Functional equations and ladders for polylogarithms

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Abstract

We give a number of new functional equations for polylogarithms and in the process we obtain the first proven ladder relations, à la Lewin, of weight 6 and 7.

1 Motivation

Polylogarithms appear in many contexts within mathematical physics, like in dimensional regularization expansions or when determining analytic solutions of various Feynman integrals in quantum field theory; e.g. the dilogarithm appeared already in the famous paper by t'Hooft and Veltman [14], and Ussyukina and Davydychev [22], eq. (30), found all k-logarithms ($n \leq k \leq 2n$) in a closed expression for the "n-box" diagram (for a more recent update cf. [21]), as well as in conformal field theory (the dilogarithm plays a crucial role in a conjecture of Nahm [20] characterizing rational CFTs) or when considering expansions of hypergeometric functions (cf. e.g. [15]). Even more closely related to our results below, (multiple) polylogarithms and their special values have occurred, among many others, in various ways in work of Broadhurst and Kreimer (e.g. [6]), occasionally even in connection with ladder relations (cf. [5]) as defined below. Very recently, when calculating the two-loop hexagon Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, Del Duca, Duhr and Smirnov [8] were led to a long expression in polylogarithms that has been subsequently enormously simplified by Goncharov, Spradlin, Vergu and Volovich [13]. In these contexts, insight into functional equations for the polylogarithms involved can be useful to reduce the ensuing typically complicated expressions considerably.

Functional equations of polylogarithms play also a pivotal role in a more abstract context when trying to define an explicit version of the (odd index) algebraic K-groups $K_{2m-1}(F)$ of a number field F. The latter can conjecturally be written as a subquotient of the free abelian group on F (pioneered by Bloch [3] in the dilogarithm case and generalized by Zagier [24] and by Goncharov [12] for higher m), and the group of relations in that description is expected to encode all the functional equations of the m-logarithm.

In 1840, Kummer [16] gave non-trivial functional equations for polylogarithms $Li_m(z) = \sum_{n\geq 1} z^n/n^m$ up to weight $m = 5$, where results had previously been known only up to $m = 3$. He mentioned "peculiar difficulties" ("eigenthümliche Schwierigkeiten") which arise when trying to extend the results to $m > 5$. In fact, Wechsung proved [23] that the type of functional equation that Kummer had found does *not* extend to $m > 5$.

In the eighties, Lewin and his coauthors ([1], [19]) tried several approaches to conquer what he called the "trans-Kummer region" $m > 5$ (cf. e.g. [1], p.11), and they indeed found new functional equations, but all results were ultimately confined to the same range $m \leq 5$. On the way, Lewin discovered interesting special relations of the form $\sum_j n_j Li_m(\alpha^j) = 0$ ($n_j \in \mathbf{Q}$), for certain algebraic numbers α. He realized that such relations, which he dubbed "ladders", were consequences of a certain intrinsic property of such an α, viz. the property that it satisfies many different "cyclotomic relations" (loc.cit.), which are equations of the form $\prod_r (1-\alpha^r)^{\nu_r} = \pm\alpha^N$ where r, ν_r and N are integers. This insight enabled him to conjecture certain ladders even up to weight $m = 9$ (he used the terminology *order* in place of the now more common notion of *weight*). By cleverly specializing and combining old and new functional equations, he was able to prove quite a number of his conjectured ladders, but was again confined to weights ≤ 5.

In our thesis [9], we gave the first functional equations beyond that range, and the largest weight, for which non-trivial such equations are known for the m-logarithm, is $m = 7$ (cf. also [11]). We provide in this note a collection of functional equations for polylogarithms up to this weight, which moreover have a very specific symmetry: the arguments (in one variable t) involve only the three factors t, $1-t$ and $1-t(1-t)$ (with roots 0, 1 and the primitive sixth roots of unity, respectively), and each given equation is invariant under the action of the symmetric group $\mathcal{S}_3$. With increasing weight, the number of independent such equations decreases, and for weight 8 and beyond we did not find any.

