Tame kernels and second regulators of number fields and their subfields

To Professor Aderemi O. Kuku
on the occasion of his 70th birthday

by

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Abstract

Assuming a version of the Lichtenbaum conjecture, we apply Brauer-Kuroda relations between the Dedekind zeta function of a number field and the zeta function of some of its subfields to prove formulas relating the order of the tame kernel of a number field $F$ with the orders of the tame kernels of some of its subfields. The details are given for fields $F$ which are Galois over $\mathbb{Q}$ with Galois group the group $\mathbb{Z}/2 \times \mathbb{Z}/2$, the dihedral group $D_{2p}$, $p$ an odd prime, or the alternating group $A_4$. We include numerical results illustrating these formulas.

Key Words: Dedekind zeta functions, Bloch-Wigner dilogarithm, Bloch group, second regulator, Brauer-Kuroda relations.

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1. Introduction

There are multiplicative relations between the Dedekind zeta functions of a number field and of some of its subfields, given by Brauer and Kuroda. The first nonvanishing coefficient of the Taylor expansion at $s = 0$ of the Dedekind zeta function of a number field is related to the class number and the first regulator of this field. Similarly, the analogous coefficient at $s = -1$ is related to the order of the tame kernel and the second regulator of the field (under the assumption of the Lichtenbaum conjecture).

We give more explicit statements in the case of a number field which is Galois over $\mathbb{Q}$ with Galois group the group $\mathbb{Z}/2 \times \mathbb{Z}/2$, the dihedral group $D_{2p}$, $p$ an odd prime, or the alternating group $A_4$.

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The paper is organized as follows. In Part I we recall known facts on Dedekind’s zeta functions and on the leading coefficients of their Taylor expansions at \( s = 0 \) and \( s = -1 \). These coefficients depend on the class number, on the order of the tame kernel and on the corresponding regulators of the field in question.

In Part II we recall the Brauer-Kuroda relations and write them in an explicit form for several groups, including the groups mentioned above. We show that, for a biquadratic field, Brauer-Kuroda relations imply a well known expression of the class number of the field by means of class numbers of its quadratic subfields.

The main results of the paper (Theorems 1, 2 and 3, and Corollaries 1, 2 and 3) give relations between the second regulator, respectively the order of the tame kernel, of a field \( F \) with that of some of its subfields, where \( F \) is Galois over \( \mathbb{Q} \) with Galois group \( \mathbb{Z}/2 \times \mathbb{Z}/2, D_{2p}, \) or \( A_4 \). These results are proved under the assumption of Conjecture 1, which is a variant of the Lichtenbaum conjecture, combined with results of Bloch and Suslin.

In Part III we include results of numerical experiments, which give the (conjectural) values of the second regulator for some fields \( F \) of small degree over \( \mathbb{Q} \), and give some evidence for Conjecture 1. We check its compatibility with the Brauer-Kuroda relations and give an example of two fields of different signature having the same second regulator numerically.

The first two parts of the paper contain an extended version of the talk given by the first author at the Conference in Nanjing University on the occasion of the 70th birthday of Professor Aderemi O. Kuku. The last part written by the second author presents numerical results giving some evidence for the conjecture mentioned above.

**Part I. Dedekind zeta functions and their values at** \( s = 0 \) **and** \( s = -1 \)**

2. The Dedekind zeta function

We recall the basic properties of the Dedekind zeta function \( \zeta_F(s) \) of a number field \( F \) of a finite degree \( n \) over \( \mathbb{Q} \).

It is a meromorphic function on \( \mathbb{C} \) with a unique single pole at \( s = 1 \). It has zeros in the strip \( \{ s \in \mathbb{C} : 0 < \Re s < 1 \} \) and possibly at nonpositive integers \(-m, m \geq 0 \). The multiplicity of zero at \( s = -m \) equals

\[
d_m = d_m(F) = \begin{cases} 
  r_1 + r_2 - 1 & \text{if } m = 0, \\
  r_1 + r_2 & \text{if } m \text{ is even, } m > 0, \\
  r_2 & \text{if } m \text{ is odd.}
\end{cases} \tag{2.1}
\]
Here $r_1 = r_1(F)$ is the number of real places of $F$, and $r_2 = r_2(F)$ is the number of complex ones. We have $n = [F : \mathbb{Q}] = r_1 + 2r_2$.

The Dedekind zeta function satisfies a functional equation. To write it we need the following notation. Let

$$A(F) := \frac{|d(F)|^{1/2}}{2^{r_2} \pi^{n/2}},$$

where $d(F)$ is the discriminant of $F$, and let

$$\Phi(s) := A(F)^s \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_F(s).$$

Then the following functional equation holds: $\Phi(s) = \Phi(1-s)$ for $s \in \mathbb{C} \setminus \mathbb{Z}$. More explicitly,

$$A(F)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_F(s) = A(F)^{1-s} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} \zeta_F(1-s). \quad (2.2)$$

Since $\Gamma(s)$ has poles at nonpositive integers and $\zeta_F(s)$ has a pole at $s = 1$, the formula (2.2) does not make sense for $s \in \mathbb{Z}$.

To overcome this difficulty we introduce the following notation. For an arbitrary function $f(s)$ whose Laurent expansion in a neighborhood of $s = s_0$ is

$$f(s) = a_r(s-s_0)^r + a_{r+1}(s-s_0)^{r+1} + \cdots, \quad \text{where} \quad r \in \mathbb{Z}, \ a_r \neq 0, \quad (2.3)$$

we denote by $f^*(s_0)$ or by $(f(s_0))^*$ the first nonvanishing coefficient $a_r$ in the expansion (2.3). Obviously $(f_1 f_2)^*(s_0) = f_1^*(s_0) \cdot f_2^*(s_0)$.

Then (2.2) implies that

$$A(F)^s \left(\Gamma\left(\frac{s}{2}\right)^{r_1}\right)^* \left(\Gamma(s)^{r_2}\right)^* (\zeta_F(s))^* = A(F)^{1-s} \left(\Gamma\left(\frac{1-s}{2}\right)^{r_1}\right)^* \left(\Gamma(1-s)^{r_2}\right)^* (\zeta_F(1-s))^*, \quad (2.4)$$

and this formula holds for every $s \in \mathbb{C}$.

When substituting in (2.4) integer values for $s$, the following well known formula will be useful:

$$\Gamma^*(-n) = \frac{(-1)^n}{n!} \quad \text{for} \quad n \in \mathbb{Z}, \ n \geq 0.$$

3. The value of $\zeta_F^*(0)$ and the first regulator

It is known (see [17], Theorem 7.3 and (6.8)) that

$$\zeta_F^*(1) = \frac{2^{r_1}(2\pi)^{r_2}}{|d(F)|^{1/2}} \cdot \frac{R_1(F) h(F)}{w_1(F)}, \quad (3.1)$$
where $w_1(F)$ is the number of roots of 1 in $F$, $h(F)$ is the class number of $F$, and $R_1(F)$ is the (first) regulator of $F$.

We recall here its definition (see e.g. [17] or [19]). Let $\mathcal{O}_F$ be the ring of algebraic integers in $F$, and let $\mathcal{O}^*_F$ be its group of units. The Dirichlet unit theorem says that $\mathcal{O}^*_F$ is the direct sum of the cyclic group of roots of unity in $F$ of order $w_1(F)$, and a free abelian group of the rank $d_0 = r_1 + r_2 - 1$.

Let $\varepsilon_1, \ldots, \varepsilon_{d_0}$ be generators of this free abelian group. It is called a system of fundamental units of $F$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{r_1 + r_2}$ be embeddings $F \rightarrow \mathbb{C}$ corresponding to the archimedean places of $F$. The absolute value of the determinant

$$R_1(F) := \left| \det(c_i \log |\sigma_i(\varepsilon_j)|)_{1 \leq i, j \leq d_0} \right|,$$

where $c_i = 1$ if $\sigma_i$ is real, and $c_i = 2$ otherwise, does not depend on the choice of the fundamental units $\varepsilon_j$, and on the order of the places $\sigma_1, \sigma_2, \ldots, \sigma_{d_0}$ chosen. Since $d_0 = r_1 + r_2 - 1$, the archimedean place $\sigma_{r_1 + r_2}$ has been omitted in (3.2).

We call $R_1(F)$ the first regulator of $F$.

From the functional equation (2.4) with $s = 0$ and (3.1) it follows that

$$\zeta_F^*(0) = -\frac{R_1(F) h(F)}{w_1(F)}. \quad (3.3)$$

In particular, if $d_0 = 0$, i.e. if $F = \mathbb{Q}$ or $F$ is quadratic imaginary, then $R_1(F) = 1$. Hence (3.3) for $F = \mathbb{Q}$ gives

$$\zeta(0) = \zeta_{\mathbb{Q}}(0) = \zeta_{\mathbb{Q}}^*(0) = -\frac{1}{2},$$

and for $F = \mathbb{Q}(\sqrt{-d})$, $d$ squarefree and $> 0$, we have

$$\zeta_F(0) = \zeta_F^*(0) = -\frac{h(F)}{w_1(F)},$$

where $w_1(F) = 4$ for $d = 1$, $w_1(F) = 6$ for $d = 3$, and $w_1(F) = 2$ otherwise.

For real quadratic fields $F$, we have $w_1(F) = 2$, and $R_1(F) = \log \varepsilon(F)$, where $\varepsilon(F) > 1$ is the fundamental unit of the field $F$. Thus (3.3) takes the form

$$\zeta_F^*(0) = -\frac{1}{2} \log \varepsilon(F) \cdot h(F).$$

4. The value of $\zeta_F^*(-1)$ and the second regulator

The results presented above concerning the case $s = 0$ are classical, and there are known effective algorithms for computing the values of the class number $h(F)$ and
of the first regulator \( R_1(F) \) of a number field \( F \). See the computer algebra package [18], where these algorithms have been implemented. See also [11].

Our knowledge in the next cases \( s = -1, -2, \ldots \) is less complete. In the present paper we do not discuss the cases \( s \leq -2 \). Instead we concentrate on the case \( s = -1 \).

By analogy with the formula (3.3) in the case \( s = 0 \), one can expect that an analogous formula holds in the case \( s = -1 \).

