Values of Zeta Functions and Their Applications

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Zeta functions of various sorts are all-pervasive objects in modern number theory, and an ever-recurring theme is the role played by their special values at integral arguments, which are linked in mysterious ways to the underlying geometry and often seem to dictate the most important properties of the objects to which the zeta functions are associated. It is this latter property to which the word "applications" in the title refers. In this article we will give a highly idiosyncratic and prejudiced tour of a number of these "applications," making no attempt to be systematic, but only to give a feel for some of the ways in which special values of zeta functions interrelate with other interesting mathematical questions. The prototypical zeta function is "Riemann's" $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and the prototypical result on special values is the theorem that

$$\zeta(k) \;=\; \text{rational number} \times \pi^k \qquad (k > 0 \;\text{ even}), \tag{1}$$

which Euler proved in 1735 and of which we will give a short proof in Section 1. (The "applications" in this case are the role which the rational numbers occurring on the right-hand side of this formula play in the theory of cyclotomic fields, in the construction of p-adic zeta functions, and in the investigation of Fermat's Last Theorem.) In Section 2 we list some of the characteristic properties of nice (= motivic) zeta functions and recall the definition of critical points, the arguments at which the value of the zeta function is expected to be given by a formula of the same general form as (1). We discuss applications of such critical values to Diophantine equations (Birch–Swinnerton–Dyer conjecture) in Section 3 and applications of the theory of periods of modular forms (which are critical values of Hecke L-series) in Section 4, while Section 5 is devoted to connections between values at non-critical integral points of Dedekind zeta functions of number fields, algebraic K-theory, and the classical polylogarithm function. In Section 6 we return to critical values to give one or two examples of special values of zeta functions at the symmetry point of their functional equation.

In Section 7 we turn to another theme, the relation of special zeta values to invariants of moduli spaces of various kinds. Finally, in Sections 8–9 we will discuss another kind of zeta value, the multiple zeta values

$$\zeta(k_1, \ldots, k_r) = \sum_{0<n_1<\ldots<n_r} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}} \qquad (k_i \geq 1,\; k_r \geq 2). \tag{2}$$

These numbers, whose investigation is only now beginning, seem to be connected with several fascinating topics, ranging from mixed motives to the knot invariants of Vassiliev–Kontsevich.

1. Elementary proofs of (1)

We start, for fun, with an ultra-simple proof of Euler's formula $\zeta(2) = \pi^2/6$ discovered a few years ago by E. Calabi. Expanding $(1 - x^2y^2)^{-1}$ in a geometric series and integrating termwise gives

$$\int\int_S (1 - x^2y^2)^{-1}\, dxdy = 1^{-2} + 3^{-2} + 5^{-2} + \cdots = \left(1 - \frac{1}{4}\right)\zeta(2)\,,$$

where S is the square $[0,1] \times [0,1]$. But the clever substitution $(x,y) = \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u}\right)$ has Jacobian precisely $1 - x^2y^2$ and maps the open triangle $T = \{u, v > 0,\ u + v < \pi/2\}$ bijectively to the interior of S, so $\int\int_S (1 - x^2y^2)^{-1}\, dxdy = \int\int_T dudv = \pi^2/8$.

Calabi found a similar proof of (1) for all k, but we give a different proof which will tie up later with the multiple zeta values (2) (in the case $r = 2$). We start with $k = 4$. Define

$$f(m,n) = \frac{1}{mn^3} + \frac{1}{2m^2n^2} + \frac{1}{m^3n}\,.$$

Then one checks directly that

$$f(m,n) - f(m+n,n) - f(m,m+n) = \frac{1}{m^2n^2}$$

and hence, summing over all m, $n > 0$, that

$$\zeta(2)^2 = \left(\sum_{m,n>0} - \sum_{m>n>0} - \sum_{n>m>0}\right) f(m,n) = \sum_{n>0} f(n,n) = \frac{5}{2}\,\zeta(4)\,.$$

Thus the formula $\zeta(2) = \pi^2/6$ implies $\zeta(4) = \pi^4/90$. Similarly, for larger k we set

$$f(m,n) = \frac{1}{mn^{k-1}} + \frac{1}{2}\sum_{r=2}^{k-2} \frac{1}{m^r n^{k-r}} + \frac{1}{m^{k-1}n}$$

and check easily that

$$f(m,n) - f(m+n,n) - f(m,m+n) = \sum_{0<j<k,\ j \text{ even}} \frac{1}{m^j n^{k-j}}\,,$$

so the same proof gives

$$\sum_{0<j<k,\ j \text{ even}} \zeta(j)\,\zeta(k-j) = \frac{k+1}{2}\zeta(k) \qquad (k \geq 4 \text{ even})$$

and hence (1) by induction.

2. Properties of motivic zeta functions

The classical zeta functions with arithmetically nice properties arise from *algebraic number theory* (Riemann zeta function, Dirichlet L-series, Dedekind zeta functions of algebraic number fields, L-series of Hecke grossencharacters), from *representation theory and the theory of automorphic forms* (Hecke L-series and Rankin–Selberg L-functions of classical modular forms and their generalizations to Hilbert and Siegel modular forms and to automorphic forms on more complicated groups), and from *algebraic geometry* (Hasse–Weil zeta functions of varieties over number fields). From a modern point of view, all of these zeta functions are, or are conjectured to be, special cases of the notion of motivic L-functions. We do not attempt to define this, but merely list some of the "nice arithmetic properties," illustrating each for the simplest case $L(s) = \zeta(s)$.

