\varepsilon^2) \). In the first case, fast–fast coupling does not appear and the results were found to be essentially identical to those of the one-wave model discussed in Section 4 (though with \( \varepsilon \) replaced by \( \varepsilon / \Omega \) for the second wave in the asymptotic scaling). The fast–fast coupling comes in first at \( \mathcal{O}(\varepsilon^2) \), where it appears in the mixed fast–slow term in the Hamiltonian; in this case the exchange-of-action behaviour emerges, together with independent drifts of the fast actions, as in our original model (2.4). We reiterate that this coupling cannot be removed by higher-order corrections to \( I_i \), so it is an inherent behaviour of systems such as these. Since direct fast–fast coupling mediated through the slow modes is also a feature of the SWE, we conclude that this behaviour is physical.
6. Lyapunov exponents

A useful diagnostic tool of a dynamical system is its Lyapunov exponents. However, it seems that Lyapunov exponents have been little used as a measure of complexity in Hamiltonian systems. This is perhaps partly caused by the difficulty in accurately computing the exponents, especially the smaller ones. Another contributing factor is the lack of a theoretical framework for the interpretation of the exponents in Hamiltonian systems: while each integral of motion implies the existence of a pair of zero exponents and each adiabatic invariant an exponentially small pair, we are not aware of any result on the converse. In this section we present numerical computation of the exponents and discuss their connection to the picture of the dynamics obtained above.

A discrete QR algorithm [29] is used for the numerical computation. As pointed out in [30], the computation can be done symplectically using a splitting scheme (as we do here); this ensures the pairing of the exponents. (A consequence of the symplectic structure in Hamiltonian systems is the property that its Lyapunov exponents occur in pairs: if \( \lambda \) is an exponent, then so is \(-\lambda\).)

In Fig. 8 we show the second- and third-largest exponents in our model for different values of \( \epsilon \) (the largest one, \( \lambda_1 \approx 0.15 \), is insensitive to \( \epsilon \)). Unlike the action drift, which is quantifiable with relatively little computation, the time scale \( T \) required for convergence of the Lyapunov exponents appears to scale as the drift time scale of the adiabatic invariants: the computation has converged for \( \epsilon^{-1} \leq 6 \), but for smaller values of \( \epsilon \) a \( 1/t \) decay is clearly visible, indicating that the computation has yet to converge [31]. This makes the computation of the smaller exponents considerably more difficult, as the effects of round-offs (especially in the re-orthogonalizations) become significant for long runs. It should also be mentioned that the computed values of the exponents differ quite significantly for
different trajectories, at least up to \( t \sim 10^5 \), although the rapidly decaying dependence on \( \epsilon \) discussed below appears to be robust.

From the figure, it is clear that both exponents drop rapidly with \( \epsilon \). As mentioned above, it is difficult to determine the functional dependence precisely since the range of \( \epsilon \) for which the computation is practical is limited and the convergence of the ensemble average is slow. Although the exponents do not generally correspond to the physical degrees of freedom in the problem, it is clear that when two independent adiabatic invariants exist (i.e., when \( \Omega \) is non-resonant), both \( \lambda_2 \) and \( \lambda_3 \) will be bounded above by the drift rates of the invariants, which are known to scale exponentially in \( \epsilon \). Our numerical evidence is consistent with this, although it does not tell us the exact \( \epsilon \)-dependence. So the interesting case is when \( \Omega \) is resonant: here we find \( \lambda_2 \) to be slightly smaller than in the non-resonant case, but \( \lambda_3 \) much smaller. That one exponent is smaller can be understood in terms of the stability of resonances [32,33]; however, the smallness of both exponents suggests that the motion on the resonance surface is also slow. The question of whether exponentially small Lyapunov exponents implies the existence of adiabatic invariants is therefore an interesting one, and it would be useful either to establish this rigorously or to find a counterexample.

Unlike the speed of the action drift, the Lyapunov exponents do not seem to be affected by the value of \( \bar{I}_i \) at all (up to statistical errors). This is true for both resonant and non-resonant \( \Omega \).