Namely, the first regulator \( R_1(F) \) will be replaced by the second dilogarithmic regulator \( \widetilde{R}_2(F) \) defined below in Section 6. It is the absolute value of the determinant of a matrix of size \( d_1 = r_2(F) \).

The class number \( h(F) \) of the field \( F \), which is equal to the order of the torsion subgroup of the group \( K_0 F \), will be replaced by the order \( k_2(F) \) of the tame kernel \( K_2 \mathcal{O}_F \) of \( F \). See [16] for a definition.

Finally, the number \( w_1(F) \) of roots of unity in \( F \) will be replaced by the number \( w_2(F) \) of roots of unity in the compositum of all quadratic extensions of \( F \).

Thus, by analogy with (3.3), (see also remarks before (12.1)) one can state the following conjecture:

**Conjecture 1** For every number field \( F \) we have

\[
|ζ_F^*(-1)| = \frac{\widetilde{R}_2(F)k_2(F)}{w_2(F)}.
\]

Let us remark that Conjecture 1 is related to the Birch-Tate and the Lichtenbaum conjectures, see [3]. For totally real fields \( F \) we have \( r_2(F) = 0 \), so \( \widetilde{R}_2(F) = 1 \) and (4.1) is the Birch-Tate conjecture.

5. The Bloch group

To define the second (or dilogarithmic) regulator, we need a definition of the Bloch group \( B(F) \) of a number field \( F \) (see [10]).

For any subfield \( E \) of \( \mathbb{C} \) let \( \mathbb{Z}[E] \) be the free abelian group with generators \([a]\), where \( a \) runs over all elements of \( E \) distinct from 0 and 1. Let \( \partial_2 = \partial_2(E) : \mathbb{Z}[E] \rightarrow E^× \wedge E^× \) be the homomorphism defined on the free generators by \( \partial_2([a]) := a \wedge (1 - a) \). Here \( \wedge \) is a modified wedge product satisfying \( u \wedge (-u) = 0 \) in place of the usual \( u \wedge u = 0 \).

Let \( \mathcal{A}(E) := \ker \partial_2(E) \). Then we have the exact sequence

\[
0 \rightarrow \mathcal{A}(E) \rightarrow \mathbb{Z}[E] \overset{\partial_2(E)}{\rightarrow} E^× \wedge E^× \overset{\nu}{\rightarrow} K_2 E \rightarrow 0.
\]

Here \( \nu \) is defined by \( \nu(a \wedge b) = \{a, b\} \), where \( \{a, b\} \in K_2 F \) is the Steinberg symbol.
Let $\mathcal{C}(E)$ be the subgroup of $\mathbb{Z}[E]$ generated by the elements $[a] + [1 - a]$, $[a] + [1/a]$, $a \neq 0, 1$, and by the elements of the form $[a_1] + [a_2] + [a_3] + [a_4] + [a_5]$, (called 5-cycles), where $a_1, ..., a_5 \in E^\times \setminus \{1\}$ satisfy $a_1a_{i+1} + a_{i+3} = 1$, for $i = 1, ..., 5$, and the indices are taken modulo 5. Hence $a_1a_{i+1} \neq 1$.

Obviously, every cyclic permutation of the elements in a 5-cycle gives the same 5-cycle. Moreover, the 5-cycle is determined by its first two arguments: If $a_1 = x, a_2 = y$, then $a_3 = \frac{1-x}{1-xy}, a_4 = 1 - xy$ and $a_5 = \frac{1-y}{1-xy}$, since $xy = a_1a_2 \neq 1$.

One can easily verify that $\partial_2([a] + [1 - a]) = \partial_2([a] + [1/a]) = 0$, and for every 5-cycle $b$ we have $\partial_2(b) = 0$. Hence $\mathcal{C}(E) \subseteq \ker \partial_2 = \mathcal{A}(E)$. Defining the Bloch group of $E$ by $\mathcal{B}(E) := \mathcal{A}(E)/\mathcal{C}(E)$ we get from (5.1) the exact sequence

$$0 \rightarrow \mathcal{B}(E) \overset{\partial_2}{\longrightarrow} \mathbb{Z}[E]/\mathcal{C}(E) \overset{\partial_3}{\longrightarrow} E^\times \mathbb{C}^\times \overset{\nu}{\longrightarrow} K_2E \rightarrow 0.$$

6. The second regulator $\widetilde{R}_2(F)$

In the definition of the first regulator we considered the matrix of size $d_0$, with elements which are logarithms of some archimedean norms of fundamental units. In the case of the second regulator, we consider an analogous matrix of size $d_1$. The role of the units will be played by the elements of the Bloch group, and the logarithm will be replaced by the dilogarithm of Wigner and Bloch normalized as follows:

$$\widetilde{D}(z) := -\text{Im} \left( \frac{1}{\pi} \int_1^z \frac{\log(1-t)}{t} \, dt \right) + \frac{\arg(1-z)}{\pi} \cdot \log |z|.$$

It differs by the factor $\frac{1}{\pi}$ from the original one $D(z)$ (see [3], Corollary 6.1.2).

It is a real analytic function $\widetilde{D} : \mathbb{C} \rightarrow \mathbb{R}$ satisfying $\widetilde{D}(\overline{z}) = -\widetilde{D}(z)$, where $\overline{z}$ is the complex conjugate of $z$. Hence $\widetilde{D}$ vanishes on $\mathbb{R}$.

The mapping $\widetilde{D}$ can be extended by linearity to a homomorphism $\mathbb{Z}[\mathbb{C}] \rightarrow \mathbb{R}$, defined on generators, by $\widetilde{D}([a]) := \widetilde{D}(a)$ for $a \in \mathbb{C}$. It can be proved that $\widetilde{D}(b) = 0$ for every element $b \in \mathcal{C}(E)$. Hence $\widetilde{D}$ induces a homomorphism

$$\widetilde{D} : \mathbb{Z}[E]/\mathcal{C}(E) \rightarrow \mathbb{R}, \quad \text{where } E \subseteq \mathbb{C},$$

called also the dilogarithm.

Now let us return to the number field $F$. Let $\sigma_j, j = 1, 2, ..., r_2$ be the complex places of $F$.

Then $\widetilde{D}_j := \widetilde{D} \circ \sigma_j$ are homomorphisms $\mathbb{Z}[F]/\mathcal{C}(F) \rightarrow \mathbb{R}$ for $j = 1, 2, ..., r_2$.

Collecting them we get a homomorphism $\mathbb{D} : \mathbb{Z}[F]/\mathcal{C}(F) \rightarrow \mathbb{R}^{r_2}$ defined by $\mathbb{D} := (\widetilde{D}_1, ..., \widetilde{D}_{r_2})$. Since $\mathcal{B}(F)$ is a subgroup of $\mathbb{Z}[F]/\mathcal{C}(F)$, we can restrict $\mathbb{D}$ to this
subgroup. It turns out that \( D(B(F)) \) is a lattice \( \Lambda_2(F) \) of maximal rank in \( \mathbb{R}^{r_2} \). We define the second dilogarithmic regulator \( \tilde{R}_2(F) \) as the covolume of this lattice.

In other words, if for some \( b_1, \ldots, b_{r_2} \in B(F) \) the vectors \( D(b_1), \ldots, D(b_{r_2}) \in \mathbb{R}^{r_2} \) generate the lattice \( \Lambda_2(F) \), then

\[
\tilde{R}_2(F) = |\text{det}(\tilde{D}(\sigma_j(b_i)))_{1 \leq i, j \leq r_2}|.
\]

Part II. Brauer–Kuroda relations

7. Brauer–Kuroda relations


Let \( F/k \) be a Galois extension of number fields with the Galois group \( G \). Then the following multiplicative relation holds.

For every cyclic subgroup \( H \) of \( G \), let

\[
c(H) := \frac{1}{(G : H)} \sum_{H^{\ast}} \mu(|H^{\ast}/H|),
\]

where \( \mu \) is the Möbius function.

Then, writing \( F^H \) for the fixed field of \( H \) in \( F \), we have

\[
\zeta_k(s) = \prod_{H \text{-cyclic}} \zeta_{F^H}^{c(H)}(s). \tag{7.1}
\]

In what follows we usually assume that \( k = \mathbb{Q} \), so that \( \zeta_k = \zeta \) is the Riemann zeta function.

Substituting \( s = 0 \) in (7.1), in view of (3.3), we get multiplicative relations between class numbers and the first regulators of corresponding fields. There are many papers devoted to this subject.

When we substitute \( s = -1 \), and apply Conjecture 1, we get conjectural relations between the orders of tame kernels and the second regulators of the fields in question.

We illustrate this by some simple examples. Let us observe that in fact the relation (7.1) depends essentially on the structure of the Galois group \( G \) of the field \( F \) only, and not on the field \( F \) itself.
Example 1 Let $G$ be the cyclic group of order $n$. Then there exists a unique cyclic subgroup $H$ of order $d$, for every $d \mid n$. The subgroups $H^*$ containing $H$ have orders $dd'$, where $d' \mid n/d$. Consequently

$$
\sum_{H^*} \mu(|H^*/H|) = \sum_{d' \mid n/d} \mu(d') = \begin{cases} 1, & \text{if } n/d = 1, \\ 0, & \text{otherwise}. \end{cases}
$$

Therefore $c(H) = 1$ if $H = G$, and $c(H) = 0$, otherwise. From (7.1) we get

$$
\zeta_k(s) = \zeta_{FG}(s),
$$

which is not interesting, since $FG = k$.

Example 2 Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle \sigma_1, \sigma_2 \rangle$.

Then $H_0 = \langle \sigma_1 \sigma_2 \rangle$, $H_1 = \langle \sigma_1 \rangle$, $H_2 = \langle \sigma_2 \rangle$, and $E = \{1\}$ are all cyclic subgroups of $G$.