As a by-product, the equations for weight 6 and 7 allow, after specialization, to prove the first ladders in that range.

2 Zagier's criterion for functional equations of polylogarithms

In his seminal papers [24] and [25], Zagier described a criterion for functional equations for polylogarithms. More precisely, he first gave a single-valued function $\mathcal{L}_m(z) = \Re_m\big(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!} \log^k |z|\, Li_{m-k}(z)\big)$ (denoted by $P_m(z)$ in [24]) attached to the (multivalued) function $Li_m(z)$, where $\Re_m$ denotes the real part for m odd and the imaginary part for m even, and the B_k denote the Bernoulli numbers. This function now satisfies "clean" functional equations, i.e. without invoking products of lower weight polylogarithms as occur typically—and in abundance—for Li_m-equations (cf. e.g. almost any functional equation in [17]). Furthermore, one can give a very useful characterization for them which we describe in the following subsection.

2.1 Higher Bloch conditions

For a field F, let β_m^F be the map

$$\beta_m^F \;:\quad \mathbf{Z}[F] \longrightarrow \bigotimes^{m-2} F^\times \otimes \bigwedge^2 F^\times\,,$$

defined as $\beta_m^F([0]) = \beta_m^F([1]) = 0$, and on generators $[x]$ $(x \neq 0, 1)$ as follows:

$$\beta_m^F([x]) = x \otimes \cdots \otimes x \otimes \big(x \wedge (1-x)\big)\,.$$

For $m = 2$, this map was related to the dilogarithm in Bloch's seminal paper [3].

We say that a combination $\xi \in \mathbf{Z}[F]$ *satisfies the m-th higher Bloch condition* simply if it lies in $\ker \beta_m^F$. This fits very well with the above one-valued function $\mathcal{L}_m(z)$:

Theorem 1 *(Zagier, [24]) Let F be a subfield of $\mathbf{C}(t)$ then for $\xi \in \mathbf{Z}[F]$ we have*

$$\xi \in \ker \beta_m^F \implies \mathcal{L}_m(\sigma(\xi)) = \text{constant}\,,$$

for any embedding $\sigma : F \hookrightarrow \mathbf{C}(t)$.

Here we extend the definition of $\mathcal{L}_m$ as well as of σ to all of $\mathbf{Z}[F]$ by linearity, i.e.

$$\mathcal{L}_m \circ \sigma\Big(\sum_i n_i[x_i]\Big) = \sum_i n_i \mathcal{L}_m\big(\sigma(x_i)\big)\,.$$

In this way, the problem of finding functional equations is reduced to a problem in linear algebra and the hard part is to find a suitable list of potentially good arguments x_i.

2.2 A rich collection of arguments

A particularly good collection of arguments for functional equations (in one variable t) turns out to be given by

$$\{\pm t^{a_1}(1-t)^{a_2}\big(1-t(1-t)\big)^{a_3} \;\mid\; a_i \in \mathbf{Z}\}\,.$$

It is convenient to introduce new variables

$$u_1(t) = \frac{-t}{1-t(1-t))}\,,\quad u_2(t) = \frac{-(1-t)}{1-t(1-t)}\,,\quad u_3(t) = \frac{t(1-t)}{1-t(1-t)}$$

and then to rewrite the above expressions as

$$\{\pm u_1(t)^{\alpha_1}u_2(t)^{\alpha_2}u_3(t)^{\alpha_3} \;\mid\; a_i \in \mathbf{Z}\}\,,$$

for suitable α_i, since then a further $\mathcal{S}_3$–symmetry becomes apparent. The two involutory automorphisms induced by $t \mapsto \frac{1}{t}$ and $t \mapsto 1-t$, respectively, generate this $\mathcal{S}_3$–action on the set of those arguments by simply permuting the exponents. Any of the arguments can hence be encoded by a triple of exponents, together with a sign. There are many functional equations for $m \le 7$, in the exponent range $|\alpha_i| \le 6$, which carry the above symmetry. All the ones that were found have arguments chosen from the following list $\mathcal{A}$ which represents 32 $\mathcal{S}_3$-orbits in $\mathbf{Z}[\mathbf{Q}(t)]$:

$$\begin{aligned}\mathcal{A} = \big\{&(-,\ 2,-2,3),\ (+,\ 0,\ 5,\ 0),\ (-,\ 6,-1,-1),\ (+,\ 3,\ 0,\ 0),\\
&(+,\ 0,-3,3),\ (-,-3,\ 6,-3),\ (-,-3,\ 3,\ 3),\ (+,\ 0,-5,\ 5),\\
&(+,\ 4,-1,0),\ (+,-3,\ 4,\ 4),\ (+,\ 3,\ 0,-2),\ (-,-1,\ 2,-1),\\
&(+,\ 0,\ 1,1),\ (-,\ 2,\ 0,-2),\ (+,\ 1,\ 0,-1),\ (-,\ 1,\ 0,-1),\\
&(+,-2,-2,3),\ (-,-1,\ 3,-1),\ (+,-4,-1,\ 4),\ (-,-2,-2,\ 5),\\
&(-,\ 2,-1,1),\ (-,-2,-1,\ 3),\ (-,\ 2,\ 0,-1),\ (+,\ 2,\ 0,-1),\\
&(-,-2,\ 2,2),\ (+,\ 2,-1,-1),\ (-,\ 2,-1,-2),\ (-,\ 0,\ 1,\ 0),\\
&(+,\ 0,\ 1,0),\ (-,-1,\ 1,\ 1),\ (+,\ 1,-1,-1),\ (-,\ 1,\ 1,\ 0)\ \big\}.\end{aligned}$$

The factors of $1-x$ where x runs through those arguments can be found in the $\mathcal{S}_3$-orbits of the following list (where $T = 1-t(1-t)$)

$$\big\{t, T, 1+t, 1+t(1-t), 1+\frac{1}{T}, 1+\frac{t}{T}, 1+\frac{(1-t)}{tT}, 1+\frac{t(1-t)^2}{T^2}, 1-\frac{t^2(1-t)}{T^2}\big\}.$$

Due to the above symmetry we focus on $\mathcal{S}_3$-invariant functional equations and introduce the shorthand

$$\big[(\pm,\alpha_1,\alpha_2,\alpha_3)\big] := \sum_{\sigma\in\mathcal{S}_3}\Big[\pm\prod_{i=1}^{3}u_i(t)^{\alpha_{\sigma(i)}}\Big]\,.$$

2.3 The functional equations

In the above notation, the functional equations can be given in concise form, with coefficients taken from the tables below. We first state the results for combinations satisfying the higher Bloch conditions.

Theorem 2 *For $m \in \mathbf{N}$, let $\kappa_m = \ker\left(\beta_m^{\mathbf{Q}(t)}\right)^{\mathcal{S}_3}$ be the space of $\mathcal{S}_3$–invariant elements in the kernel of the map $\beta_m^{\mathbf{Q}(t)}$. Then we have the following bounds on ranks of κ_m for $m = 4, 5, 6, 7$.*

m	4	5	6	7
rank κ_m	≥ 11	≥ 9	≥ 4	≥ 2

Explicitly, the corresponding elements are given by

$$\sum_{a \in \mathcal{A}} c_j^{(m)}(a)\,[a]\,,$$

with the coefficients $c_j^{(m)} = c_j^{(m)}(a)$ as in Tables 1–3 below.

The proof that the given elements are indeed in the kernel of β_m^F is a tedious and mechanical task, which is best left to a computer. One determines all the factors occurring in a factorization of x and $1-x$, where x runs through all the corresponding arguments in an equation and then checks that all the terms in the ensuing image under β_m^F do cancel. Using the $\mathcal{S}_3$-symmetry involved, one can cut down on the actual calculations, but they are still too cumbersome to give in detail.