7. Summary and discussion

We have reported numerical results from a study of a simple dynamical system consisting of a chaotic slow component coupled to one or two linear fast oscillators. This system is intended to model large-scale atmospheric motion, with the slow and fast components representing vortical motion and (inertia-) gravity waves, respectively. In the one-frequency case, we find an adiabatic invariant \( \bar{I}_1 \) corresponding to the fast action in the decoupled system whose variance \( \Delta \bar{I}_1^2 \) grows linearly in time at a speed that scales as \( \exp(-\kappa/\epsilon) \), in the ensemble average sense, where \( \epsilon \ll 1 \) is the ratio of the fast and slow time scales and \( \kappa \) is a constant [Figs. 4 and 5]. In the two-frequency case, the variations of the fast actions \( \bar{I}_1 \) and \( \bar{I}_2 \) are often found to almost cancel each other, giving rise to an 'exchange of action’ behaviour [Fig. 7]; this is attributed to the \( O(\epsilon^2) \) fast–fast coupling term in the Hamiltonian, although its precise mechanism is still unclear. We also find that the ensemble-averaged distance of solutions from the slaving manifold, \( \langle \text{Imb}_i \rangle \), grows linearly in time at a speed that scales exponentially with \( \epsilon \) [Fig. 6]; unlike action-like adiabatic invariants, this is true regardless of the value of the frequency ratio \( \Omega \).

The fact that the fast actions drift is of course to be expected in Hamiltonian systems with more than two degrees of freedom [20]. Such drift is generically known as Arnol’d diffusion, after [34], although it should be pointed out that the mechanism for this drift in our system, whose slow part is strongly chaotic, may be different from that suggested by Arnol’d for his near-integrable system. The speed of this drift is bounded, by virtue of Neishtadt’s or Nekhoroshev’s theorems, by an exponentially small quantity. Our numerical observation that certain quantities such as \( \bar{I}_1 \) and \( J \) do in fact drift at an exponentially small rate shows that this bound is not only qualitatively optimal (as is widely suspected, cf. [35]), but also in a sense the typical drift speed, defined as \( v = \langle \langle \Delta \bar{I}_1^2 \rangle \rangle / \langle \Delta \bar{I}_1 \rangle \). When the drift of action is modelled as a random walk, then \( v \) scales as the square of the speed of the action drift \( |d\bar{I}_i/dt| \) for each realization (which is the quantity bounded by the theorems), and both quantities scale exponentially with \( \epsilon \). From the point of view of physical applications, an interesting aspect of our observation is that the exponential scaling is observed for modest values of \( \epsilon \), which are many orders of magnitude larger than the \( \epsilon_0 \) required for the validity of the bound (3.3).

While the numerical results for the one-frequency case agree with the bound of Neishtadt’s theorem, in the two-frequency case we find no evidence, for the values of \( \epsilon \) and \( \Omega \) we have used, that the drift speed of each fast
action scales qualitatively as the bound provided by Nekhoroshev’s theorem, \( \exp(-\kappa / \varepsilon^d) \) where \( d < 1 \). This and the fact that \( J \) does drift exponentially slowly suggest that a different scaling regime may be present in multi-frequency systems for larger \( \Omega \); this may require introducing other small parameters based on \( \varepsilon \) and \( \Omega \). Similarly, we are not aware of any theory, even heuristic ones, that explains the action-exchange phenomenon.

The finding that the resonance property of \( \Omega \) is relatively weak may be related to the fact that the exponential scaling in \( \varepsilon \) becomes effective so early in the perturbation expansion [as seen in Fig. 3]: for large \( \varepsilon \), anything beyond the first few terms in the perturbation theory is inherent drift, so higher-order resonances in \( \Omega \) cannot matter in this case. When the system is near-integrable, Nekhoroshev’s conjecture [18] that higher-order resonances do not matter for small values of \( \tilde{I} \) has recently been proved [36,37]. Whether these recent results have analogues for systems, such as ours, that are far from (fully) integrable remains to be investigated.

Our observations that the drift is slower near the \( \tilde{I} = 0 \) surface (the ‘slow manifold’) and that the slaving solution is exponentially stable regardless of \( \Omega \) also await theoretical explanation. Going back to our original motivation of finding balance dynamics, this is of particular interest since in realistic atmospheric models many gravity-wave modes have resonant frequencies.

As expected, the existence of an adiabatic invariant was found numerically to correspond to an exponentially small Lyapunov exponent. It is interesting however to note that the smaller exponents (the largest exponent corresponds, roughly speaking, to the chaotic slow dynamics) also appear to be exponentially small even in resonant cases where the adiabatic invariants do not exist (or at least are not computable by direct averaging). This raises the tantalizing possibility of an exponentially slow drift in these cases as well, although to prove it we would have to rule out algebraic drifts.

Acknowledgements

The work reported here represents a part of DW’s Ph.D. thesis at the University of Toronto, which was supported by an Ontario Graduate Scholarship, the University of Toronto Open Fellowship, and an Atmospheric Environment Service Fellowship. TGS is supported by the Natural Sciences and Engineering Research Council and the Atmospheric Environment Service of Canada. We thank the reviewers for constructive comments and criticisms.

