Hyperbolic polyhedra from Bianchi groups

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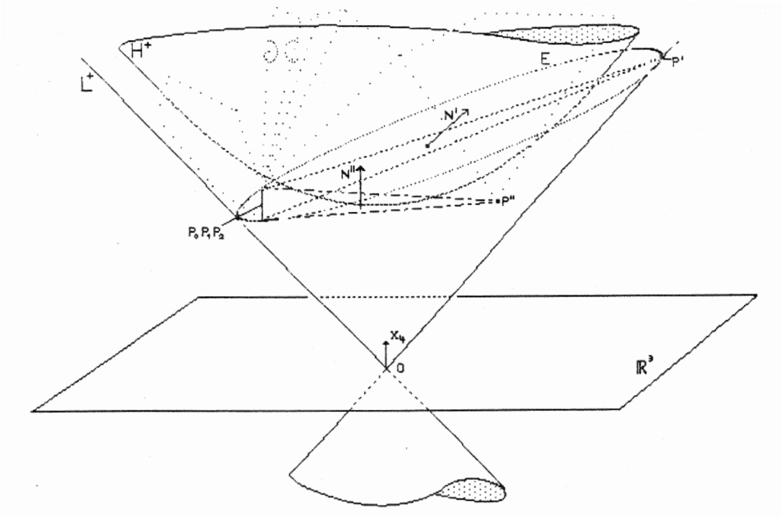
Our setting.

We consider two models of hyperbolic space
1) the usual upper half space model $\mathbb{H}^3$;
2) the upper sheet H_d^+ of a hyperboloid in 4-space
here we vary the latter one with d squarefree:
put $f_d(x, y, z, w) = x^2 + dy^2 + z^2 - w^2$, then

$$H_d^+ = \{(x, y, z, w) \in \mathbb{R}^4 \mid f_d(x, y, z, w) = 1\}$$

with its light cone L_d^+ defined by $f_d(x, y, z, w) = 0$ (rather than 1);
it corresponds to the boundary $\partial\mathbb{H}^3$ (topologically an S^2) of $\mathbb{H}^3$:
each point on $\partial\mathbb{H}^3$ corresponds to a *ray* through the origin in L_d^+.
(More freedom!)

For clarification, here is an analogous picture of 2) in the 3-dimensional case.



Isometry groups. Recall the isometry groups
1) for $\mathbb{H}^3$ we have $\mathrm{PSL}_2(\mathbb{C})$;
2) for H_d^+ we have $\mathrm{SO}^+(f_d) \subset \mathrm{GL}(4, \mathbb{R})$, a close relative of the Lorentz group $\mathrm{O}(3,1)$.

Rational structure. For 2) choose particular one:
for d squarefree, consider the number ring $\mathbb{O}_d$ of $F = \mathbb{Q}(\sqrt{-d})$.
For H_d^+ we have in principle a choice for every (equiv. class of) cusp(s) $\xleftrightarrow{1:1}$ ideal classes in $\mathbb{O}_d$.

Our choice is uniform for all d: take "roughly primitive" solutions of $f_d(x, y, z, w)$, i.e. on the light cone L_d^+:
(x, y, z, w) is **roughly primitive** if

$$\gcd(x, y, z, w) = 1 \text{ and } z \equiv w \pmod 2$$

or

$$\gcd(x, y, z, w) = 2 \text{ and } z \not\equiv w \pmod 4.$$

Examples. $(0,0,1,1)$ and $(2,2,1,5)$ are roughly primitive solutions (the latter for $d=5$) while $(1,0,0,1)$ is not: we need to take $(2,0,0,2)$ instead.

Proposition. $\mathrm{SO}(f_d,\mathbb{Z})$ acts (discretely) on the roughly primitive solutions on L_d^+.

Epstein–Penner: Take the *Euclidean* convex hull of these points. (This constitutes a piecewise linear approximation of H_d^+.)

Then we pass to a torsionfree subgroup of $SO(f_d,\mathbb{Z})$.