Corollary 3 *There are ≥ 2 (resp., ≥ 4, ≥ 9, ≥ 11) linearly independent functional equations for $\mathcal{L}_7$ (resp., $\mathcal{L}_6$, $\mathcal{L}_5$, $\mathcal{L}_4$), with arguments encoded by $\mathcal{A}$, up to permutation, which are $\mathcal{S}_3$-symmetric.*

We remark that the two functional equations for $\mathcal{L}_7$ do not seem to follow from the 2-variable equations for $\mathcal{L}_7$ given in [11], but a suitable linear combination of the two which cancels the constant terms is a specialization of that equation.

Example. We spell out some equations corresponding to the columns of Table 1. The last one, $c_{11}^{(4)}$, gives

$$2\left[(+,\ 2,-1,-1)\right]\ +6\left[(-,\ 0,\ 1,\ 0)\right]\ +3\left[(+,\ 1,-1,-1)\right] \in \ker \beta_4^F$$

for $F = \mathbf{Q}(t)$. This amounts essentially to our 9-term equation for $\mathcal{L}_4$ cited in [25], §7. The second-to-last column gives another element in that kernel,

$$\begin{aligned}\big[(-,-2,2,2)\big] \;&+\; 4\big[(-,2,-1,-2)\big] \;-\; 6\big[(-,0,1,0)\big]\\ &-\; 12\big[(+,0,1,0)\big] \;-\; 2\big[(+,1,-1,-1)\big]\,,\end{aligned}$$

which explicitly (and with less apparent symmetry) is written as

$$\begin{aligned}
&2\Big(\mathcal{L}_4\big(-\tfrac{t^4}{(1-t+t^2)^2}\big) \;+\; \mathcal{L}_4\big(-\tfrac{(1-t)^4}{(1-t+t^2)^2}\big) + \mathcal{L}_4\big(-\tfrac{1}{(1-t+t^2)^2}\big)\Big)\\
&+4\Big(\mathcal{L}_4\big(-\tfrac{(1-t)(1-t+t^2)}{t^3}\big) \;+\; \mathcal{L}_4\big(-\tfrac{t(1-t+t^2)}{(1-t)^3}\big) \;+\; \mathcal{L}_4\big(\tfrac{1-t+t^2}{t^3}\big)\\
&\qquad + \mathcal{L}_4\big(\tfrac{1-t+t^2}{(1-t)^3}\big) \;+\; \mathcal{L}_4\big((1-t)(1-t+t^2)\big) \;+\; \mathcal{L}_4\big(t(1-t+t^2)\big)\Big)\\
&-12\Big(\mathcal{L}_4\big(-\tfrac{t(1-t)}{1-t+t^2}\big) \;+\; \mathcal{L}_4\big(\tfrac{t}{1-t+t^2}\big) \;+\; \mathcal{L}_4\big(\tfrac{1-t}{1-t+t^2}\big)\Big)\\
&-24\Big(\mathcal{L}_4\big(\tfrac{t(1-t)}{1-t+t^2}\big) \;+\; \mathcal{L}_4\big(-\tfrac{t}{1-t+t^2}\big) \;+\; \mathcal{L}_4\big(-\tfrac{1-t}{1-t+t^2}\big)\Big)\\
&-4\Big(\mathcal{L}_4\big(\tfrac{1-t+t^2}{t^2}\big) \;+\; \mathcal{L}_4\big(\tfrac{1-t+t^2}{(1-t)^2}\big) \;+\; \mathcal{L}_4\big(1-t+t^2\big)\Big) \quad = \; 0\,.
\end{aligned}$$

The constant of Theorem 1 is zero for each $c_j^{(m)}$ for even m, while for $c_j^{(5)}$ the constants can be obtained by specialising t to 1, say, and turn out to be $\zeta(5)$ times $0, 0, 0, 0, 1662, 378, 4230, -126$ and 414, respectively, for $c_1^{(7)}$ we get the constant $-\frac{25461}{4}\zeta(7)$ and for $c_2^{(7)}$ we find $-\frac{54495}{4}\zeta(7)$. Note that certain $a \in \mathcal{A}$ only occur with non-trivial coefficient for odd m, as the inversion relation annihilates the sum over the corresponding orbit.