(i) *Algebraicity.* The L-functions in question should be Dirichlet series $L(s) = \sum a_n n^{-s}$ with all a_n integers in a fixed algebraic number field. For $L(s) = \zeta(s)$, $a_n = 1$ for all n.

(ii) *Euler product.* We should have $L(s) = \prod \phi_p(p^{-s})$, where the product runs over all prime numbers p and the $\phi_p(X)$ are rational functions of bounded degree with $\phi_p(0) = 1$. In particular, $n \mapsto a_n$ is multiplicative. For $L(s) = \zeta(s)$, $\phi_p(X) = 1/(1-X)$ for all p.

(iii) *Functional equation.* There is a "gamma factor" $\gamma(s)$ (equal to an exponential function A^s times a finite product of terms $\Gamma(\frac{1}{2}(s+m))$ with $m \in \mathbb{Z}$) such that the product $L^*(s) = \gamma(s)L(s)$ has a meromorphic continuation with only finitely many poles in $\mathbb{C}$ and satisfies $L^*(s) = wL^*(h-s)$ for some integer $h > 0$ and sign $w = \pm 1$. (Sometimes the functional equation relates $L(s)$ to $L_1(h-s)$ for a different L-series $L_1(s)$, and w is only a complex number of absolute value 1, but we will not look at such cases.) For $L(s) = \zeta(s)$, $\gamma(s) = \pi^{-s/2}\Gamma(s/2)$, $h = 1$, and $w = +1$; the functional equation in this case was discovered by Euler in 1749 and proved by Riemann in 1859.

(iv) *Special values.* Given the properties in (iii), we call an integer m *critical* (following Deligne) if neither m nor $h-m$ is a pole of $\gamma(s)$. Then the

corresponding *critical value* $L(m)$ should have the form

$$L(m) = A(m)\,\Omega(m) \qquad (m \text{ critical}) \tag{3}$$

where $\Omega(m)$ is a predictable period (the integral over some algebraic cycle of an algebraic differential form defined over a number field) and $A(m)$ an algebraic number belonging to a predictable number field. For $L(s) = \zeta(s)$, the critical integers are $m = k$ and $m = 1 - k$, where k is positive and even, and the corresponding values are given by (1) and by the equivalent formula $\zeta(1-k) \in \mathbb{Q}$.

Of course, some of these properties are only conjectural in general, and there are other properties shared (or conjectured to be shared) by these zeta functions which we have not mentioned, such as the Riemann hypothesis or p-adic interpolation properties of the algebraic numbers $A(m)$ in (3). Another property is that if the critical point m in (3) is the symmetry point of the functional equation (i.e., if h is even and $m = h/2$), and if $w = +1$, then the number $A(m)$ in (3) is supposed to be—possibly up to some simple, predictable factors — a perfect square in the number field in which it naturally lies. Examples of this phenomenon will be given in Section 6. First, however, we will give in the next three sections "applications" of special values of zeta functions from each of these three domains mentioned above, though in the reverse order.

3. First "application": Diophantine equations

The simplest zeta functions from algebraic geometry are the Hasse–Weil zeta functions of elliptic curves over $\mathbb{Q}$. If

$$E : y^2 = x^3 - 3Ax + 2B \qquad (A, B \in \mathbb{Z},\ A^3 \neq B^2) \tag{4}$$

is such a curve, then $L(E, s)$ is defined by an Euler product as in (ii) above with $\phi_p(X)^{-1}$ a polynomial of degree ≤ 2 for each prime number p, equal to $1 - (p - N_p)X + pX^2$ for $p \nmid 6(A^3 - B^2)$, where N_p is the number of solutions of (4) over $\mathbb{Z}/p\mathbb{Z}$. It is known in some cases, and conjectured in general ("Taniyama–Weil conjecture"), that $L(E, s)$ satisfies a functional equation as in (iii) with $h = 2$ and $\gamma(s) = (2\pi)^{-s} N_E^{s/2} \Gamma(s)$ for a certain natural number N_E, the conductor of E. Thus $\gamma(s)$ has poles at all non-positive integers and $\gamma(h - s)$ at all integers ≥ 2, so the only critical point of $L(E, s)$ is $s = 1$.

The conjecture of Birch and Swinnerton–Dyer in its weakest form says that the value of the L-function at the critical point is non-zero if and

only if the group $E(\mathbb{Q})$ of rational solutions of (4) (including the point at infinity) is finite, and thus gives a criterion for the solvability in rational numbers of a large class of Diophantine equations. For instance, a more than 1000-year old problem is to determine which natural numbers are "congruent," i.e., are the area of a right-angled triangle with rational sides or, equivalently, the common difference of three rational squares in arithmetic progression. The Birch–Swinnerton–Dyer conjecture leads to the conjectural answer (Tunnell) that an odd (resp. even) square-free number m is congruent if and only if it has equally many representations as $a^2+b^2+c^2$ with $a+b$ (resp. a) congruent to 0 modulo 8 and to 4 modulo 8. (This condition is satisfied by any $m \equiv 5, 6, 7 \pmod 8$ and by $m = 34$, 41, 65, 137,....) The "only if" part of this assertion is in fact known to be true, as a consequence of the theorem of Coates–Wiles and a theorem of Waldspurger.