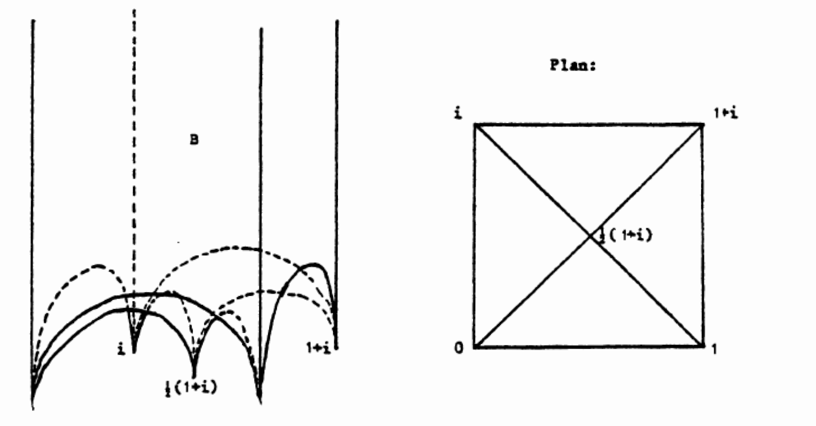
(Technical aside: we aim for such a subgroup of minimal index: presumably a divisor of 24 is sufficient [A. Rahm, referring to a corresponding result for $\mathrm{PSL}(2,\mathbb{O}_d)$ due to Hurwitz and/or Bianchi(?)].)

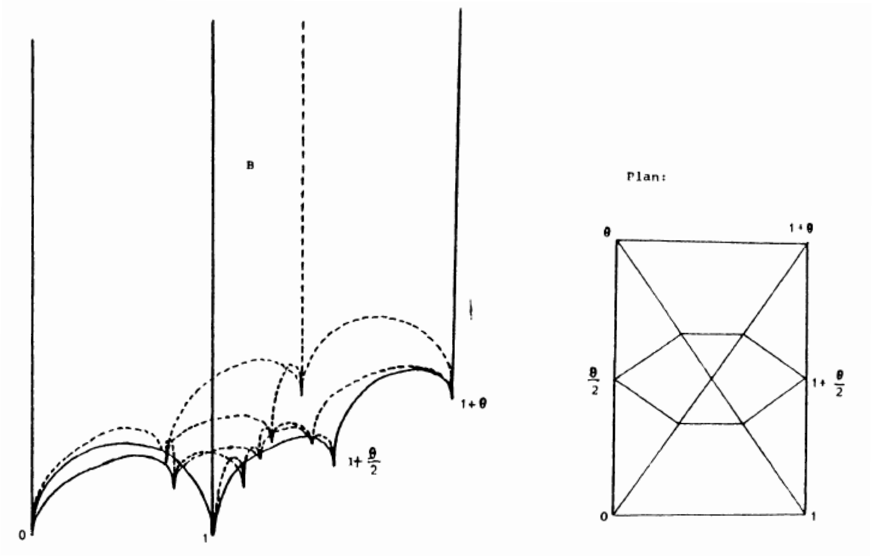
Upshot: We arrive at an **ideal** (i.e. all vertices lie on the boundary, here the light cone) fundamental domain for that subgroup.

Results.

Agree for small d with known ideal tessellations found earlier by Grunewald–Gushoff–Mennicke ('82) and Cremona ('85).
Sometimes, especially for $\mathbb{O}_d$ of class number 1, also agree with ideal hyperbolic tessellations found by D. Yasaki.

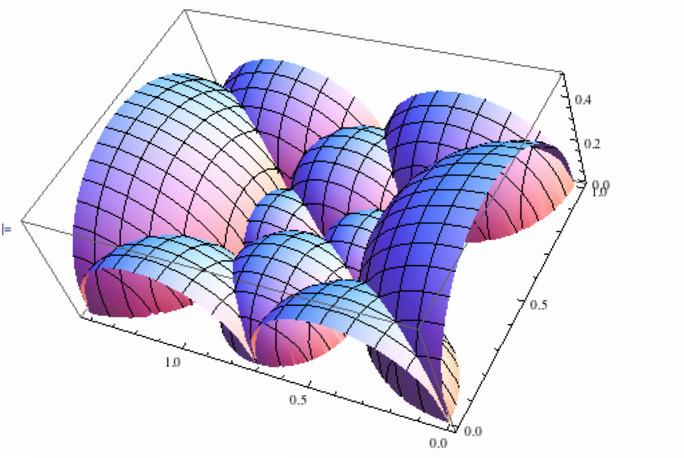
Examples. $\mathrm{SL}_2(\mathbb{Z}[i])$ $(d = 1)$ gives a tessellation with octahedra.