2.4 The tables

2.4.1 Functional equations for $m = 4$

a	$c_1^{(4)}$	$c_2^{(4)}$	$c_3^{(4)}$	$c_4^{(4)}$	$c_5^{(4)}$	$c_6^{(4)}$	$c_7^{(4)}$	$c_8^{(4)}$	$c_9^{(4)}$	$c_{10}^{(4)}$	$c_{11}^{(4)}$
$(-,\ 2,-2,\ 3)$	2	0	0	0	0	0	0	0	0	0	0
$(+,\ 0,\ 5,\ 0)$	0	1	0	0	0	0	0	0	0	0	0
$(-,\ 6,-1,-1)$	0	0	1	0	0	0	0	0	0	0	0
$(+,\ 3,\ 0,\ 0)$	0	0	0	3	0	0	0	0	0	0	0
$(-,-3,\ 6,-3)$	0	0	0	0	3	0	0	0	0	0	0
$(-,-3,\ 3,\ 3)$	0	0	0	0	0	3	0	0	0	0	0
$(+,\ 4,-1,\ 0)$	0	0	0	0	0	0	2	0	0	0	0
$(+,-3,\ 4,\ 4)$	0	0	1	1	-1	-1	0	0	0	0	0
$(+,\ 3,\ 0,-2)$	0	-10	0	0	0	0	-6	0	0	0	0
$(-,-1,\ 2,-1)$	0	0	0	0	-81	0	0	0	0	0	0
$(+,\ 0,\ 1,\ 1)$	0	-30	0	0	0	0	-6	0	0	0	0
$(+,-2,-2,\ 3)$	0	0	7	6	-15	3	0	0	0	0	0
$(-,-1,\ 3,-1)$	0	0	-14	-8	-10	-1	0	0	0	0	0
$(+,-4,-1,\ 4)$	0	0	0	0	0	0	0	1	0	0	0
$(-,-2,-2,\ 5)$	0	0	0	0	0	0	0	0	1	0	0
$(-,\ 2,-1,\ 1)$	-6	0	-42	-30	30	30	-6	-1	2	0	0
$(-,-2,-1,\ 3)$	2	0	0	6	-6	-6	-2	-3	-6	0	0
$(-,\ 2,\ 0,-1)$	6	0	-28	20	-20	-20	-12	6	-6	0	0
$(+,\ 2,\ 0,-1)$	-12	0	-14	-30	30	30	6	3	-12	0	0
$(-,-2,\ 2,\ 2)$	0	0	0	0	0	0	0	0	0	1	0
$(+,\ 2,-1,-1)$	0	0	0	0	0	0	0	0	0	0	2
$(-,\ 2,-1,-2)$	6	0	0	0	0	0	0	-4	0	4	0
$(-,\ 0,\ 1,\ 0)$	6	-120	42	78	30	-132	-42	-9	-6	-6	6
$(+,\ 0,\ 1,\ 0)$	-48	-125	0	-81	0	0	0	-3	0	-12	0
$(-,-1,\ 1,\ 1)$	0	0	0	0	0	-81	0	0	0	0	0
$(+,\ 1,-1,-1)$	15	-15	-21	36	45	18	-27	-2	-1	-2	3
$(-,\ 1,\ 1,\ 0)$	0	0	-21	-24	15	6	0	0	0	0	0

Table 1 Generators for $\ker\left(\beta_4^{\mathbf{Q}(t)}\right)^{\mathcal{S}_3}$ **in Thm 1**