Since $H_0, H_1, H_2$ are maximal cyclic subgroups, we get

$$
c(H_i) = \frac{1}{2} \mu(1) = \frac{1}{2} \quad \text{for } i = 0, 1, 2.
$$

Next,

$$
c(E) = \frac{1}{4}(3\mu(2) + \mu(1)) = -\frac{1}{2}.
$$

For $i = 0, 1, 2$, let $F_i := F^{H_i}$. Then (7.1) gives

$$
\zeta_k(s) = (\zeta_{F_0}(s)\zeta_{F_1}(s)\zeta_{F_2}(s))^{1/2} \zeta_F(s)^{-1/2}.
$$

Hence

$$
\zeta_F(s)\zeta_k(s)^2 = \zeta_{F_0}(s)\zeta_{F_1}(s)\zeta_{F_2}(s). \quad (7.2)
$$

Example 3 Let $G = S_3$.

Let $H_0$ be the subgroup of $G$ of order 3, and let $H_1, H_2, H_3$ be subgroups of order 2. They are conjugate. To these subgroups there correspond subfields of $F$.

The quadratic subfield $F_0$, and the cubic ones $F_1, F_2, F_3$, which are isomorphic.

We have

$$
c(E) = \frac{1}{6}(3\mu(2) + \mu(3) + \mu(1)) = -\frac{1}{2},
$$

$$
c(H_i) = \frac{1}{3} \mu(1) = \frac{1}{3} \quad \text{for } i = 1, 2, 3,
$$

$$
c(H_0) = \frac{1}{2} \mu(1) = \frac{1}{2}.
$$

Then (7.1) gives

$$
\zeta_F \zeta_k^2 = \zeta_{F_0}(\zeta_{F_1} \zeta_{F_2} \zeta_{F_3})^{2/3}.
$$

Since zeta functions of isomorphic fields are equal, we get

$$
\zeta_F \zeta_k^2 = \zeta_{F_0} \zeta_{F_1}^2. \quad (7.3)
$$
In the following examples, we leave the details to the reader. Denote by $\zeta_\sigma$, the zeta function of the subfield of $F$ fixed by the automorphism $\sigma$.

**Example 4** Let $G = A_4$.

The formula (7.1) gives

$$\zeta_F \zeta_k^2 = \zeta_{(12)(34)} \zeta_{(234)}.$$ 

**Example 5** Let $G = S_4$.

Then

$$\zeta_F \zeta_k^2 = \zeta_{(12)} \zeta_{(1234)} \zeta_{(123)}.$$ 

**Example 6** Let $G = S_5$.

Then

$$\zeta_F \zeta_k^4 = \zeta_{(123)(45)} \zeta_{(1234)} \zeta_{(12345)}.$$ 

Let us remark that other cyclic subgroups of $G$ do not contribute to this formula.

**Example 7** Let $G = D_{2p}$ be the dihedral group of order $2p$, where $p$ is an odd prime. Let $H_p$ be its unique subgroup of order $p$, and $H_2$ a subgroup of order 2. There are $p$ subgroups of order 2 and they are conjugate.

Then

$$\zeta_F \zeta_k^2 = \zeta_{F H_p} \zeta_{F H_2}.$$ 

The case $p = 3$ has been treated above in Example 3 with more details, since $D_6 \cong S_3$.

**Example 8** Let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternions. It has one cyclic subgroup $\langle -1 \rangle$ of order 2, and three cyclic subgroups $\langle i \rangle$, $\langle j \rangle$, $\langle k \rangle$ of order 4.

We have

$$c(E) = \frac{1}{8} (\mu(1) + \mu(2) + 3\mu(4)) = 0,$$

$$c(\langle -1 \rangle) = \frac{1}{4} (\mu(1) + 3\mu(2)) = -\frac{1}{2},$$

$$c(\langle i \rangle) = c(\langle j \rangle) = c(\langle k \rangle) = \frac{1}{2} \mu(1) = \frac{1}{2}.$$ 

Here the Brauer-Kuroda relation takes the form

$$\zeta_{F \langle -1 \rangle}(s) \zeta_k(s)^2 = \zeta_{F \langle i \rangle}(s) \zeta_{F \langle j \rangle}(s) \zeta_{F \langle k \rangle}(s).$$ 

The Dedekind zeta function $\zeta_F(s)$ of the field $F$ does not appear in this relation, because $c(E) = 0$.

Since $Q/\langle -1 \rangle = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $F^{(i)}$, $F^{(j)}$, $F^{(k)}$ are quadratic subfields of $F^{(-1)}$, (7.5) is simply the relation (7.2) from Example 2 for the extension $F^{(-1)}/k$. 


We call a finite group $G$ exceptional, if the coefficient $c_G(E) = c(E) = 0$. Thus cyclic groups and the group of quaternions are exceptional. In [7] it has been proved that the only exceptional $p$-groups are cyclic and generalized quaternion. Consequently a nilpotent group is exceptional iff some of its Sylow subgroups are cyclic or generalized quaternion. In [7] there are given also many other examples of exceptional groups, e.g. $\text{SL}(2, \mathbb{F}_q)$ with $q$ odd is exceptional.

In all the above examples, except the first and the last, the Brauer-Kuroda relation is of the form $\zeta_k \zeta_m = \text{a product of zeta functions of some proper subfields of } F$, with $m > 0$, because the value of $c(E)$ is negative, and $c(H) \geq 0$ for nontrivial cyclic subgroups $H$ of $G$.

In general this is not the case. For example for $G = S_7$ and $\mathbb{Z}/6 \times \mathbb{Z}/6$, we have $c(E) > 0$. Hence $\zeta_k$ and $\zeta_F$ are on different sides of the Brauer-Kuroda equation.

If the Galois group of a number field $F$ is exceptional, then the Dedekind zeta function of the field does not appear in the Brauer-Kuroda relation (7.1). One may expect that then $\zeta_F$ is, in a sense, independent of $\zeta_k$ and of the Dedekind zeta functions of some other proper subfields of $F$.

This can be illustrated by the following observation. There are known Galois extensions $F$ of $\mathbb{Q}$ with the quaternion Galois group such that $\zeta_F(\frac{1}{2}) = 0$ and $\zeta_{F_j}(\frac{1}{2}) \neq 0$ for proper subfields $F_j$ of $F$.

Hence $\zeta_F$ is multiplicatively independent of $\zeta$ and $\zeta_{F_j}$. (See [12]).

8. The case of a biquadratic field and $s = 0$

Let $F$ be a biquadratic extension of $\mathbb{Q}$, and let $F_0, F_1, F_2$ be its quadratic subfields, with $F_0$ real. Then (7.2) holds.

First we substitute $s = 0$ in (7.2) and we get some well known relations between the class numbers and the first regulators of $F$ and of its subfields.

Next we substitute $s = -1$ and assuming Conjecture 1, we discuss analogous relations between orders of the tame kernels and of the second regulators of the fields in question.

We are looking for some analogies in these two situations.

The most interesting case is when $F$ is imaginary. Then $F_1$ and $F_2$ are quadratic imaginary, so their first regulators are trivial, $R_1(F_1) = R_1(F_2) = 1$. Moreover $r_2(F) = 2$ and $r_1(F_0) = 2$, so $d_0(F) = d_0(F_0) = 1$. Thus in $F$ and in $F_0$, there is only one fundamental unit. Denote it by $\varepsilon$ and by $\varepsilon_0$, respectively. Then $R_1(F_0) = \log |\varepsilon_0|$ and $R_1(F) = 2\log |\varepsilon|$. We may assume that $|\varepsilon| > 1$ and $|\varepsilon_0| > 1$.

Moreover, since $\varepsilon_0$ is a unit of $F$, we get $|\varepsilon_0| = |\varepsilon|Q_1(F)$ for some $Q_1(F) \in \mathbb{N}$. 

Consequently we obtain a regulator relation

\[ R_1(F) = \frac{2}{Q_1(F)} R_1(F_0). \]  

(8.1)

It is known that there are exactly two possibilities: \( \varepsilon_0 = \varepsilon \) (then \( Q_1(F) = 1 \)) and \( \varepsilon_0 = \zeta \varepsilon^2 \), where \( \zeta \) is a root of unity (then \( Q_1(F) = 2 \)). Some sufficient conditions for \( Q_1(F) = 1 \) are known. E.g.

- (i) If \( N\varepsilon_0 = -1 \), then \( Q_1(F) = 1 \).
- (ii) Let \( F = \mathbb{Q}(\sqrt{-d_1}, \sqrt{-d_2}) \), where \( d_1, d_2 \) are positive and squarefree. If \( \gcd(d_1, d_2) \) has an odd prime factor, or both \( d_1, d_2 \) are even and \((d_1d_2)/4 \equiv 1 \) (mod 4), then \( Q_1(F) = 1 \).

A more precise description of conditions equivalent to \( Q_1(F) = 1 \) is given in [14].

One can easily verify that for the biquadratic field \( F \) and its quadratic subfields \( F_0, F_1, F_2 \), we have

\[ 4w_1(F) = w_1(F_0)w_1(F_1)w_1(F_2) \]

with only one exception: \( F = \mathbb{Q}(\sqrt{4}, \sqrt{2}) = \mathbb{Q}(\zeta) \).

Taking into account the value \( \zeta(0) = -\frac{1}{2} \), the formula (7.2) for \( s = 0 \) gives

\[ R_1(F)h(F) = R_1(F_0)h(F_0)h(F_1)h(F_2). \]  

(8.2)

Hence for \( F \neq \mathbb{Q}(\zeta) \), we get from (8.1) and (8.2) that

\[ h(F) = \frac{Q_1(F)}{2} \prod_{j=0}^{2} h(F_j) \quad \text{with} \quad Q_1(F) = 1 \text{ or } 2. \]  

(8.3)

If \( F \) is a totally real biquadratic field, then an analogous formula holds (see [14]):

\[ h(F) = \frac{Q_1(F)}{4} \prod_{j=0}^{2} h(F_j) \quad \text{with} \quad Q_1(F)|4. \]

9. The case of a biquadratic field and \( s = -1 \)

We assume that \( F \) is a complex biquadratic extension of \( \mathbb{Q} \), and we use the notation from the last section. Substituting \( s = -1 \) into (7.2) and assuming Conjecture 1, we get because \( \zeta(-1) = -\frac{1}{12} \) that

\[ \frac{\tilde{R}_2(F)k_2(F)}{w_2(F)} \frac{1}{12^2} = \frac{\tilde{R}_2(F_1) \tilde{R}_2(F_2)}{w_2(F_1)w_2(F_2)} \prod_{j=0}^{2} \frac{k_2(F_j)}{w_2(F_j)}. \]  

(9.1)
One can easily verify that

\[
    w_2(F) = w_2(F_0) = \begin{cases} 
        2 \cdot 24 & \text{if } \sqrt{2} \in F, \\
        5 \cdot 24 & \text{if } \sqrt{5} \in F, \\
        24 & \text{otherwise,}
    \end{cases}
\]

and \( w_2(F_1) = w_2(F_2) = 24 \). Hence (9.1) implies

\[
    \bar{R}_2(F)k_2(F) = \frac{1}{4} \bar{R}_2(F_1)\bar{R}_2(F_2) \prod_{j=0}^{2} k_2(F_j). \tag{9.2}
\]

To proceed further we need a regulator formula for second regulators analogous to (8.1). We shall prove that \( 2\bar{R}_2(F_1) \cdot \bar{R}_2(F_2) \) is an integer multiple of \( \bar{R}_2(F) \).