The full version of the Birch–Swinnerton–Dyer conjecture relates the behavior of $L(E,s)$ at the critical point to the arithmetic invariants of E. If $L(E,1) \neq 0$, then, in accordance with the general expectation (3)), it predicts that

$$L(E,1) = c\frac{S}{|E(\mathbb{Q})|^2}\Omega, \quad \Omega = \int_{x^3-3Ax+2B\geq 0} \frac{dx}{\sqrt{x^3-3Ax+2B}}, \qquad (5)$$

where c is an easy rational number and S a difficult integer (the order of the Shafarevich–Tate group of E over $\mathbb{Q}$), while if $L(E,s)$ vanishes at $s=1$, then the order of vanishing there should be equal to r, the rank of the finitely generated abelian group $E(\mathbb{Q})$, and the rth derivative at $s=1$ should be related in a specific way to the size of the rational solutions of (4). All of this makes sense, of course, only for E satisfying the Taniyama–Weil conjecture, since otherwise not even the analytic continuation of $L(E,s)$ to $s=1$ is known, and under that assumption some partial results are known. In particular, we know that if $L(E,1) \neq 0$ then $E(\mathbb{Q})$ is indeed finite and formula (5) is essentially true (Kolyvagin), while if $L(E,s)$ has a simple zero at $s=1$ then $E(\mathbb{Q})$ is indeed infinite and $L'(E,1)$ is related to the size of a certain non-trivial rational solution of (4) (Gross–Zagier). These partial results often suffice to determine whether or not $E(\mathbb{Q})$ is finite, since if the order of $L(E,s)$ at $s=1$ is less than 2 they decide the question and if it is ≥ 2 then the rank of $E(\mathbb{Q})$ is (presumably) at least 2 and we can often find a rational solution of infinite order by a straightforward search. For instance, the congruence nature of all numbers ≤ 2000 was decided in a few minutes of computer time (Kramarz), whereas previously each individual case represented a non-trivial research problem.

4. Second "application": periods of modular forms

The simplest zeta functions from the theory of automorphic forms are the Hecke L-series of holomorphic modular forms on $SL(2,\mathbb{Z})$. Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ ($z \in \mathfrak{H} =$ upper half-plane) be such a form, of (necessarily even) weight $k > 0$, i.e., $f(-1/z) = z^k f(z)$, and $L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$ the associated L-series. We assume that f is a normalized eigenform of all Hecke operators T_n, i.e., $T_n f = a_n f$ for all n. Then $L(f,s)$ has the properties (i)–(iii), with the rational function $\phi_p(X)$ in (ii) of the form $1/(1 - a_p X + pX^2)$ and $\gamma(s) = (2\pi)^{-s}\Gamma(s)$, $h = k$, $w = (-1)^{k/2}$ in (iii). The critical points are thus the finitely many points $m = 1, 2, \ldots, k-1$, and for these points (3) is indeed true with $A(m)$ an element of the field generated by the coefficients $a(n)$ and $\Omega(m)$ the product of π^m with one of two basic periods $\Omega_{\pm}(f)$ (depending on the parity of m) defined by suitable integrals of f over a closed geodesic in $\mathfrak{H}/SL(2,\mathbb{Z})$. If f is an Eisenstein series, then $L(f,s) = \zeta(s)\zeta(s-k+1)$, so one usually restricts to the case of cusp forms ($a_0 = 0$), for which the special values $L^*(f,m)$ ($1 \le m \le k-1$) have a particularly simple description as $\int_0^{\infty} f(it)\, t^{m-1}\, dt$.

The Eichler–Shimura–Manin theory of periods says that the map assigning to each cusp form its $(k-1)$-tuple of critical values is an injective map from the space of cusp forms to $\mathbb{C}^{k-1}$, and gives a partial description of the image. The group $GL(2,\mathbb{C})$ acts on $\mathbb{C}^{k-1}$ in a natural way (think of $\mathbb{C}^{k-1}$ as the space of binary forms of degree $k-2$), and if $g \in GL(2,\mathbb{C})$ is a matrix of trace t and determinant n, then the trace of g on $\mathbb{C}^{k-1}$ is given by a simple polynomial $p_k(t,n)$ (e.g., $p_2(t,n) = 1$, $p_4(t,n) = t^2 - n$). Extending ideas of Manin, one can find a splitting of $\mathbb{C}^{k-1}$, roughly into the image of the space of cusp forms and a complementary subspace, and for each $n \in \mathbb{N}$ a finite formal linear combination of elements $g \in M(2,\mathbb{Z})$ of determinant $\pm n$ whose operation on the cusp form part corresponds to the action of the nth Hecke operator T_n and whose action on the complementary subspace has trace zero. In this way one immediately obtains an explicit formula for the trace of T_n on the space of modular forms of weight k on $SL(2,\mathbb{Z})$, as a finite linear combination of expressions $p_k(t, \pm n)$, e.g., for $n = 2$ the trace is $\frac{1}{2}[p_k(3,2) - p_k(2,2) - 2p_k(1,2) - p_k(0,2) + p_k(1,-2)]$. This is of course the well-known Eichler–Selberg trace formula, but with a far easier and more elementary proof than the usual one. (Details will appear later.)

As a second application of special values of L-series of modular forms, now for congruence subgroups of $SL(2,\mathbb{Z})$, we mention the celebrated result of Waldspurger (1981) which expresses the Fourier coefficients of the Shimura lift of a Hecke eigenform of even weight k (this is another modular form, of half-integral weight $(k+1)/2$) as essentially the square-roots of

the values at the central critical point $s = k/2$ of the L-series of the form and its twists by quadratic characters.