2.4.2 Functional equations for $m = 5$

a	$c_1^{(5)}$	$c_2^{(5)}$	$c_3^{(5)}$	$c_4^{(5)}$	$c_5^{(5)}$	$c_6^{(5)}$	$c_7^{(5)}$	$c_8^{(5)}$	$c_9^{(5)}$
$(-, 2, -2, 3)$	18	0	0	0	0	0	0	0	0
$(+, 0, 5, 0)$	0	3	0	0	0	0	0	0	0
$(-, 6, -1, -1)$	0	0	3	0	0	0	0	0	0
$(+, 3, 0, 0)$	0	0	0	3	0	0	0	0	0
$(+, 0, -3, 3)$	0	0	0	0	10	0	0	0	0
$(-, -3, 6, -3)$	0	0	0	0	0	3	0	0	0
$(-, -3, 3, 3)$	0	0	-63	-12	0	15	0	0	0
$(+, 0, -5, 5)$	0	0	0	0	0	0	3	0	0
$(+, 4, -1, 0)$	0	-18	0	0	0	0	6	0	0
$(+, -3, 4, 4)$	0	0	15	3	0	-3	0	0	0
$(+, 3, 0, -2)$	0	12	0	0	0	0	-54	0	0
$(-, -1, 2, -1)$	0	0	0	0	0	-243	0	0	0
$(+, 0, 1, 1)$	0	-144	0	0	0	0	-27	0	0
$(-, 2, 0, -2)$	0	0	0	0	0	0	0	90	0
$(+, 1, 0, -1)$	0	0	0	0	-810	0	-1875	0	0
$(-, 1, 0, -1)$	0	0	0	0	0	0	0	0	180
$(+, -2, -2, 3)$	0	0	105	18	0	-45	0	0	0
$(-, -1, 3, -1)$	0	0	-105	-12	0	-15	0	0	0
$(+, -4, -1, 4)$	-8	-10	-56	-14	6	14	20	-8	2
$(-, -2, -2, 5)$	5	4	35	8	3	-8	7	-4	1
$(-, 2, -1, 1)$	-80	80	-1190	-224	6	224	-10	-8	2
$(-, -2, -1, 3)$	0	0	0	0	-54	0	-150	72	-18
$(-, 2, 0, -1)$	180	360	1050	336	-54	-336	-270	72	-18
$(+, 2, 0, -1)$	-360	-180	-2310	-504	-54	504	-90	72	-18
$(-, -2, 2, 2)$	-40	-20	-112	-28	12	28	40	-1	4
$(+, 2, -1, -1)$	0	0	0	0	-108	-162	-150	-96	-6
$(-, 2, -1, -2)$	190	200	1120	280	-120	-280	-400	40	-40
$(-, 0, 1, 0)$	540	-540	9450	1980	-108	-2142	-720	-36	-36
$(+, 0, 1, 0)$	-360	-1425	2520	387	-270	-630	-900	0	-90
$(-, -1, 1, 1)$	0	0	5103	972	0	-1215	0	0	0
$(+, 1, -1, -1)$	-544	-560	1085	32	39	-194	370	8	13
$(-, 1, 1, 0)$	0	0	-315	-72	0	45	0	0	0

Table 2 Generators for $\ker\left(\beta_5^{\mathbf{Q}(t)}\right)^{\mathcal{S}_3}$ **in Thm 1**