The lattices \( \Lambda_2(F_1) \) and \( \Lambda_2(F_2) \) corresponding to the Bloch groups \( B(F_1) \) and \( B(F_2) \) are 1-dimensional, since \( r_2(F_1) = r_2(F_2) = 1 \).

For \( i = 1, 2 \) let \( b_i \in B(F_i) \) define a generator \( \bar{D}(b_i) \) of the lattice \( \Lambda_2(F_i) \). Hence \( \bar{R}(F_i) = \bar{D}(b_i) \).

From \( r_2(F) = 2 \) it follows that the lattice \( \Lambda_2(F) \) is 2-dimensional. Obviously \( b_1, b_2 \in B(F) \), hence \( D(b_1), D(b_2) \in \Lambda_2(F) \).

Let \( G = \text{Gal}(F/\mathbb{Q}) \). For \( i = 1, 2 \), denote by \( \tau_i \in G \) the nontrivial automorphism of \( F \) trivial on \( F_i \). Then \( \tau_1 \tau_2 \) is trivial on the real subfield \( F_0 \), hence it is complex conjugation. Consequently the two complex places of \( F \) are represented by \( \sigma_1 = \text{id} \) and \( \sigma_2 = \tau_2 \).

The sublattice \( \Lambda'_2 \) generated by \( D(b_1) \) and \( D(b_2) \) in \( \Lambda_2(F) \) has covolume equal to the absolute value of the determinant of the matrix

\[
    \begin{pmatrix}
        D(b_1) \\
        D(b_2)
    \end{pmatrix} = \begin{pmatrix}
        \bar{D}(\sigma_1(b_1)) & \bar{D}(\sigma_2(b_1)) \\
        \bar{D}(\sigma_1(b_2)) & \bar{D}(\sigma_2(b_2))
    \end{pmatrix} = \begin{pmatrix}
        \bar{D}(b_1) & -\bar{D}(b_1) \\
        \bar{D}(b_2) & \bar{D}(b_2)
    \end{pmatrix},
\]

since \( \sigma_2(b_1) = \tau_2(b_1) = \tau_2 \tau_1(b_1) \) is the complex conjugate of \( b_1 \).

Thus

\[
    \text{covol}(\Lambda'_2) = 2\bar{D}(b_1)\bar{D}(b_2) = 2\bar{R}_2(F_1)\bar{R}_2(F_2).
\]

The covolume of a sublattice is an integer multiple of the covolume of the lattice. Therefore \( \text{covol}(\Lambda'_2) = Q_2(F)\text{covol}(\Lambda_2(F)) \) for some \( Q_2(F) \in \mathbb{N} \). Thus we have proved

**Theorem 1** If \( F \) is a complex biquadratic extension of \( \mathbb{Q} \) with imaginary quadratic subfields \( F_1 \) and \( F_2 \), then

\[
    \bar{R}_2(F) = \frac{2}{Q_2(F)} \bar{R}_2(F_1)\bar{R}_2(F_2), \quad \text{for some } Q_2(F) \in \mathbb{N}. \tag{9.3}
\]
From (9.2) and (9.3) we get

**Corollary 1** Assume Conjecture 1 for the fields in question. Then in the notation of Theorem 1, we have

\[ k_2(F) = \frac{Q_2(F)}{8} \prod_{j=0}^{2} k_2(F_j) \quad \text{for some } Q_2(F) \in \mathbb{N}. \quad (9.4) \]

Zhou Haiyan [23] proved that the odd parts of the \( k_2 \)'s on both sides of (9.4) are equal. It follows that \( Q_2(F) \) is \(^1\) a power of 2.

On the basis of numerical evidence given below in Section 12.3, we expect that \( Q_2(F) \) is always 1 or 2.

If \( F \) is a real biquadratic field not containing \( \sqrt{2} \) nor \( \sqrt{5} \), and if \( F_0, F_1, F_2 \) are its quadratic subfields, then \( w_2 \) of all these fields equals 24, and their second regulators are equal 1.

Applying the Birch–Tate conjecture, we get

\[ k_2(E) = w_2(E) |\zeta_E(-1)|, \]

which has been proved already for all totally real abelian fields \( E \) (see [22] Theorem 1.5 and note on p.499). From the Brauer–Kuroda relation (7.2) and the fact that \( \zeta(-1) = -\frac{1}{12} \), we get

\[ k_2(F) = \frac{1}{4} \prod_{j=0}^{2} k_2(F_j). \quad (9.5) \]

Now, assuming Conjecture 1, we give a numerical example, which indicates that the regulator index \( Q_2(F) \) in (9.3) can be even, and thus can be greater than 1.

**Example 9** (cf. [6]). Let \( F = \mathbb{Q}(\sqrt{-6}, \sqrt{-15}) \). Then \( F_0 = \mathbb{Q}(\sqrt{10}) \), \( F_1 = \mathbb{Q}(\sqrt{-15}) \), \( F_2 = \mathbb{Q}(\sqrt{-6}) \) are quadratic subfields of \( F \). For all these fields, \( w_2 \) equals 24 and \( \tilde{R}_2(F_0) = 1 \), since \( F_0 \) is real.

1) The number \( a = \frac{1 + \sqrt{-15}}{4} \in F_1 \) satisfies \( a^2 - \frac{1}{2}a + 1 = 0 \), and hence \( 1 - a^3 = -(1-a)^3 \). Taking \( b_1 := 18[a] - 2[a^3] \) for \( \partial_{21} := \partial_2(F_1) \) we get

\[ \partial_{21}(b_1) = 18(a \tilde{\zeta}(1-a)) - 2(a^3 \tilde{\zeta}(1-a^3)) = 18(a \tilde{\zeta}(1-a)) - (a^3 \tilde{\zeta}(1-a)^6) = 0. \]

Hence \( b_1 \in \mathcal{A}(F_1) \) and \( \tilde{D}(b_1) \in \Lambda_2(F_1) \).

---

\(^1\)Note added on January 27, 2013: She proved recently that \( Q_2(F) = 1, 2 \) or 4 , see [24].
Assuming Conjecture 1 for the field $F_1$, we get

$$|\zeta^*_F_1(-1)| = \frac{\tilde{R}_2(F_1)k_2(F_1)}{w_2(F_1)}.$$  \hfill (9.6)

Since $k_2(F_1) = 2$, $w_2(F_1) = 24$, $\zeta^*_F_1(-1) = -0.499525$ and $\tilde{D}(b_1) = 5.99431$, it follows from (9.6) that $\tilde{R}_2(F_1) = \tilde{D}(b_1)$, i.e. $\tilde{D}(b_1)$ generates the lattice $\Lambda_2(F_1)$.

2) We have $a = a_1^2$, where $a_1 = \frac{\sqrt{10} + \sqrt{-6}}{4} \in F$. Therefore in $F \times F \times$ we have

$$a^3 \tilde{\alpha}(-1-a)^3) = a_1^6 \tilde{\alpha}(1-a)^3) = a_1^6 \tilde{\alpha}(1-a)^3) = a^3 \tilde{\alpha}(1-a)^3).$$

Consequently, for $\partial_2 := \partial_2(F)$ we get

$$\partial_2(b_1/2) = 9(a \tilde{\alpha}(1-a) - (a^3 \tilde{\alpha}(1-a)^3) = 9(a \tilde{\alpha}(1-a) - (a^3 \tilde{\alpha}(1-a)^3) = 0.$$

Hence $b_1/2 \in A(F)$ and $\tilde{D}(b_1/2) = \frac{1}{2} \tilde{D}(b_1) \in \Lambda_2(F)$.

Let $\tilde{D}(b_2)$ be a generator of the lattice $\Lambda_2(F_2)$, where $b_2 \in A_2$. Then $\mathcal{D}(b_1/2)$ and $\mathcal{D}(b_2)$ generate a sublattice of $\Lambda_2(F)$ of covolume

$$\left| \det \begin{pmatrix} \mathcal{D}(b_1/2) \\ \mathcal{D}(b_2) \end{pmatrix} \right| = \frac{1}{2} \left| \det \begin{pmatrix} \mathcal{D}(b_1) \\ \mathcal{D}(b_2) \end{pmatrix} \right| = \tilde{R}_2(F_1)\tilde{R}_2(F_2).$$

Therefore $\tilde{R}_2(F_1)\tilde{R}_2(F_2)$ is an integer multiple of $\tilde{R}_2(F)$ and from (9.3) it follows that $Q_2(F)$ is even.

10. The case of the dihedral Galois group $D_{2p}$ and $s = -1$

Let $D_{2p}$ be the dihedral group of order $2p$, where $p$ is an odd prime. It is the group of isometries of a regular $p$-gon with $p$ vertices $1,2,\ldots,p$.

The group has a unique subgroup of order $p$ generated by the rotation $\tau = (123\ldots p)$, and $p$ subgroups of order 2 generated by symmetries.

Let $\sigma = \sigma_p$ be the symmetry fixing the vertex $p$,

$$\sigma = (1,p-1)(2,p-2)\cdots(\frac{p-1}{2},\frac{p+1}{2}).$$

Other symmetries are $\sigma_j := \tau^j \sigma \tau^{-j}$, $j = 1,2,\ldots,p-1$. Then $\sigma_j$ fixes the vertex $j$.