5. Third "application": algebraic K-theory and polylogarithms

The simplest zeta functions from algebraic number theory are the Dedekind zeta functions of algebraic number fields. Let $F = \mathbb{Q}(\alpha)$ where α is the root of an irreducible monic polynomial $P(T)$ of degree n with integral coefficients. Then the Dedekind zeta function $\zeta_F(s)$ of F is given by an Euler product as in (ii) where each $\phi_p(X)$ is the reciprocal of a polynomial of degree at most n. (More precisely, for p not dividing the discriminant D of $P(T)$, $\phi_p(X)^{-1} = (1 - X^{n_1}) \cdots (1 - X^{n_r})$ where $n_1, \ldots, n_r$ are the degrees of the irreducible factors of the reduction of P modulo p.) Thus (i) and (ii) hold, as does (iii) with $h = 1$, $w = +1$ and $\gamma(s) = A^s \Gamma(s/2)^{r_1} \Gamma(s)^{r_2}$ for a certain $A > 0$, r_1 and r_2 being the number of linear and quadratic irreducible factors into which $P(T)$ decomposes over $\mathbb{R}$. Thus either $\gamma(s)$ or $\gamma(1 - s)$ has a pole at every integer if r_2 is positive, so there are no critical values in this case. If $r_2 = 0$ (totally real case), then we have the same critical points k and $1 - k$ ($k > 0$ even) as in the case of the Riemann zeta function, corresponding to $F = \mathbb{Q}$, and the Siegel–Klingen theorem gives the analogue of (1): $\zeta_F(1 - k)$ is rational and $\zeta_F(k)$ is a rational multiple of $\pi^{nk}/\sqrt{D}$ for every positive even integer k.

We now look for the first time also at non-critical values. Essentially nothing is known about the arithmetic nature of these numbers, even for $F = \mathbb{Q}$, but a theorem of Borel relates them to the algebraic K-theory of the field F. More precisely, algebraic K-theory associates to F a finitely generated abelian group $K_i(F)$ for each $i > 1$. Borel showed that the rank of this group is 0 if i is even and $r_1 + r_2$ or r_2, depending whether m is odd or even, if $i = 2m - 1$, and that in the latter case $K_i(F)/$ (torsion) is canonically isomorphic to a certain lattice ("regulator lattice") in a Euclidean space of the same dimension whose covolume is a simple multiple of the number $\zeta_F(m)$. This relationship with zeta values, which for many fields can be computed numerically, made it possible to formulate and provide extensive numerical evidence for a conjecture which completely describes the higher K-groups and the regulator lattices of arbitrary number fields. Roughly, the points of the regulator lattice should have coordinates which are finite rational linear combinations of values of the classical mth *polylogarithm function* $Li_m(z) = \sum_{n=1}^{\infty} z^n/n^m$ at arguments in F. A consequence of this conjecture is that the value of $\zeta_F(m)$ for any number field F and any integer $m > 1$ can be expressed as a simple multiple of the determinant of a matrix (of size $r_1 + r_2$ or r_2, depending on the parity of m) whose

coefficients are linear combinations of polylogarithms of numbers in F, a typical formula being

$$\zeta_{\mathbb{Q}(\sqrt{5})}(3) = \frac{24}{25\sqrt{5}}\, Li_3(1)\,\Big[Li_3(\alpha) + Li_3(-\alpha) + \frac{1}{2}\log^3(\alpha) - \frac{\pi^2}{6}\log(\alpha)\Big]\,, \quad \alpha = \frac{\sqrt{5}-1}{2}\,.$$

The case $m = 2$ of the conjecture follows from work of Bloch, Suslin and Merkuriev about algebraic K-theory (specifically, about K_3 of fields), and the case $m = 3$ has been proved in beautiful recent work of A. Goncharov.

One also expects that the values of partial zeta functions associated to ideal classes in a number field F would be related to polylogarithms with arguments in the Hilbert class field of F. An exciting special case is the one where F is imaginary quadratic–the only case besides $\mathbb{Q}$ where the class field can be constructed effectively by analytic functions–would be the existence of a *generalized Kronecker limit formula*. Recall that the classical Kronecker limit formula says that the value (after subtracting the pole and taking the limit) at $s = 1$ of the Eisenstein series

$$E(z,s) = {\sum_{(p,q)\in\mathbb{Z}}}' \frac{\Im(z)^s}{|pz+q|^{2s}} \qquad (z \in \mathbb{C},\ \Im(z) > 0,\ \Re(s) > 1)$$

is essentially equal to the logarithm of the absolute value of the classical discriminant function $\Delta(z) = e^{2\pi i z}\prod(1-e^{2\pi i n z})^{24}$. This and the theory of complex multiplication then imply that the value at $s = 1$ of the difference of $E(z,s)$ for two quadratic irrationalities z having the same discriminant is, up to a simple factor, the logarithm of a unit in the class field of the quadratic field in which they lie. The generalization would express the difference of two values of $E(z,m)$ for an integer $m > 1$ similarly in terms of polylogarithms of order m of numbers in the class field. A typical formula here is

$$E\big(\frac{1+i\sqrt{23}}{2},\,2\big) - E_2\big(\frac{1+i\sqrt{23}}{4},\,2\big) \;=\; \frac{4\pi^2}{\sqrt{23}}\,D(\theta)\,,$$

where θ is the root with positive imaginary part of the cubic equation $x^3 - x - 1 = 0$ and $D(z) = \Im\big(Li_2(z) + \log|z|\,\log(1-z)\big)$ is the Bloch–Wigner modified dilogarithm function.