2.4.3 Functional equations for $m = 7$ and $m = 6$

a	$c_1^{(7)}$	$c_2^{(7)}$	$c_1^{(6)}$	$c_2^{(6)}$	$c_3^{(6)}$	$c_4^{(6)}$
$(-,\ 2,-2,\ 3)$	50	126	6	0	0	0
$(+,\ 0,\ 5,\ 0)$	-3	0	0	3	0	0
$(-,\ 6,-1,-1)$	0	3	0	0	3	0
$(+,\ 3,\ 0,\ 0)$	0	-35	0	0	0	5
$(+,\ 0,-3,\ 3)$	0	140	0	0	0	0
$(-,-3,\ 6,-3)$	0	7	0	0	21	4
$(-,-3,\ 3,\ 3)$	0	28	0	0	-84	-20
$(+,\ 0,-5,\ 5)$	4	0	0	0	0	0
$(+,\ 4,-1,\ 0)$	50	0	0	-30	0	0
$(+,-3,\ 4,\ 4)$	0	-3	0	0	12	3
$(+,\ 3,\ 0,-2)$	-300	0	0	60	0	0
$(-,-1,\ 2,-1)$	0	-5103	0	0	-5103	-972
$(+,\ 0,\ 1,\ 1)$	900	0	0	-360	0	0
$(-,\ 2,\ 0,-2)$	-2750	-4410	0	0	0	0
$(+,\ 1,\ 0,-1)$	-62500	-102060	0	0	0	0
$(-,\ 1,\ 0,-1)$	-68000	-112140	0	0	0	0
$(+,-2,-2,\ 3)$	0	-210	0	0	-420	-90
$(-,-1,\ 3,-1)$	0	-420	0	0	-420	-60
$(+,-4,-1,\ 4)$	50	70	8	50	224	70
$(-,-2,-2,\ 5)$	25	35	5	20	140	40
$(-,\ 2,-1,\ 1)$	-1000	-140	-40	200	-2380	-560
$(-,-2,-1,\ 3)$	-1000	-1386	-120	-600	-3402	-1008
$(-,\ 2,\ 0,-1)$	-4500	-7140	180	1800	4200	1680
$(+,\ 2,\ 0,-1)$	-4500	-6720	-360	-900	-9240	-2520
$(-,-2,\ 2,\ 2)$	-125	-343	-20	-50	-224	-70
$(+,\ 2,-1,-1)$	3000	-420	20	600	0	360
$(-,\ 2,-1,-2)$	-250	490	-190	-1000	-4480	-1400
$(-,\ 0,\ 1,\ 0)$	27000	-22680	540	-2700	37800	9900
$(+,\ 0,\ 1,\ 0)$	26625	-22995	-360	-7125	10080	1935
$(-,-1,\ 1,\ 1)$	0	-20412	0	0	20412	4860
$(+,\ 1,-1,-1)$	400	-20272	544	2800	-4340	-160
$(-,\ 1,\ 1,\ 0)$	0	630	0	0	-630	-180

Table 3 Generators for $\ker\left(\beta_7^{\mathbf{Q}(t)}\right)^{\mathcal{S}_3}$ **and** $\ker\left(\beta_6^{\mathbf{Q}(t)}\right)^{\mathcal{S}_3}$ **in Thm 1**

2.5 Specializing to Ladders

A *polylogarithmic ladder* is a (finite) linear combination $\sum_i n_i Li_m(\alpha^i)$ for some algebraic number α, some positive integer m, and integers n_i, which can be written as a rational linear combination of $\log^j(\alpha)$ products of logarithms. Lewin gave examples up to weight $m = 9$ (cf. [1], [18], Chapters 1–6). Cohen, Lewin and Zagier were able to push the set-up in Zagier's polylogarithm conjecture [24] to produce an example of a ladder up to weight $m = 16$ (cf. [7]), but they had missed a relation which was eventually detected by Bailey and Broadhurst, allowing the latter to "climb" one weight higher to the current ladder record $m = 17$ (cf. [2], where they also give ladders for other Salem numbers up to weight 13). The algebraic number α involved in this ladder is a very distinguished one: it is the so-called *Lehmer number* (the unique root of $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$ of absolute value > 1) which conjecturally has the smallest Mahler measure among algebraic numbers.

The originally quite surprising occurrence of such ladders seems now well understood in the context of Zagier's polylogarithm conjecture (see, e.g., [24], §7C and [25], §4).

Specific ladders

From the functional equations above, we can deduce two linearly independent ladders of weight 7 and four such of weight 6.

With the notation of [1], we let ω be a root of the equation

$$x^3 + x^2 = 1.$$

Zagier's conjecture implies that there should be at least 4 linearly independent ladders for weight 7 for ω (cf. [7], §3, and [25], §4).