Let $F$ be a Galois extension of $\mathbb{Q}$ with the Galois group $G = D_{2p}$. It has a unique quadratic subfield $F_0$ fixed by $\tau$, and $p$ subfields $F_j$ fixed by $\sigma_j$, $j = 1,2,\ldots,p$, of degree $p$.

We have $\tau(F_j) = F_{j+1}$ for $j = 1,2,\ldots,p-1$ and $\tau(F_p) = F_1$. 
Tame kernels and second regulators

Namely, if \( a \in F_j \) and \( \sigma_j(a) = a \) then \( \tau(a) \) satisfies

\[
\sigma_{j+1}(\tau(a)) = \tau^{j+1} \sigma \tau^{-j}(a) = \tau(\sigma_j(a)) = \tau(a).
\]

Therefore \( \tau(a) \in F_{j+1} \) and \( F_j = \tau^j(F_p) \) for \( j = 1, 2, \ldots, p \).

Assume that the field \( F \) is complex. Then complex conjugation belongs to \( G \), we may assume that \( \sigma = \sigma_p \) is the complex conjugation.

Then the field \( F_p \) fixed by \( \sigma \) is the unique maximal real subfield of \( F \). We have

\[
r_2(F_0) = 1, \quad r_2(F_j) = \frac{p-1}{2}, \quad j = 1, 2, \ldots, p, \quad r_2(F) = p.
\]

We determine the complex places of the fields \( F_0, F_1, F_p \) and \( F \).

Obviously, \( \text{id} \) is the complex place of \( F_0 \), and \( \tau^j, j = 0, 1, \ldots, p-1 \) are complex places of \( F \).

Since \( \sigma \tau^j \) is a symmetry, we get \( \sigma \tau^j \sigma \tau^j = \text{id} \), hence \( \sigma \tau^j = \tau^{-j} \sigma \). Consequently

\[
\sigma(F_j) = \sigma \tau^j(F_p) = \tau^{-j} \sigma(F_p) = \tau^{p-j}(F_p) = F_{p-j}
\]

for \( j = 1, 2, \ldots, p-1 \). It follows that the fields \( F_j \) and \( F_{p-j} \) are complex conjugate. Therefore the complex places of \( F_p \) are

\[
\tau, \tau^2, \ldots, \tau^t, \quad \text{where} \quad t = \frac{p-1}{2},
\]

and the complex places of \( F_1 = \tau(F_p) \) are

\[
\text{id}, \tau, \tau^2, \ldots, \tau^{t-1}.
\]

Now we describe the dilogarithmic lattices of the fields \( F_0, F_p \) and \( F_1 \).

The dilogarithmic lattice \( \Lambda_2(F_0) \) of rank 1 is generated by \( \widetilde{D}(b_0) \) for some \( b_0 \in \mathcal{B}(F_0) \). Hence

\[
\widetilde{R}_2(F_0) = \widetilde{D}(b_0). \tag{10.1}
\]

The dilogarithmic lattice \( \Lambda_2(F_p) \) of rank \( t \) is generated by the following vectors \( \mathbb{D}_{F_p}(b_1), \ldots, \mathbb{D}_{F_p}(b_t) \) for some \( b_1, \ldots, b_t \in \mathcal{B}(F_p) \), where

\[
\mathbb{D}_{F_p}(b_j) = (\widetilde{D}(\tau(b_j)), \widetilde{D}(\tau^2(b_j)), \ldots, \widetilde{D}(\tau^t(b_j))).
\]

Consequently,

\[
\widetilde{R}_2(F_p) = |\det(U_1, U_2, \ldots, U_t)|, \quad \text{where} \quad U_j = \begin{pmatrix}
\widetilde{D}(\tau^j(b_1)) \\
\vdots \\
\widetilde{D}(\tau^j(b_t))
\end{pmatrix}, \quad j = 1, 2, \ldots, t. \tag{10.2}
\]
Similarly, the dilogarithmic lattice $\Lambda_2(F_1)$ of rank $t$ is generated by the following vectors $\mathbb{D}_{F_1}(b_{t+1}), \ldots, \mathbb{D}_{F_1}(b_{2t})$ for some $b_{t+1}, \ldots, b_{2t} \in \mathcal{B}(F_1)$, where
\[
\mathbb{D}_{F_1}(b_j) = (\widetilde{D}(b_j), \widetilde{D}(\tau(b_j)), \ldots, \widetilde{D}(\tau^{t-1}(b_j))).
\]
Consequently,
\[
\mathbb{R}_2(F_1) = |\det(V_1, V_2, \ldots, V_t)|, \quad \text{where} \quad V_j = \begin{pmatrix} \widetilde{D}(\tau^{t+j}(b_{t+1})) \\ \cdots \\ \widetilde{D}(\tau^{t+j}(b_{2t})) \end{pmatrix}, \quad j = 1, 2, \ldots, t.
\]
(10.3)
Since the Bloch groups $\mathcal{B}(F_0), \mathcal{B}(F_p), \mathcal{B}(F_1)$ can be mapped canonically into $\mathcal{B}(F)$, the elements $b_0, b_1, \ldots, b_{2t}$ defined above can be considered as elements of $\mathcal{B}(F)$.

Therefore the lattice $\Lambda'_2$ generated by elements $\mathbb{D}_F(b_j), j = 0, 1, \ldots, 2t$, where $\mathbb{D}_F(b) = (\widetilde{D}(b), \widetilde{D}(\tau(b)), \widetilde{D}(\tau^2(b)), \ldots, \widetilde{D}(\tau^{p-1}(b)))$ for $b \in \mathcal{B}(F)$, is a sublattice of the dilogarithmic lattice $\Lambda_2(F)$.

We determine the covolume of $\Lambda'_2$. By the definition of $\Lambda'_2$ we have
\[
\text{covol}(\Lambda'_2) = |\det \begin{pmatrix} \mathbb{D}_F(b_0) \\ \mathbb{D}_F(b_1) \\ \vdots \\ \mathbb{D}_F(b_{2t}) \end{pmatrix} |.
\]
(10.4)
The first row of this matrix is simply
\[
(\widetilde{D}(b_0), \widetilde{D}(b_0), \ldots, \widetilde{D}(b_0)) = \widetilde{D}(b_0)(1, 1, \ldots, 1).
\]
The $(j + 1)$st row, where $1 \leq j \leq t$, is
\[
(\widetilde{D}(\tau(b_j)), \widetilde{D}(\tau^2(b_j)), \ldots, \widetilde{D}(\tau^t(b_j)), \widetilde{D}(\tau^{t+1}(b_j)), \ldots, \widetilde{D}(\tau^{2t}(b_j)), \widetilde{D}(\tau^{2t+1}(b_j)))
\]
\[
= (\widetilde{D}(\tau(b_j)), \widetilde{D}(\tau^2(b_j)), \ldots, \widetilde{D}(\tau^t(b_j)), -\widetilde{D}(\tau^t(b_j)), \ldots, -\widetilde{D}(\tau(b_j)), 0),
\]
since $\tau^k(b_j)$ and $\tau^{p-k}(b_j)$ are complex conjugate and $\tau^{2t+1}(b_j) = \tau^p(b_j) = b_j$ is real.

The $(j + 1)$st row, where $t + 1 \leq j \leq 2t$, is
\[
(\widetilde{D}(\tau(b_j)), \widetilde{D}(\tau^2(b_j)), \ldots, \widetilde{D}(\tau^{t-1}(b_j)), \widetilde{D}(\tau^t(b_j)), \widetilde{D}(\tau^{t+1}(b_j)), \ldots, \widetilde{D}(\tau^{2t}(b_j)), \widetilde{D}(\tau^{2t+1}(b_j)))
\]
\[
= (\widetilde{D}(\tau(b_j)), \widetilde{D}(\tau^2(b_j)), \ldots, \widetilde{D}(\tau^{t-1}(b_j)), -\widetilde{D}(\tau^{t-1}(b_j)), \ldots, -\widetilde{D}(\tau(b_j)), -\widetilde{D}(b_j), 0, \widetilde{D}(b_j)),
\]
since $\tau^k(b_j)$ and $\tau^{2t-k}(b_j)$ are complex conjugate and $\tau^{2t}(b_j) = \tau^{p-1}(b_j)$ is real.
Hence covol($\Lambda'_2$) is the absolute value of the determinant of the matrix

$$
\tilde{D}(b_0) \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ U_1 & U_2 & \cdots & U_{p-1} & U_p \\ V_2 & V_3 & \cdots & V_p & V_1 \end{pmatrix} = \tilde{D}(b_0) \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 \\ U_1 & U_2 & \cdots & U_{t-1} & U_t & \cdots & -U_2 & -U_1 & 0 \\ V_2 & V_3 & \cdots & V_t & -V_t & \cdots & -V_1 & 0 & V_1 \end{pmatrix},
$$

where $U_j, V_j$ are given by (10.2) and (10.3).

**Lemma 1** (On the "circulant" matrix) *Let $M$ be the last matrix in (10.6), where $U_j$ and $V_j$ are arbitrary column vectors of height $t$. Then*

$$
|\det(M)| = (2t + 1)|\det(U_1, \ldots, U_t) \cdot \det(V_1, \ldots, V_t)|.
$$

**Proof:** We operate on columns of $M$ as follows:

1) Add the column containing $U_j$ to the column containing $-U_j$ for $j = 1, \ldots, t$.