6. Central values

Returning now to critical values, we discuss briefly the phenomenon mentioned at the end of Section 2: that the algebraic factor $A(m)$ in (3) is

essentially a square when $m = h - m$, i.e. at the point of symmetry of the functional equation. One example of this, for $m = 1$, is given by the Birch–Swinnerton–Dyer formula (5), since the integers S and $|E(\mathbb{Q})|^2$ are both squares. As a second illustration we take the series

$$L_k(s) = \sum \frac{\alpha^{k-1}}{N(\alpha)^s} \qquad (k > 0 \text{ even}),$$

where the sum is over all integers $\alpha = (r + s\sqrt{-7})/2$ of $\mathbb{Q}(\sqrt{-7})$ with $r \equiv 1, 2$ or 4 modulo 7 and $N(\alpha) = (r^2 + 7s^2)/4$ is the norm of α. This series belongs to all three of the domains of mathematics mentioned in Section 2: to algebraic number theory because it is the L-series associated to a Hecke grossencharacter of weight $k - 1$ on the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$; to the theory of automorphic forms because it is the Hecke L-series of a holomorphic modular form of weight k on the congruence subgroup $\Gamma_0(49)$ of $SL(2, \mathbb{Z})$; and to algebraic geometry because it is the $(k-1)$st symmetric power of the Hasse–Weil zeta function of the elliptic curve $y^2 = x^3 - 35x - 98$. The functional equation is $L_k^*(s) := (7/2\pi)^s \Gamma(s) L_k(s) = (-1)^{k/2-1} L_k^*(k-s)$. Here F. Rodriguez Villegas and I, using a method invented by him for the case $k = 2$ (*Inv. Math.* **106**, 1991), showed that

$$L_k^*(k/2) = \frac{\sqrt{7}}{2} \left(\frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{4\pi^2} \right)^{k-1} c_k^2$$

with $c_k \in \mathbb{Z}$, as had been conjectured by B. Gross and myself some ten years earlier on the basis of numerical evidence. The c_k turn out to be the constant terms of a sequence of polynomials $\{c_k(x)\}$ satisfying a 3-term linear recurrence.

7. Zeta values, invariants of moduli spaces, and the Witten zeta function

It is an old and well-known phenomenon that the volumes or Euler characteristics of moduli spaces are often expressible in terms of special values of zeta functions. For instance, the volume of the moduli space for elliptic curves over $\mathbb{C}$, which is the quotient of the upper half-plane $\mathfrak{H}$ by the action of $SL(2, \mathbb{Z})$, can be evaluated simply by integrating $y^{-2} dx\, dy$ over a fundamental domain of this action and equals $\pi/3$, but the "true" explanation of this number is that it is $2\pi^{-1}\zeta(2)$ or $-4\pi\,\zeta(-1)$, where $\zeta(s)$ is the Riemann zeta function. Similarly, the volume (or, what is by virtue of the Gauss–Bonnet formula essentially the same thing, the Euler characteristic) of a Hilbert modular variety $\mathfrak{H}^{[K:\mathbb{Q}]}/SL(2, \mathcal{O}_K)$ for a totally real number

field K is essentially given by $\zeta_K(2)$ or $\zeta_K(-1)$, a fact which played a key role in Hirzebruch's study of the geometry of Hilbert modular surfaces in the 1970's. More generally, results of Siegel and others which from a modern point of view correspond to the evaluation of Tamagawa numbers show that the volumes of quotients by arithmetic subgroups of $SL(n)$, $Sp(n)$ and other algebraic groups can be expressed as special values or finite products of special values of zeta functions. In a different direction, Harer and I showed in 1984 that the Euler characteristic of the moduli space of Riemann surfaces of genus g, which is the quotient of a contractible space (Teichmüller space) by a non-arithmetic discrete group (the mapping class group), is equal up to a simple factor to the value of the Riemann zeta function $\zeta(s)$ at $s = 2g$ (or $s = 1 - 2g$).

Recently, because of their appearance in physics (Verlinde formulas), there has been much interest in certain moduli spaces of vector bundles of curves. On the basis of arguments coming from theoretical physics, Witten gave a formula expressing the volumes of these spaces in terms of special values of a new type of zeta function. Specifically, let $\mathfrak{g}$ be a semi-simple Lie algebra and define $\zeta_{\mathfrak{g}}(s)$ be the zeta function $\sum \dim(\rho)^{-s}$, where the sum runs over all finite-dimensional representations of $\mathfrak{g}$. Their dimensions can be calculated by a well-known formula of Hermann Weyl; thus $\zeta_{\mathfrak{sl}(2)}(s)$ is the Riemann zeta function while

$$\zeta_{\mathfrak{sl}(3)}(s) = \sum_{m,\,n=1}^{\infty} \frac{2^s}{m^s n^s (m+n)^s},$$

$$\zeta_{\mathfrak{so}(5)}(s) = \sum_{m,\,n=1}^{\infty} \frac{6^s}{m^s n^s (m+n)^s (m+2n)^s}.$$

A consequence of Witten's volume formula is the following purely number-theoretical fact:

Theorem. *The values of $\zeta_{\mathfrak{g}}(s)$ at positive even integers are always rational multiples of powers of π; more precisely, $\zeta_{\mathfrak{g}}(s) \in \mathbb{Q}\pi^{rs}$ for $s = 2, 4, 6, \ldots$ where r denotes the number of positive roots of $\mathfrak{g}$.*

Direct proofs of this statement were given by myself and by Stavros Garoufalidis (private communication). In the special case $\mathfrak{g} = \mathfrak{sl}(3)$ one can even give a closed formula, namely for even $s \geq 2$ one has

$$\sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s} = \frac{4}{3} \sum_{\substack{0 \leq r \leq s \\ r \text{ even}}} \binom{2s-r-1}{s-1} \zeta(r)\zeta(3s-r) \qquad (6)$$