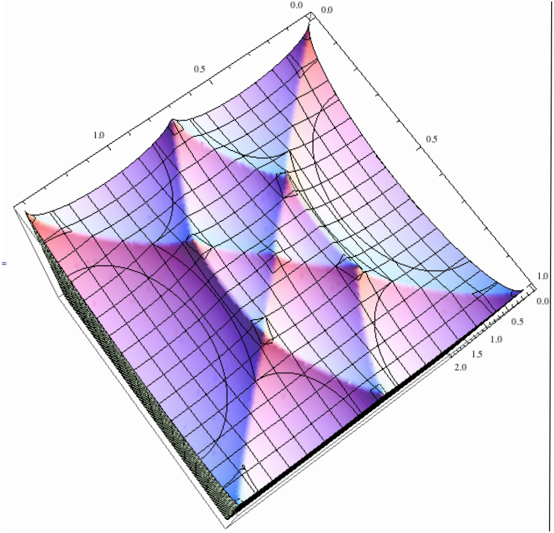


Tessellation of the hyperbolic plane using $SL_2(\mathbb{Z}[\sqrt{-2}])$

A schematic 3D-picture of the half-spheres bounding the polyhedron looks as follows:

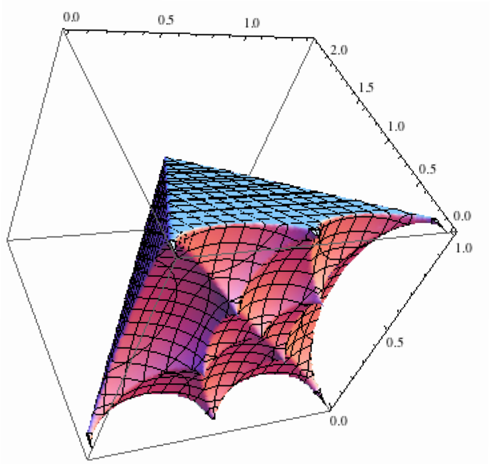


Here's how that fundamental domain looks from below:



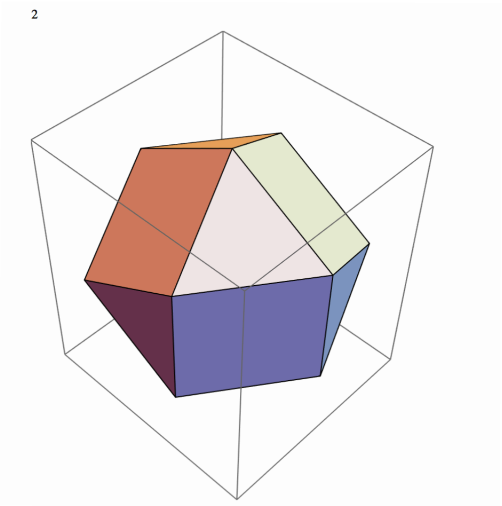
Ideal fundamental domain arising from $SL_2(\mathbb{Z}[\sqrt{-2}])$ [uses Mathematica]

Different viewing angle, 'pushing down' vertex at ∞ to finite height.



Approximate ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$

Euclidean counterpart for $d = 2$: a cuboctahedron



Implementations.
Fundamental domain implementation using the above recipe was originally done in diploma thesis (Atari ST+, 1MB RAM...), cases $d \equiv 3(4)$, $d \leq 51$; a few larger ones (83, 123,..) worked out, too. M. Dutour-Sikirič recently also implemented it with far more sophisticated tools ($d < 700$). Yasaki's variant: $d < 5000$.

A mathematical puzzle app. If you are tempted by this 'puzzle game' deducing a polytope from its projection, here is an app for you (by former Durham student Josh Yaxley): *Polyshadow*...

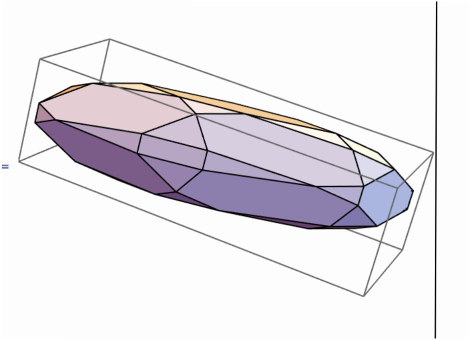
A comparison game. When comparing our ideal tessellations (many combinatorially different polytopes) to those arising from Yasaki's work (few types as building blocks) we get 'scissors congruence challenges'.
First such for $d = 6$:
1) we obtain a rhombicuboctahedron (Rubik's snake);
2) Dan obtains a truncated tetrahedron and a couple of hexagonal caps.