We obtain

$$
\begin{pmatrix}
\text{first } t \text{ columns} & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 1 \\
\text{as in } M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-V_t - V_{t-1} & V_t - V_{t-2} & V_{t-1} - V_{t-3} & \cdots & V_4 - V_2 & V_3 - V_1 & V_2 & V_1
\end{pmatrix}
$$

2) Add the $j$th column to the $(j-2)$nd consecutively for $j = 2t + 1, 2t, 2t-1, \ldots, t + 3$, i.e. we begin with the last column. We get

$$
\begin{pmatrix}
\text{first } t \text{ columns} & t + 1 & t & t - 1 & \cdots & 4 & 3 & 2 & 1 \\
\text{as in } M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-V_t & V_t & V_{t-1} & \cdots & V_4 & V_3 & V_2 & V_1
\end{pmatrix}
$$

3) Adding the $(t + 2)$nd column to the $(t + 1)$st, we get

$$
M' := \begin{pmatrix}
\text{first } t \text{ columns} & 2t + 1 & t & t - 1 & \cdots & 4 & 3 & 2 & 1 \\
\text{as in } M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & V_t & V_{t-1} & \cdots & V_4 & V_3 & V_2 & V_1
\end{pmatrix}
$$

From the above it follows that $\det M = \det M'$. Now we apply to $\det M'$ the Laplace formula with respect to the $(t + 1)$st column. We get

$$
|\det M'| = (2t + 1)|\det(U_1 U_2 \cdots U_t 0 0 \cdots 0 0 V_t V_{t-1} \cdots V_2 V_1)|.
$$

Hence the lemma follows. $\square$
Consequently, by (10.4), (10.5) and Lemma 1, we get

$$\text{covol}(\Lambda'_2) = p \tilde{R}_2(F_0) \tilde{R}_2(F_p) \tilde{R}_2(F_1) = p \tilde{R}_2(F_0) \tilde{R}_2(F_1)^2,$$  (10.5)

since the isomorphic fields $F_1$ and $F_p$ have equal second regulators, and $2t + 1 = p$. Thus we have proved

**Theorem 2** If $F$ is a Galois extension of $\mathbb{Q}$ with the dihedral Galois group $D_{2p}$, where $p$ is an odd prime, and if $F$ is not totally real and satisfies $w_2(F) = 24$, then

$$\tilde{R}_2(F) = \frac{p}{Q_2(F)} \tilde{R}_2(F_0) \tilde{R}_2(F_1)^2 \quad \text{for some} \quad Q_2(F) \in \mathbb{N},$$

where $F_0$ is the unique quadratic subfield of $F$ and $F_1$ is a subfield of degree $p$.

**Corollary 2** In the notation of Theorem 2, assume that Conjecture 1 holds for all fields in question and that they satisfy $w_2(\cdot) = 24$. Then

$$k_2(F) = \frac{Q_2(F)}{4p} k_2(F_0) k_2(F_1)^2 \quad \text{for some} \quad Q_2(F) \in \mathbb{N}.$$

11. The case of the alternating Galois group $A_4$ and $s = -1$

Let $F$ be a complex field Galois over $\mathbb{Q}$ with the Galois group $G = A_4$. We can assume that $\sigma := (12)(34)$ is complex conjugation. Then $F_6 := F^\sigma$ is the maximal real subfield of $F$ of degree 6 over $\mathbb{Q}$ and $F_4 := F^{(234)}$, $F'_4 := F^{(124)}$ are isomorphic but not complex conjugate subfields of $F$ of degree 4 over $\mathbb{Q}$. Let $\tau := (13)(24)$. Then $F_3 := F^{(\sigma, \tau)}$ is a totally real cubic cyclic subfield of $F_6$.

We have $r_2(F_6) = r_2(F_4) = r_2(F'_4) = 2$ and $r_2(F) = 6$. Let $\rho := (123)$. One can verify that the complex places for these fields are

- $\rho$ and $\rho^{-1}$ for $F_6$,
- $\text{id}$ and $\tau$ for $F_4$ and $F'_4$,
- $\text{id}, \tau, \rho, (142), \rho^{-1}$ and $(234)$ for $F$.

Let $b_1, b_2 \in \mathcal{B}(F_6)$, $b_3, b_4 \in \mathcal{B}(F_4)$, and $b_5, b_6 \in \mathcal{B}(F'_4)$ define the lattices $\Lambda_2(F_6)$, $\Lambda_2(F_4)$, and $\Lambda_2(F'_4)$, respectively. Then

$$\tilde{R}_2(F_6) = |\det \left( \begin{array}{c} \mathbb{D}(b_1) \\ \mathbb{D}(b_2) \end{array} \right) |, \tilde{R}_2(F_4) = |\det \left( \begin{array}{c} \mathbb{D}(b_3) \\ \mathbb{D}(b_4) \end{array} \right) |, \text{ and } \tilde{R}_2(F'_4) = |\det \left( \begin{array}{c} \mathbb{D}(b_5) \\ \mathbb{D}(b_6) \end{array} \right) |.$$

Note added on January 27, 2013: Zhou Haiyan proved recently that in the case $p = 3$ we have $Q_2(F) = 1, 3, 9$ or 27, see [24].
Since the elements \( b_j, j = 1, \ldots, 6 \), belong to \( B(F) \), they define a sublattice \( \Lambda'_2(F) \) of the lattice \( \Lambda_2(F) \). We have

\[
\text{covol}(\Lambda'_2(F)) = \left| \det \begin{pmatrix}
0 & 0 & \tilde{D}(b_3) & \tilde{D}(b_4) & \tilde{D}(b_5) & \tilde{D}(b_6) \\
0 & \tilde{D}(\rho b_1) & \tilde{D}(\rho b_2) & \tilde{D}(\tau b_3) & \tilde{D}(\tau b_4) & \tilde{D}(\tau b_5) \\
-\tilde{D}(b_1) & -\tilde{D}(b_2) & -\tilde{D}(b_3) & -\tilde{D}(b_4) & -\tilde{D}(b_5) & -\tilde{D}(b_6) \\
-\tilde{D}(\rho^{-1} b_1) & -\tilde{D}(\rho^{-1} b_2) & -\tilde{D}(b_3) & -\tilde{D}(b_4) & -\tilde{D}(b_5) & -\tilde{D}(b_6)
\end{pmatrix} \right|.
\]

After computing this determinant we get

\[
\text{covol}(\Lambda'_2(F)) = 4\tilde{R}_2(F_6)\tilde{R}_2(F_4)\tilde{R}_2(F'_4) = 4\tilde{R}_2(F_6)\tilde{R}_2(F_4)^2,
\] (11.1)

since isomorphic fields have equal second regulators.

Let \( Q_2(F) \) be the index of \( \Lambda'_2(F) \) in \( \Lambda_2(F) \). Then

\[
\text{covol}(\Lambda'_2(F)) = Q_2(F)\text{covol}(\Lambda_2(F)) = Q_2(F)\tilde{R}_2(F).
\]

Thus, by (11.1), we have proved

**Theorem 3** If \( F \) is a complex Galois extension of \( \mathbb{Q} \) with Galois group \( A_4 \), satisfying \( w_2(F) = 24 \), then in the above notation we have

\[
\tilde{R}_2(F) = \frac{4}{Q_2(F)}\tilde{R}_2(F_6)\tilde{R}_2(F_4)^2, \quad \text{for some } Q_2(F) \in \mathbb{N}.
\]

**Corollary 3** Assuming Conjecture 1 for the fields in question and that \( w_2(F) = 24 \), we have in the notation of Theorem 3 that

(i)  \[ k_2(F) = \frac{Q_2(F)}{16}k_2(F_6)k_2(F_4)^2 \quad \text{for some } Q_2(F) \in \mathbb{N}. \]

and

(ii)  \[ k_2(F_6) = \frac{\tilde{R}_2(F_4)}{2R_2(F_6)}k_2(F_3)k_2(F_4). \]

**Proof:** From the Brauer-Kuroda relation given in Example 4, we get

\[
\zeta_F \xi^2 = \xi_{F_6} \zeta_{F_4}^2.
\] (11.2)

Then, by Conjecture 1, we obtain

\[
4\tilde{R}_2(F)k_2(F) = \tilde{R}_2(F_6)\tilde{R}_2(F_4)^2k_2(F_6)k_2(F_4)^2
\]
and by Theorem 3, we get the first part of the corollary.

From the Brauer-Kuroda relation given in Example 2 applied to the Galois extension $F/F_3$ with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$, we get

$$\xi_F^2 \xi_{F_3}^2 = \xi_{F_6}^3,$$

(11.3)

since all three sextic subfields of $F$ are isomorphic.

Eliminating $\xi_F$ from (11.2) and (11.3), we obtain

$$\xi_{F_6}^2 = \xi_{F_3}^2 \xi_{F_4}^2.$$

Hence $\xi_{F_6} \xi = \xi_{F_3} \xi_{F_4}$, since for real $s > 1$ every Dedekind zeta function of a number field takes positive values.

Then, by Conjecture 1, we get

$$2\tilde{R}_2(F_6)k_2(F_6) = \tilde{R}_2(F_4)k_2(F_4)k_2(F_4),$$

(11.4)

since $\tilde{R}_2(F_3) = 1$. Thus the regulators $\tilde{R}_2(F_6)$ and $\tilde{R}_2(F_4)$ differ by a rational factor. From (11.4) we obtain the second part of the corollary. 

\[ \square \]

\section*{Part III. Numerical Examples}

\subsection*{12.1. Introduction}

For a number of Galois extensions $F$ of $\mathbb{Q}$ with dihedral Galois groups (i.e. of type $D_{2p}$, for some $p \geq 3$ not necessarily a prime), we compare the Brauer-Kuroda relation at $s = -1$ with the associated numerical regulator values.

We also consider biquadratic extensions and $A_4$-extensions of $\mathbb{Q}$, and find in the latter case a surprising coincidence of regulators, which result from different lattices of certain subfields.

Our set-up is the following: We assume that (a version of) the Lichtenbaum Conjecture in weight 2 holds and combine it with results of Bloch and Suslin which relate $K_3(F)$ to the Bloch group and the Borel regulator to the Bloch-Wigner dilogarithm function $D(z)$ given above.