Appendix A. Accuracy estimate for slaving solutions

Here we derive an asymptotic accuracy estimate for the slow solution of the slaving method, valid for time scales scaling as \( \log \varepsilon \) as \( \varepsilon \to 0 \). The method is fairly standard; we include it here for completeness. Consider a (finite-dimensional) dynamical system of the form

\[
\frac{df}{dt} + \frac{L}{\varepsilon} f = F(f, s), \quad \frac{ds}{dt} = G(f, s), \tag{A.1}
\]

where the eigenvalues of \( L \) are purely imaginary.

Suppose an invariant manifold exists on which the fast variables \( f \) are slaved to the slow ones \( s \), \( f = U_\infty(s; \varepsilon) \). Substituting this relation into the equations of motion (A.1) gives the so-called “superbalance” equation

\[
\frac{\partial U_\infty}{\partial s} G(U_\infty(s; \varepsilon), s) + \frac{L}{\varepsilon} U_\infty(s; \varepsilon) = F(U_\infty(s; \varepsilon), s). \tag{A.2}
\]
Suppose now that we have found a slaving relation $U(s; \varepsilon)$ that satisfies (A.2) to $O(\varepsilon^n)$, or in other words,

$$\left| \frac{\partial U}{\partial s} G(U, s) + \frac{L}{\varepsilon} U - F(U, s) \right| \leq C_1 \varepsilon^{n+1}$$

(A.3)

for all relevant values of $s$ (here and henceforth the $C_i$s denote $O(1)$ constants). Let $\tilde{f} := f - U(s)$, then we have

$$\frac{d}{dt} \tilde{f} + \frac{L}{\varepsilon} \tilde{f} = F(\tilde{f} + U, s) - \frac{L}{\varepsilon} U - \frac{\partial U}{\partial s} G(\tilde{f} + U, s)$$

$$= F(\tilde{f} + U, s) - F(U, s) - \frac{\partial U}{\partial s} G(\tilde{f} + U, s) + \frac{\partial U}{\partial s} G(U, s) + F(U, s) - \frac{L}{\varepsilon} U - \frac{\partial U}{\partial s} G(U, s).$$

(A.4)

If now $F$ and $G$ are Lipschitz near $f = U(s)$, namely,

$$|F(\tilde{f} + U, s) - F(U, s)| \leq C_F |\tilde{f}|, \quad |G(\tilde{f} + U, s) - G(U, s)| \leq C_G |\tilde{f}|$$

(A.5)

for $|\tilde{f}| \leq f_{\text{max}}$, then we have the estimate

$$\frac{d}{dt} |\tilde{f}| \leq \frac{d}{dt} e^{L/\varepsilon} |\tilde{f}| = \left| \frac{d}{dt} \tilde{f} + \frac{L}{\varepsilon} \tilde{f} \right| \leq \left[ C_F + \left| \frac{\partial U}{\partial s} \right| C_G \right] |\tilde{f}| + C_1 \varepsilon^{n+1} \leq C_2 |\tilde{f}| + C_1 \varepsilon^{n+1},$$

(A.6)

where for the first inequality we have used the fact that the eigenvalues of $L$ have zero real parts. From here one can use Gronwall’s lemma: Integrating the last inequality, and taking $\tilde{f}(t = 0) = 0$ for convenience, we find

$$|\tilde{f}(t)| \leq \frac{C_1 \varepsilon^{n+1}}{C_2} (e^{C_2 t} - 1).$$

(A.7)

Now take $T = (\log \varepsilon_0 - \log \varepsilon)/C_2$ for some constant $\varepsilon_0$; with this,

$$|\tilde{f}(t)| \leq \frac{\varepsilon_0 C_1}{C_2} \varepsilon^n$$

(A.8)

for $\varepsilon \leq \varepsilon_0$ and $0 \leq t \leq T$. Then taking $\varepsilon_0 = C_2/C_1$, we have

$$|\tilde{f}(t)| \leq \varepsilon^n$$

(A.9)

which provides an asymptotic bound on the departure of the true solution from the slaved solution over time scales of order $\log \varepsilon$.

References

[7] T. Warn, Nonlinear balance and quasi-geostrophic sets, Atmos.–Ocean 35 (1997) 135–145. (This paper was written in 1983 but not published. It was finally published in its original form in 1997 after having been referred to in print many times.)
[34] V.I. Arnol’d, Instability of dynamical systems with several degrees of freedom, Soviet Math.–Dokl. 5 (1964) 581–585.