Student Josh Inoue has realised this as a puzzle, using a 3D-printer.

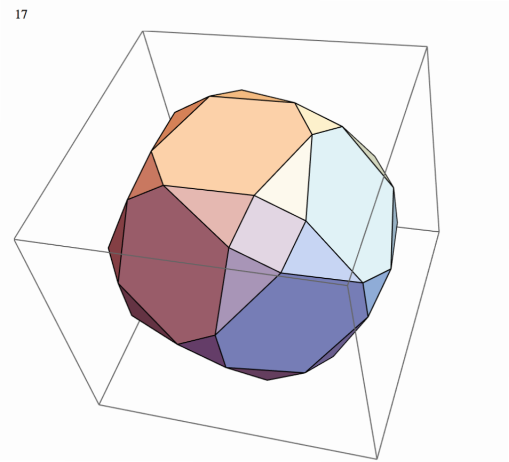
Surprising hidden symmetry:

The ‘first’ polyhedron arising this way contains the triangular face with vertices $0, 1, \infty$ which correspond on the light cone to $(0, 0, -1, 0), (2, 0, 0, 2)$ and $(0, 0, 1, 1)$, resp. (note the independence of d as the second coordinate for these 3 vanishes). It often exhibits surprisingly many somewhat hidden symmetries (had emerged from Maple images produced by former DH student Matthew Spencer).

Example. The first polytope for $d = 17$:



Inoue made the symmetries more explicit: he found affine transformations which map the vertices of that first polytope on a sphere centred at the origin.



Far more pleasing to the eye...

The affine transformation here is not obvious (for $d = 17$):

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3d-2}}{2\sqrt{3}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}}\sqrt{3d-2} \\ 0 \end{pmatrix}$$

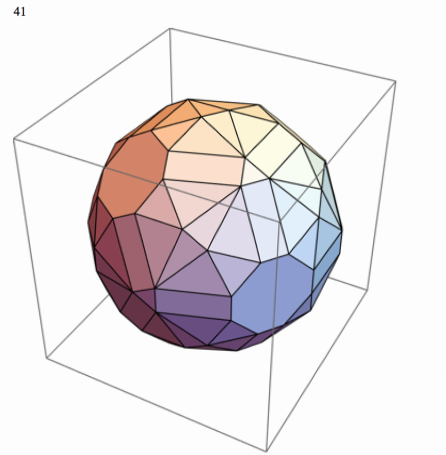
A nice family.

Via cluster computations I determined most first polytopes with $d < 23000$ squarefree.

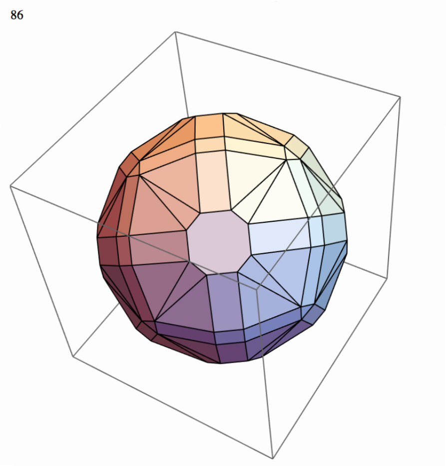
The arguably most interesting polytopes arise for d of the form $d = \frac{n^2+1}{8}$, $n \not\equiv 0 \pmod 3$ (condition singled out by Inoue). They all have a octahedral (i.e. extended A_4-)symmetry, and for $d > 2$ the number of vertices is a multiple of 24.

Apart from the first few, most of these do not seem to be in any classification.