Numerous experiments support and suggest the following formulation (cf. (4.1)):

$$\zeta^*_F(-1) \equiv \pm \frac{k_2(F)\tilde{R}_2(F)}{w_2(F)},$$

(12.1)

where we put

- $k_2(F) = \#K_2\mathcal{O}_F$, the order of the $K$-group of the number ring, as in the text above,
• $w_2(F)$ also as in the text above,

• $\widetilde{R}_2(F) = \text{covol}(\Lambda_2(F))$, where the lattice

$$\Lambda_2(F) = \left\{ \left( \frac{1}{\pi} D \circ \sigma_j(\xi) \right)_j | \xi \in \ker(\partial_2 : \mathbb{Z}[F] \rightarrow F^\times \backslash F^\times) \right\}$$

is generated by the images, under the normalized Bloch-Wigner dilogarithm function $\widetilde{D}(z) = \frac{1}{\pi} D(z)$, of all the elements in the Bloch group $\mathcal{B}(F)$, where the $\sigma_j : F \rightarrow \mathbb{C}$ represent $r_2$ complex embeddings of $F$ (one for each pair of complex conjugate ones).

12.2. What we compute

Our program, written with the computer algebra package GP-PARI [18], finds a set of elements in the Bloch group, and in many cases sufficiently many of them to generate a sublattice of full rank of the lattice $\Lambda_2(F)$ and hence we get a meaningful covolume. With these data we can form the quotient of covolumes of the regulator lattices for the subfields of a given field and compare it to the theoretical prediction in the text above. Moreover, we can compare it to the corresponding Dedekind zeta values at $s = -1$, which we can conveniently obtain via Magma [4], and get conjectural values for the $K_2$-orders of the number rings involved. For more details see e.g. [9].

Caveat: We will use the notation $\doteq$ below to indicate that two sides are equal up to several digits (usually we work with a minimum precision of 30 digits), but often under the further assumption that we have found the actual lattice generated by the Bloch group. Note that we cannot prove in any single case that the elements found will suffice to generate the full group. In fact for fields of degree $> 10$, say, this assumption is presumably overly optimistic.

12.3. Biquadratic cases

The Galois group of a biquadratic extension $F_4$ of signature $(0,2)$ is the Klein 4-group (note that this can be viewed as a dihedral group with 4 elements, i.e. as $D_{2p}$ with $p = 2$, but its behaviour is rather different from the other $D_{2p}$, $p > 2$).

In a nutshell, the experiments hint at a strong correlation: if we denote the two imaginary quadratic subfields of $F_4$ by $F_1$ and $F_2$, then we find for the regulator quotients (see Theorem 1)

$$\frac{2}{Q_2(F)} = \frac{\widetilde{R}_2(F_4)}{\widetilde{R}_2(F_1)\widetilde{R}_2(F_2)} \doteq 1 \quad \text{or} \quad \doteq 2.$$
Moreover, the pattern evolving is that this quotient seems to be equal to 1 if the "Birch-Tate-Lichtenbaum quotients"

\[
\frac{w_2(F)\xi_F(-1)}{\tilde{R}_2(F)}
\]

for \( F = F_1 \) and \( F = F_2 \) (which are conjecturally equal to the orders of the \( K \)-groups \( \#K_2(O_{F_1}) \) and \( \#K_2(O_{F_2}) \), respectively) are odd. Otherwise it is usually equal to 2, with few exceptions (for \( d_1 = -4 \) and \( d_2 = -51, -123, \) or \( d_2 = -132 \) we couldn’t find a better regulator lattice for \( F_4 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \) to make the index quotient 2).

We give an explicit example for \( F = \mathbb{Q}(\sqrt{-11}, \sqrt{-19}) \). With subfields \( F_1 = \mathbb{Q}(\sqrt{-11}) \) and \( F_2 = \mathbb{Q}(\sqrt{-19}) \), we get

\[
\tilde{R}_2(F) = 528.23, \quad \tilde{R}_2(F_1) = 16.59, \quad \text{and} \quad \tilde{R}_2(F_2) = 31.83,
\]

with quotient 1. Here we know (from Tate [21] and Skalba [20]) that both groups \( K_2\mathcal{O}_{F_j} (j=1,2) \) are trivial.

Similarly, if \( F_1 = \mathbb{Q}(\sqrt{-571}) \) (still considering \( F_2 = \mathbb{Q}(\sqrt{-19}) \)), where we know from [2] that \( K_2\mathcal{O}_{F_1} = \mathbb{Z}/5\mathbb{Z} \), then

\[
\tilde{R}_2(F) = 32315.473, \quad \tilde{R}_2(F_1) = 1015.004, \quad \text{and} \quad \tilde{R}_2(F_2) = 31.83,
\]

still leaving the quotient 1.

If we consider instead \( F_1 = \mathbb{Q}(\sqrt{-23}) \), for which \( K_2\mathcal{O}_{F_1} = \mathbb{Z}/2\mathbb{Z} \) has even order, we find

\[
\frac{\tilde{R}_2(F)}{\tilde{R}_2(F_1)} = \frac{2463.935}{38.695} \approx 2 \cdot \tilde{R}_2(F_2),
\]

whereas for \( F_1 = \mathbb{Q}(\sqrt{-51}) \), for which also \( K_2\mathcal{O}_{F_1} = \mathbb{Z}/2\mathbb{Z} \), we obtain

\[
\frac{\tilde{R}_2(F)}{\tilde{R}_2(F_1)} = \frac{2409.997}{75.695} = \tilde{R}_2(F_2).
\]

The former case is the more common one when considering that quotient in the case that at least one of the \( K_2 \)-orders is even, but the latter case also occurs for certain distinguished discriminants.

Let us recall that for an imaginary quadratic field \( F = \mathbb{Q}(\sqrt{-d}) \), the order of \( K_2(O_F) \) is odd iff \( d = 1, 2, p \) or \( 2p \), where \( p \equiv \pm 3 \pmod{8} \) is a prime, see [8].

12.4. Dihedral cases

The case \( p = 3 \). We looked at several number fields of the form \( F = \mathbb{Q}(\sqrt{3d}, \sqrt{-3}) \) which have Galois group \( D_6 \cong S_3 \). As a typical case, take \( d = 2 \) : we find, using
the notation $F_0 = \mathbb{Q}(\sqrt{-3})$ for the unique quadratic subfield and $F_1 = \mathbb{Q}(\sqrt[3]{d})$ for one of the three isomorphic cubic subfields of $F$:

$$\widetilde{R}_2(F) = \frac{3}{\varphi_2(F)} \approx 389.3591874/\pi^3 \approx 12.55743124,$$
$$\widetilde{R}_2(F_1) \approx 13.84967835/\pi \approx 4.408489539,$$
$$\widetilde{R}_2(F_0) \approx 2.029883212/\pi \approx 0.646131894,$$

from which we find for the quotient (see Theorem 2)

$$\frac{3}{\varphi_2(F)} = \frac{\widetilde{R}_2(F)}{\widetilde{R}_2(F_1)^2 \widetilde{R}_2(F_0)} \approx 1.$$ 

The corresponding quotient for the fields $F = \mathbb{Q}(\sqrt[3]{d}, \sqrt{-3})$ with squarefree $d < 50$ is either 3 (for $d = 17, 19, 22, 23, 33, 34, 37$) or 1 (for the remaining $d$). The elements found by the program which conjecturally generate the corresponding Bloch group are typically too complicated to write down here.

The case $p \geq 5$. For $5 \leq p \leq 14$, $p \neq 12, 13$, we considered one polynomial each defining a Galois extension $F_{2p}$ of $\mathbb{Q}$ with Galois group $D_{2p}$, as linked from the GP-PARI website.

- 1. The case $p = 5$. We consider the field $F_{10}$ defined by the polynomial

$$\sum_{i=0}^{10} c_i x^i$$

where $(c_i)_i = (1, 1, 2, -1, 10, -18, 20, -18, 12, -5, 1)$, of discriminant $-47^5$ and signature $(r_1, r_2) = (0, 5)$, which is Galois over $\mathbb{Q}$ with Galois group $D_{10}$.

Its (up to an isomorphism) unique degree 5 subfield $F_5$ can be described by

$$\sum_{i=0}^{5} c_i x^i$$

where $(c_i)_i = (-11, 5, -2, 7, -5, 1)$, and is of discriminant $47^2$ and signature (1,2).

Its unique degree 2 subfield $F_2$ is the imaginary quadratic field of discriminant $-47$.

Magma gives $\zeta_{F_{10}}^*(-1) \approx -3.75562$ and our program, written in GP-PARI, finds a conjectural dilogarithm regulator

$$\tilde{R}_2(F_{10}) \approx 13791.5413/\pi^5 \approx 45.06749724.$$ 

Hence, since $w_2(F_{10}) = 24$, we are led to

$$2\tilde{R}_2(F_{10}) \approx -w_2(F_{10})\zeta_{F_{10}}^*(-1).$$

For explicit (and short) elements in the Bloch group of $F_{10}$, see [9], §5.1.1.
Similarly, we get $\zeta^*_{F_5}(-1) \doteq -0.091823$ and $\zeta^*_{F_2}(-1) \doteq -3.09308$, while the corresponding (conjectural) regulators found are

$$
\tilde{R}_2(F_5) \doteq 10.8753/\pi^2 \doteq 1.101898268,
\tilde{R}_2(F_2) \doteq 116.606541/\pi \doteq 37.11701479,
$$

so that

$$2\tilde{R}_2(F_k) \doteq -w_2(F_k)\zeta^*_{F_k}(-1), \quad \text{for} \quad k = 2 \text{ or } 5.
$$

(Again, $w_2(F_k) = 24$ in both cases.)

As a consequence, we obtain (see Theorem 2)

$$
\frac{5}{Q_2(F_{10})} = \frac{\tilde{R}_2(F_{10})}{\tilde{R}_2(F_5)^2\tilde{R}_2(F_2)} \doteq 1,
$$

from which we should expect that $k_2(F) = k_2(F_5) = k_2(F_2) = 2$ (the latter identity of which has been shown, cf. [2]).

A slightly more interesting case is the $D_{10}$-field $F'_{10}$ of discriminant $-2^{15}11^8$ given by $x^{10} + 6x^8 + 21x^6 + 12x^4 - 28x^2 + 32$ and its subfields $F'_{5}$ and $F'_{2}$.