(for example, $2^{-6}\zeta_{\mathfrak{sl}(3)}(6) = 1031(2\pi)^{18}/126 \cdot 19!$), as was shown by both

of us and also by Leonard Weinstein (private communication). The reason that the theorem holds is that, although sums over cones like the ones occurring in $\zeta_{\mathfrak{g}}(s)$ are in general very difficult to evaluate, the symmetry coming from the Weyl group allows us to evaluate these particular sums at positive even integers by summing over the *whole* lattice (minus the hyperplanes on which one of the linear forms occurring in the denominator vanishes), rather than only over one conical chamber. For instance, if s is a positive even integer then $\zeta_{\mathfrak{sl}(3)}(s)$ is 1/6 times the sum of $[\frac{1}{2}mn(m+n)]^{-s}$ over *all* pairs of integers (m,n) with $mn(m+n) \neq 0$, rather than only over pairs of positive integers. The theorem is then an immediate consequence of the following general result, which can be proved easily by induction or by partial fraction decompositions or by using Fourier expansions:

Proposition. *Let $\ell_1(x), \ldots, \ell_N(x)$ $(x = (x_1, \ldots, x_r))$ be N linear forms in r variables with rational coefficients. Then the sum $\sum \ell_1(x)^{-1} \cdots \ell_N(x)^{-1}$, where the summation is over all $x \in \mathbb{Z}^r$ for which no $\ell_i(x)$ vanishes, is, if absolutely convergent, a rational multiple of π^N.*

8. Witten's zeta function for $\mathfrak{sl}(3)$ and modular forms

We just saw that the proofs of Witten's statement about special values of $\zeta_{\mathfrak{g}}(s)$, and in particular of such special cases as (6), are in fact easy because the sums over cones are equivalent to sums over entire lattices. However, the story does not end there. For instance, formula (6) is true also for odd values of s if the factor 4/3 is replaced by 4 (for example, $\zeta_{\mathfrak{sl}(3)}(3) = 20\zeta(9) - 2\pi^2\zeta(7)$), yet the proof sketched before does not work at all since the sum over the whole lattice would vanish for parity reasons. To understand the situation better we introduce the more general sums

$$S(a,b,c) = \sum_{m,n=1}^{\infty} \frac{1}{m^a n^b (m+n)^c},$$

which in general cannot be expressed in terms of values of the Riemann zeta function. Using the Pascal triangle-like recurrence $S(a,b,c) = S(a-1,b,c+1) + S(a,b-1,c+1)$ we can express the $k^2/2 + \mathrm{O}(k)$ numbers $S(a,b,c)$ of given weight $k = a+b+c$ as integral linear combinations of the numbers $\zeta(a,c) = S(a,0,c)$ defined by equation (2) with $r = 2$. There are $k + \mathrm{O}(1)$ such numbers of weight k, and we find $k + \mathrm{O}(1)$ relations among them, namely those coming from $S(a,b,0) = \zeta(a)\zeta(b)$ $(a+b=k,\ a \leq b)$ and from $\zeta(a,c) + \zeta(c,a) = \zeta(a)\zeta(c) - \zeta(k)$ $(a+c=k,\ a \leq c)$. Moreover, the two "O(1)"'s (whose value depends on the parity of k) are the same,

so we have just as many equations as variables. We must thus solve a non-homogeneous system of linear equations of the form $A\mathbf{x} = \mathbf{c}$ where A is a square matrix with integral entries, the components of the vector $\mathbf{c}$ are rational linear combinations of $\zeta(k)$ and of products $\zeta(r)\zeta(k-r)$ $(2 \le r \le k/2)$, and the components of $\mathbf{x}$ are the unknowns $\zeta(a, k-a)$.

But now something quite different happens depending whether k is even or odd. If k is odd, then it turns out that $\det A \neq 0$, so we can solve uniquely for $\mathbf{x}$; thus *every number $S(a,b,c)$, and in particular every double zeta value $\zeta(a,c)$, of odd total weight k is a rational linear combination of the numbers $\zeta(k)$ and $\zeta(r)\zeta(k-r)$*, the formula given above for $S(a,a,a)$ being a special case. But if k is even, then A turns out to be highly singular, of corank $[(k-2)/6]$. This means, first of all, that the double zeta values $\zeta(a,c)$ of even weight $a+c \ge 8$ cannot be expressed in terms only of usual zeta values, but also that the components of $\mathbf{c}$ must satisfy certain linear relations over $\mathbb{Q}$ in order to ensure that $A\mathbf{x} = \mathbf{c}$ has a solution. These extra relations are precisely the relations of the form

$$\sum_{a \text{ even},\ 2 \le a \le k/2} c_a \zeta(a)\zeta(k-a) = c_0 \zeta(a) \qquad (c_a \in \mathbb{Q})$$

which can be obtained by the partial fraction decomposition method that we used in Section 1 to give an elementary proof that $\zeta(k) \in \mathbb{Q}\,\zeta(2)^{k/2}$, i.e., they correspond to the homogeneous polynomials $f(m,n) \in \mathbb{Q}[m^{-1}, n^{-1}]$ of degree k for which $f(m+n,n) + f(m,m+n) \in \mathbb{Q}[m^{-1}, n^{-1}]$. Moreover, there turns out to be a surprising and beautiful connection with the theory adumbrated in Section 4 of modular forms on $SL(2,\mathbb{Z})$ and their periods: the relations $\sum c_a \zeta(a)\zeta(k-a) = c_0\zeta(a)$ which one can prove by this method are exactly those obtained by taking the constant terms of modular form identities $\sum c_a G_a(z) G_{k-a}(z) = c_0 G_k(z)$, where

$$G_k(z) = \frac{1}{2} {\sum_{p,q \in \mathbb{Z}}}' \frac{1}{(pz+q)^k} = \zeta(k) + \ \mathrm{O}(e^{2\pi i z}) \qquad (z \in \mathfrak{H})$$