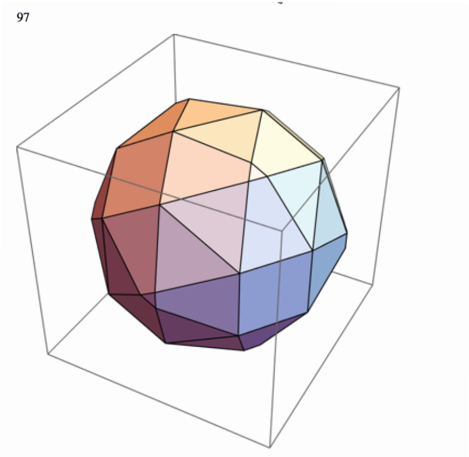
$d = 41$ (96 vertices):



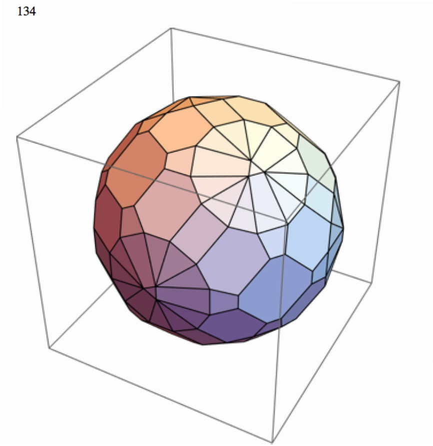
$d = 86$ (120 vertices):



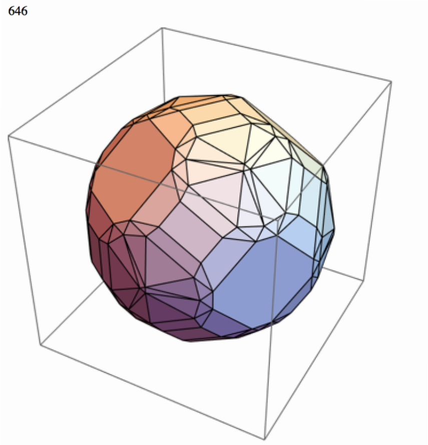
$d = 97$ (48 vertices):



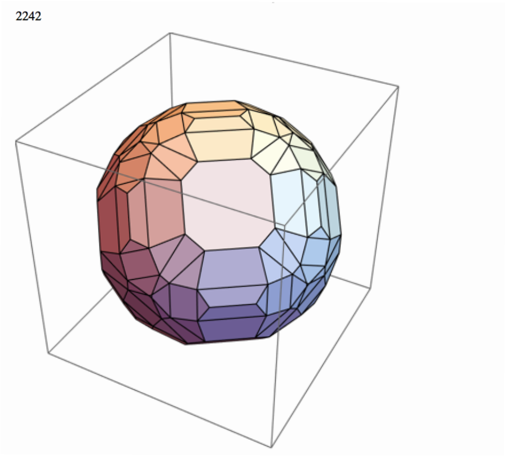
$d = 134$ (168 vertices):



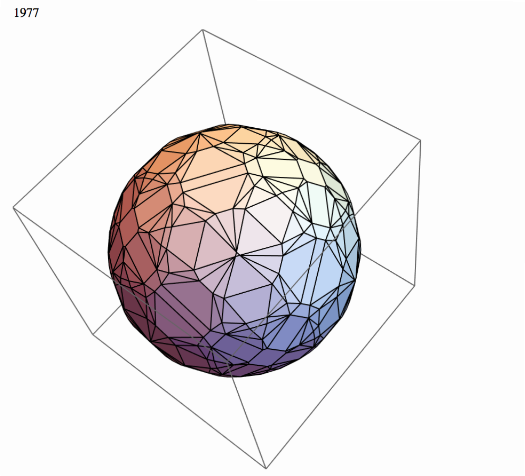
$d = 646$ (192 vertices):



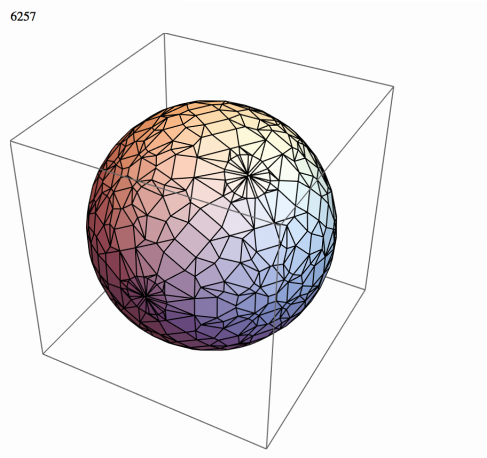
$d = 2242$ (192 vertices):



$d = 1977$ (432 vertices):



$d = 6257$ (912 vertices):



Largest number of vertices found so far: 2496 (for $d = 20009$).

Possible application?

(NYT from today.)

Amazon is working on a collection of buildings in downtown Seattle that will be arrayed around three transparent, conjoined structures that the company calls spheres.

Ian C. Bates for The New York Times