In this case we find

$$80\tilde{R}_2(F'_{10}) \doteq -w_2(F'_{10})\zeta^*_{F'_{10}}(-1)
$$

and

$$40\tilde{R}_2(F'_{5}) \doteq -w_2(F'_{5})\zeta^*_{F'_{5}}(-1),
$$

as well as

$$\tilde{R}_2(F'_{2}) \doteq -w_2(F'_{2})\zeta^*_{F'_{2}}(-1).
$$

This would suggest that $k_2(F'_{10}) = 80$, $k_2(F'_{5}) = 40$ and $k_2(F'_{2}) = 1$ (again, the latter identity is known; it has been proved long ago by Tate [21]).

Furthermore, this constitutes the first candidate of a $D_{10}$-field whose second regulator quotient is different from 1:

$$
\frac{5}{Q_2(F'_{10})} = \frac{\tilde{R}_2(F'_{10})}{\tilde{R}_2(F'_{2})\tilde{R}_2(F'_{5})^2} = 5.
$$

This example already arose a few years ago in discussion with A. Bartel who, in collaboration with de Smit, investigated related questions from a more elaborate point of view [1].
2. The case $p = 7$ : We consider the field $F_{14}$ of discriminant $-223^7$, given by the coefficients

$$(9,-87,353,-819,1301,-1618,1648,-1379,971,-566,276,-107,33,-7,1),$$

of its minimal polynomial and its subfields $F_7$ and $F_2$. Their respective zeta values at $-1$ are given by $\zeta_{F_{14}}^*(-1) \doteq -8687878.53919$, $\zeta_{F_7}^*(-1) \doteq 152.73946$ and $\zeta_{F_2}^*(-1) \doteq -25.861205$. We find for their tentative regulators the approximate values

$$\widetilde{R}_2(F_{14}) \doteq 18631911242.299834/\pi^7 \doteq 6168908.53919,$$

$$\widetilde{R}_2(F_7) \doteq 4371.583358/\pi^3 \doteq 140.9902711 \quad \text{and}$$

$$\widetilde{R}_2(F_2) \doteq 974.944466/\pi \doteq 310.334462,$$

which results in the formulas

$$2 \cdot 13^2 \widetilde{R}_2(F_{14}) \doteq -w_2(F_{14})\zeta_{F_{14}}^*(-1),$$

$$2 \cdot 13 \widetilde{R}_2(F_7) \doteq -w_2(F_7)\zeta_{F_7}^*(-1),$$

$$2 \widetilde{R}_2(F_2) \doteq -w_2(F_2)\zeta_{F_2}^*(-1).$$

From this we obtain (cf. Theorem 2)

$$\frac{7}{Q_2(F_{14})} = \frac{\widetilde{R}_2(F_{14})}{R_2(F_7)^2R_2(F_2)} \doteq 1.$$

Moreover, we expect that $k_2(F_{14}) = 2 \cdot 13^2$ and $k_2(F_7) = 2 \cdot 13$.

3. The case $p = 8$ : In this case, we get $w_2(F_r) = 48$ for $r \in \{4,8,16\}$ and $w_2(F_r) = 24$ for $r = 2$. Still the formulas agree with the ones for $p = 5$, i.e. putting

$$q(F_r) := w_2(F_r)\zeta_{F_r}^*(-1)/\widetilde{R}_2(F_r),$$

(which conjecturally agrees with $k_2(F_r)$), we find $q(F_r) = 2$ for all four fields $F_r$, $r|16$ ($r > 1$) in question.

4. The case $p = 9$ : Consider the field $F_{18}$ of degree 18 and discriminant $-2^{12} \cdot 107^9$. In this case we get $\widetilde{R}_2(F_{18}) \doteq 1507145405664649.50892/\pi^9 \doteq 50559910878.40792$ and $\zeta_{F_{18}}^*(-1) \doteq -227519598952.835$, so that

$$-w_2(F_{18})\zeta_{F_{18}}^*(-1)/\widetilde{R}_2(F_{18}) \doteq 2 \cdot 3^3,$$

and the corresponding quotients $q(F_r)$ for the subfields of $F_{18}$ of degree 9,6,3 and 2 are conjecturally given by $2^2 \cdot 3$, $2^2 \cdot 3$, $2^2$ and 3, respectively.
5. The case $p = 10$ : For the field $F_{20}$ of discriminant $2^{20} \cdot 47^{10}$, we find a regulator

$$\widetilde{R}_2(F_{20}) \doteq 601489603356159.6482/\pi^{10} \doteq 6422873936.691262$$

and a zeta value $\zeta_{F_{20}}^*(-1) \doteq 1.49867 \cdot 10^{10}$, resulting in

$$w_2(F_{20}) \zeta_{F_{20}}^*(-1)/\widetilde{R}_2(F_{20}) \doteq 2^3 \cdot 7.$$  

The same factor $2^3 \cdot 7$ occurs for the corresponding quotient for its subfield $F_4$ of degree 4, but apparently not for the other degrees, for which the quotient becomes equal to 2.

Hence we expect $k_2(F_{20}) = k_2(F_4) = 2^3 \cdot 7$, so the non-trivial part of $K_2(\mathcal{O}_{F_{20}})$ should be induced from $K_2(\mathcal{O}_F)$.

6. The case $p = 11$ : This case is for the field $F_{22}$ of discriminant $167^{11}$ and its subfields $F_{11}$ and $F_2$. It is very similar to the one for $p = 5$ above and yields the analogous formula

$$\widetilde{2R}_2(F_r) \doteq \pm w_2(F_r) \zeta_{F_r}^*(-1), \quad r = 2, 11 \text{ or } 22$$

with respective regulator values

$$\doteq 77805299818597772.5399, \quad 9899632.6249 \quad \text{and} \quad 793.9095.$$  

Hence the regulator quotient equals 1.

7. The case $p = 14$ : The corresponding quotients for Galois extension $F_{28}$ with Galois group $D_{28}$ and discriminant $2^{28} \cdot 101^{14}$ and for its subfields $F_k$ of degree $k$ are as follows: for $F_{28}, F_{14}, F_7, F_4$ and $F_2$, we find $\zeta(F_k) = 2^{12} \cdot 19, 2^6 \cdot 2^4 \cdot 19 \text{ and } 1$, respectively.

12.5. The alternating group $A_4$

We considered several complex fields with Galois group $A_4$, the alternating group on 4 letters. It turns out that in each case there is a degree 4 and a degree 6 subfield $F_4$ and $F_6$ (both with $r_2 = 2$) which have exactly the same regulator. The fact that their regulators differ by a rational factor is a consequence of the Brauer-Kuroda relations and Conjecture 1, see (11.4).

More precisely, one seems to have

$$\zeta_{F_3}(-1) \cdot \widetilde{R}_2(F_6) = -2 \zeta_{F_6}^*(-1),$$
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where $F_3$ is the totally real field of degree 3 which is a subfield of $F_6$ and

$$
\zeta_{Q}(-1) \cdot \tilde{R}_2(F_4) = 2\zeta^*_{F_4}(-1).
$$

Nevertheless, although the covolumes agree, the actual lattices do not – instead, e.g. the elements of $\Lambda_2(F_6)$ arise from taking $\mathbb{Q}$-linear combinations (with very small denominators) of the rows of $\Lambda_2(F_4)$, where different rows correspond to different embeddings (of the Bloch elements whose dilogarithm values generate the lattice).

For example, $F_{12}$ given by $x^{12} + 6x^{10} - 11x^8 + 42x^6 - 30x^4 + 40x^2 + 1$ has the subfields $F_6$ defined by $x^6 - 4x^5 + 4x^4 - 4x^3 + 20x^2 - 16x - 8$ and $F_4$ defined by $x^4 - 4x^3 + 14x^2 - 28x + 21$. They have both dilogarithmic regulator $\tilde{R}_2(F_r) \doteq 1127.145385/\pi^2 \doteq 114.203704545 (r = 4, 6)$, while their special values at $-1$ are given by $\zeta^*_{F_6}(-1) \doteq 2.7191358$ and $\zeta_{F_4}(-1) \doteq 4.7584876$, respectively. We thus get the quotients $\tilde{R}_2(F_6)/\zeta^*_{F_6}(-1) \doteq 42$ and $\tilde{R}_2(F_4)/\zeta^*_{F_4}(-1) \doteq 24$, which matches the above displayed formulas because $\zeta_{F_3}(-1) = -1/21$.

The lattice $\Lambda_2(F_4)$ is generated by the two (column) vectors $(27.115...;14.087...)$ and $(-1.9487...;40.556...)$, each vector being indexed by the two complex places of $F_4$, while $\Lambda_2(F_6)$, also equipped with two complex places, is generated by $(21.2526...;19.3039...)$ and $(34.2806...;-21.8981...)$. The lattices are (numerically) related as follows:

$$
\Lambda_2(F_6) \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \Lambda_2(F_4).
$$

The entries for both lattices arise from entries of (a specific column $c_{12} = (9.9747...;-3.1379...;-16.1659...;0.9743...;20.278...;31.227...)$ of) the rank 6 lattice for $F_{12}$ as linear combinations with coefficients of modulus $\leq 2$ (e.g. the first entry 27.115... for $F_4$ equals $c_{12}[1] + c_{12}[2] + c_{12}[5]$).

This suggests a kind of symmetrization of the Bloch groups via descent on number fields (note that this should be a more general phenomenon than the Galois descent which is known for the associated $K$-groups, as $F_6$ is not a Galois extension).

We find similar such lattice correspondences for other $A_4$-extensions.

12.6. The symmetric group $S_4$

Moreover, we obtain a further relationship for certain $S_4$-extensions: for example, for the Galois closure (of degree 24 and Galois group $S_4(6d) = [2^2]S_3$ in GAP
notation [13]) of the field with minimal polynomial $x^6 - x^4 - x^3 - x^2 + 1$, we find two subfields of signature (0,4) and (4,4), respectively, whose regulators $\approx 788598.76/\pi^4$ agree while their lattices are different but can be transformed into each other using integer matrices, in a more complicated manner than for the previous example.

An observation: In all the examples above the regulator quotients $\frac{m}{Q_2(F)}$, when computed, are integers, where $m = 2, p$ or 4 for the Galois groups $\mathbb{Z}/2 \times \mathbb{Z}/2, D_{2p}$ and $A_4$, respectively. Thus $Q_2(F)$ takes only values dividing $m$, and in these examples it holds that $m|\#G$. Is this true in general?

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