denotes the holomorphic Eisenstein series of weight k. Thus for $k < 12$ one gets every relation $\zeta(a)\zeta(k-a) \sim_{\mathbb{Q}} \zeta(k)$, but for $k = 12$ or $k \ge 16$ one "misses" certain relations, because of the existence of cusp forms; and dually, *starting at weight 12 there are relations over $\mathbb{Q}$ among the numbers $\zeta(a, k-a)$ (a odd, $2 \le a \le k/2$) and the number of such relations is equal to the number of linearly independent cusp forms of weight k.* The reason for the connection with modular forms is that the formal proof given in Section 1 remains true by sums over the "positive half" $(\mathbb{Z}_{>0} z + \mathbb{Z}) \cup \mathbb{Z}_{>0}$ if the summations over $m, n > 0$ are replaced by sums over the "positive half"

$(\mathbb{N}z+\mathbb{Z})\cup\mathbb{N}$ of the lattice $\mathbb{Z}z+\mathbb{Z}\subseteq\mathbb{C}$ and hence automatically "lift" from zeta identities to Eisenstein series identities. The connection to the theory of periods arises because, by a theorem of Rankin, the Petersson scalar product (f, G_aG_{k-a}) for a cusp form $f\in S_k(SL(2,\mathbb{Z}))$ is proportional to the ath period $L(f,a)$, so that a relation $\sum c_aG_aG_{k-a}\in\langle G_k\rangle$ is equivalent to a relation among the periods of cusp forms of weight k.

9. Multiple zeta values

Having been led by the investigation of $\zeta_{\mathfrak{sl}(3)}(s)$ and its generalization $S(a,b,c)$ to the study of the sums (2) for the special case $r=2$, it is natural to consider the general case. These numbers have in fact arisen sporadically in mathematics and physics in the past, starting (at least) with Euler, but it is only recently that it has become clear how basic—and interesting—they are. We will call them *multiple zeta values* since they generalize the classical special values $\zeta(k)$. (We do not, however, speak of "multiple zeta functions," both because this term has been used with another meaning by N. Kurokawa and because the function obtained by replacing the k_i by complex variables s_i in (2) does not seem to be very nice: in particular, it does not have a unique analytic continuation to values outside the domain of absolute convergence.) The first property of the multiple zeta values is that the set $\mathcal{A}$ of integral linear combinations of them is a ring, since the product of any two values can be expressed as a (positive) integral linear combination of others, e.g.,

$$\zeta(a,b)\,\zeta(c)=\zeta(a,b,c)+\zeta(a,b+c)+\zeta(a,c,b)+\zeta(a+c,b)+\zeta(c,a,b)\,.$$

Next, by using combinatorial manipulations like those in Section 1 and Section 8, one can derive many relations over $\mathbb{Q}$ among the multiple zeta values. Therefore, although there are 2^{k-2} tuples $\mathbf{k}=(k_1,\dots,k_r)$ as in (2) with given total weight $k=k_1+\cdots+k_r$, the dimension d_k of the $\mathbb{Z}$-module $\mathcal{A}_k$ spanned by the corresponding $\zeta(\mathbf{k})$ is much smaller. It is given conjecturally (after many discussions with Drinfel'd, Kontsevich and Goncharov!) by the Fibonacci-like recurrence $d_k=d_{k-2}+d_{k-3}$, with initial values $d_1=0$, $d_2=d_3=d_4=1$, ...; this has been checked numerically up to $k=12$ ($d_{12}=12$). Among the many multiple zeta identities there are some very pretty ones like

$$\zeta(1,k-1)\;=\;\frac{k-1}{2}\,\zeta(k)-\frac{1}{2}\sum_{r=2}^{k-2}\zeta(r)\,\zeta(k-r)\,,$$

(and in general, $\zeta(\underbrace{1,\dots,1}_{a},b) \in \mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\dots]$ for any a and b),

$$\zeta(\underbrace{8,\dots,8}_{n}) = \frac{2^{6n+2}\,\pi^{8n}}{(8n+4)!}\left[(\sqrt{2}+1)^{4n+2}+(\sqrt{2}-1)^{4n+2}\right]$$

(and in general, $\zeta(\underbrace{k,\dots,k}_{n}) \in \mathbb{Q}\,\pi^{kn}$ for any even k), or the as yet unproved

$$\zeta(\underbrace{1,3,1,3,\dots,1,3}_{n}) \stackrel{?}{=} \frac{2\,\pi^{4n}}{(4n+2)!} \quad \left(=\ 4^{-n}\,\zeta(\underbrace{4,4,\dots,4}_{n})\right).$$

A different type of identity comes from the observation, due to M. Kontsevich, that the multiple zeta values are at the same time the values of the Drinfel'd integrals

$$I(\varepsilon_1,\dots,\varepsilon_k) = \int\cdots\int_{0<t_1<\cdots<t_k<1} \frac{dt_1}{A_{\varepsilon_1}(t_1)}\cdots\frac{dt_k}{A_{\varepsilon_k}(t_k)}$$
$$(\varepsilon_i = 0 \text{ or } 1,\ \varepsilon_1 = 1,\ \varepsilon_k = 0)$$

(where $A_0(t) = t$, $A_1(t) = 1-t$), namely

$$\zeta(k_1,\dots,k_r) = I(\underbrace{1,0,\dots,0}_{k_1},\underbrace{1,0,\dots,0}_{k_2},\dots,\underbrace{1,0,\dots,0}_{k_r}).$$

The easy proof consists of expanding each factor $A_1(t_i)^{-1}$ in the integral as a power series $1+t_i+t_i^2+\cdots$ and integrating term by term. A nice consequence is that there is a non-trivial duality $\mathbf{k}\mapsto\mathbf{k}'$ with $\zeta(\mathbf{k}) = \zeta(\mathbf{k}')$, corresponding to the obvious identity $I(\varepsilon_1,\dots,\varepsilon_k) = I(1-\varepsilon_k,\dots,1-\varepsilon_1)$. But other identities, like the one for $\zeta(1,3,1,3,\dots,1,3)$ mentioned above, are still not understood.