By substituting $-\omega$ for t in the two independent functional equations in one variable stated in Table 3 in terms of the coefficients $c_j^{(7)} = c_j^{(7)}(a)$ $(j = 1, 2)$, we arrive at the first proven ladder relations for weight 7 (we divide the coefficients by a suitable power of 2):

Corollary 4 *The following two ladder relations hold for the 7–logarithm:*

$$\mathcal{L}_7\Big(\tfrac{35397}{256}[1] + \tfrac{1475}{8}[\omega] - \tfrac{166525}{1024}[\omega^2] - \tfrac{3825}{16}[\omega^3] - \tfrac{55025}{512}[\omega^4] + 127[\omega^5]$$
$$+\tfrac{34575}{512}[\omega^6] - \tfrac{5225}{256}[\omega^8] + \tfrac{475}{16}[\omega^9] - \tfrac{4117}{1024}[\omega^{10}] - \tfrac{1375}{512}[\omega^{12}] - \tfrac{75}{8}[\omega^{14}]$$
$$-\tfrac{29}{16}[\omega^{15}] - \tfrac{475}{1024}[\omega^{18}] - \tfrac{133}{512}[\omega^{20}] + \tfrac{25}{256}[\omega^{28}] + \tfrac{29}{1024}[\omega^{30}]\Big) \;=\; 0\,,$$

$$\mathcal{L}_7\Big(\tfrac{194355}{512}[1] + \tfrac{6265}{16}[\omega] - \tfrac{479395}{1024}[\omega^2] - \tfrac{2317}{4}[\omega^3] - \tfrac{146125}{1024}[\omega^4] + \tfrac{5005}{16}[\omega^5]$$
$$+\tfrac{84455}{512}[\omega^6] - 9[\omega^7] - \tfrac{6769}{128}[\omega^8] + \tfrac{497}{16}[\omega^9] - \tfrac{9835}{1024}[\omega^{10}] - \tfrac{5523}{1024}[\omega^{12}] - \tfrac{1551}{128}[\omega^{14}]$$
$$-\tfrac{35}{16}[\omega^{15}] - \tfrac{497}{1024}[\omega^{18}] - \tfrac{245}{1024}[\omega^{20}] + \tfrac{65}{512}[\omega^{28}] + \tfrac{35}{1024}[\omega^{30}]\Big) \;=\; 0\,.$$

We note that from the 2-variable equation for the 7-logarithm in [11] we do not obtain an independent ladder, but instead a linear combination of these two, viz.

$$\mathcal{L}_7\Big(-\tfrac{476217}{512}[1] - \tfrac{10675}{16}[\omega] + \tfrac{307825}{256}[\omega^2] + \tfrac{19565}{16}[\omega^3] - \tfrac{39725}{1024}[\omega^4] - \tfrac{10801}{16}[\omega^5]$$
$$-\tfrac{90125}{256}[\omega^6] + 45[\omega^7] + \tfrac{31115}{256}[\omega^8] + \tfrac{105}{2}[\omega^9] + \tfrac{5089}{256}[\omega^{10}] + \tfrac{8365}{1024}[\omega^{12}] - \tfrac{645}{128}[\omega^{14}]$$
$$-\tfrac{7}{4}[\omega^{15}] - \tfrac{105}{128}[\omega^{18}] - \tfrac{637}{1024}[\omega^{20}] + \tfrac{25}{512}[\omega^{28}] + \tfrac{7}{256}[\omega^{30}]\Big) \;=\; 0\,.$$

This seems to suggest that the 2-variable equation just mentioned may not specialize (at least not directly) to the individual 1-variable equations for $\mathcal{L}_7$ in Table 3.

We can observe here a certain "correlation" of exponents and coefficients which had already been detected by Lewin in connection with other ladders: denoting the coefficient of $[\omega^k]$ in any of the two combinations above by c_k, then one verifies, for $p = 5$ in the first combination and for $p = 7$ in the second, that $p \mid c_k \;\Leftrightarrow\; p \nmid k$, at least for $k > 0$.

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