The fundamental importance of the multiple zeta values is that, apparently, all iterated integrals à la Chen defined over $\mathbb{Z}$ can be written as linear combinations of them, or equivalently, that the graded ring $\mathcal{A} = \bigoplus \mathcal{A}_k$, after adding $2\pi i$, is exactly the ring of periods of the pro-nilpotent completion of $\pi_1\left(\mathbb{P}^1 \smallsetminus \{0,1,\infty\}\right)$. Thus understanding this ring is equivalent to constructing the category of mixed Tate motives over $\mathbb{Z}$. (Goncharov now informs me that he has succeeded in doing this.) In another direction, the integrals discovered by Kontsevich for the Vassiliev knot invariants are essentially iterated integrals and can be related directly to the numbers $\zeta(\mathbf{k})$, so that the structure of the ring $\mathcal{A}$ should be directly related to the

structure of the set of these knot invariants. However, I understand almost nothing of any of the topics mentioned in this paragraph and will stop here.

Further reading

Since the purpose of this article is neither to prove new results nor to give a survey of a well-defined area of mathematics, it is more or less impossible to give a bibliography in the usual sense. References to a few specific results which were mentioned in the main text were given as they occurred. We will content ourselves here with indicating a few places where one can find more information about some of the more general topics touched on, with no pretense of completeness. The interested reader can then use the bibliographies of the articles and books cited for the analytic continuation of the path to be followed.

(Section 1) Calabi's ultra-short proof of "$\zeta(2) = \pi^2/6$" as reproduced at the beginning of this section, as well as its generalization to $\zeta(2n)$ for $n > 1$, have now been written up; they are contained in the article "Sums of generalized harmonic series and volumes" by F. Beukers, E. Calabi and J. Kolk, *Nieuw Archief voor Wiskunde* **11** (1993), 217–224.

(Section 2) A general survey for the many kinds of zeta functions arising from motivic contexts is the book *Conjectures in Arithmetic Algebraic Geometry* by W. Hulsbergen (Aspects of Mathematics No. 18, Vieweg 1992). It contains information about the zeta functions coming from number fields, elliptic curves, and modular forms as well as about the material discussed in Section 5 of this paper, and gives many further references for these topics. The notion of critical value was introduced by Pierre Deligne in "Valeurs de fonctions L et périodes d'intégrales" (Proc. Symp. Pure Math. **33** Part II, AMS (1979), 313–346).

(Section 3) The Birch–Swinnerton–Dyer conjecture and its application to the problem of congruent numbers are described in N. Koblitz's book *Introduction to Elliptic Curves and Modular Forms* (Graduate Texts in Math. No. 97, Springer 1984), while the numerical results mentioned in the text are given in G. Kramarz, "All congruent numbers less than 2000," *Math. Ann.* **273** (1986), 337–340.

(Section 4) The afore-mentioned book by Koblitz is also a good reference for the theory of modular forms (including Waldspurger's theorem). A brief exposition of this theory is given in my survey article in *From Number Theory to Physics* (Springer 1992, 238–291). Another introductory text, also covering periods and the Eichler-Selberg trace formula, is Serge Lang's

Introduction to Modular Forms (Grundlehren No. 222, Springer 1976).

(Section 5) Borel's paper relating special values of Dedekind zeta-functions to algebraic K-theory is "Cohomologie de SL_n et valeurs de fonctions zêta aux points entiers," *Ann. Sc. Norm. Pisa* **4** (1977), 613–636. A survey of what is known about the algebraic K-theory of fields (especially K_3) is given in A. Suslin's article in the proceedings of the 1986 International Mathematical Congress, 222–244. My conjectures relating to the polylogarithm are presented in the article "Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields" in *Arithmetic Algebraic Geometry* (eds. G. van der Geer and J. Steenbrink, Progress in Math. No. 89, Birkhäuser 1991, 391–430) and in an appendix to the book *The Structural Properties of Polylogarithms* by L. Lewin (AMS 1991). Goncharov's proof for the relation between $\zeta_F(3)$ and $K_5(F)$ is given in his 1992 MPI preprint "Geometry of configurations, polylogarithms and motivic cohomology."

(Section 6) The joint paper with Villegas has appeared in *Advances in Number Theory* (eds. F. Gouvea and N. Yui, Oxford Univ. Press 1993, 81–99).

(Section 7) A survey of the theory of Hilbert modular surfaces is given in G. van der Geer's book of the same name (Ergebnisse No. 16, Springer 1988; cf. in particular pp. 19–20 for the zeta-values of real quadratic fields). The article of Witten cited in the text appeared in Commun. Math. Phys. **141** (1991) 153-209.

(Sections 8–9) The results on multiple zeta values are not yet written up in detail. The connection with the Vassiliev–Kontsevich knot invariants, as well as further references for the latter, can be found in several recent preprints by T. Q. T. Le and J. Murakami, especially "Kontsevich's integral for Homfly polynomial and relation between values of multiple zeta functions" (MPI 93-26) and "The universal Vassiliev–Kontsevich invariant for framed oriented links" (MPI 93-89).

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