**Some Computations in Weight 4 Motivic Complexes**

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**Abstract** We perform computations using the candidates for motivic cohomology as proposed by Goncharov, thereby giving evidence for his general conjecture about motivic complexes which would imply (among others) Zagier's conjecture on polylogarithms and values of Dedekind zeta functions. As a by-product, we obtain a family of functional equations for the 4-logarithm.

**1. Motivation.** One of the important problems in algebraic $K$-theory is to find an explicit description of higher $K$-groups in terms of generators and relations.

After Bloch's pioneering work [Bl] for the case of $K_3(F)$, $F$ a number field, Zagier [Z1] gave conjectural candidates for all $K_{2m-1}(F)\otimes\mathbf{Q}$ (for number fields, $m\geq 2$). Goncharov [G1] went further and proposed explicit candidates for motivic complexes whose cohomology groups for *any* field should coincide (at least rationally) with appropriate pieces of $K$-groups. These complexes not only led him to a stronger (and more conceptual) conjecture, but they also enabled him to prove a consequence of Zagier's conjecture for $m=3$ to the effect that $\zeta_F(3)$ for any number field $F$ is expressible in terms of modified trilogarithms $\mathcal{L}_3(x_i)$ (see below) where the arguments $x_i$ lie in $F$ (the corresponding result for $m=2$ had been shown earlier by Suslin [Su], and a weaker version also by Zagier [Z2]).

The fact that Goncharov's conjectures are stronger than Zagier's original one can be seen already in the case $m=4$ and $F=\mathbf{Q}$, where $K_7(\mathbf{Q})\otimes\mathbf{Q}=0$ and Zagier's conjecture is true for trivial reasons: the modified 4-logarithm is identically zero on the real line. Nevertheless the corresponding conjectured property for Goncharov's motivic complex $\mathcal{B}(F;4)^\bullet$ (see below), namely being exact in degree 2, constitutes a non-trivial statement. This also gives some evidence for Zagier's conjecture.

Goncharov suggested to investigate the latter statement which should give evidence in favor of his conjecture and is outlined as "key question" below. Roughly, it claims that the image of certain degenerate configurations (of 8 points in 3-space) should coincide with the image of generic configurations.

In order to state the problem formally, we need to introduce some notation. Let $F$ be a field. Let $\mathbf{Z}[F^\times]$ be the free abelian group generated by the elements of $F^\times$. In [G2] Goncharov constructed certain subgroups

$$\mathcal{R}_n(F)\subset\mathbf{Z}[F^\times]\,.$$

In particular the modified $n$-logarithm function (cf. [Z1])

$$\mathcal{L}_n(z)=\begin{matrix}\Re\\ \Im\end{matrix}\Bigg(\sum_{j=0}^{n-1}\frac{2^j\,B_j}{j!}\,\log^j|z|\,\mathrm{Li}_{n-j}(z)\Bigg),\ \begin{matrix}(n\,\text{odd})\\ (n\,\text{even})\end{matrix}\qquad n\geq 2\,,$$

annihilates the group $\mathcal{R}_n(\mathbf{C})$. Here

$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(z)\, d\log z\,, \qquad \mathrm{Li}_1(z) = -\log(1-z)\,,$$

is the classical polylogarithm function and $B_j$ the $j$-th Bernoulli number. Set

$$\mathcal{B}_n(F) = \frac{\mathbf{Z}[F^\times]}{\mathcal{R}_n(F)}\,.$$

One can show that $\mathcal{R}_n(F)$ belongs to the kernel of the map

$$\delta_n : \quad \mathbf{Z}[F^\times] \to \mathcal{B}_{n-1}(F) \otimes F^\times\,,$$

given on generators by $\delta_n : \{x\} \mapsto \{x\}_{n-1} \otimes x\,, \quad (n > 2)$, and by $\delta_2 : \{x\} \mapsto (1-x) \wedge x$ for $n = 2$. Consider the following cohomological complex (cf. [G1], [G2])

$$\mathcal{B}(F;4)^\bullet : \qquad \mathcal{B}_4(F) \xrightarrow{\delta^{(1)}} \mathcal{B}_3(F) \otimes F^\times \xrightarrow{\delta^{(2)}} \mathcal{B}_2(F) \otimes \bigwedge^2 F^\times \xrightarrow{\delta^{(3)}} \bigwedge^4 F^\times$$

where $\mathcal{B}_4(F)$ is placed in degree 1, and the differentials $\delta^{(i)}$ are given on generators by

$$\delta^{(1)}\{x\}_4 \mapsto \{x\}_3 \otimes x\,,$$

$$\delta^{(2)} : \{x\}_3 \otimes y \mapsto \{x\}_2 \otimes x \wedge y\,,$$

$$\delta^{(3)} : \{x\}_2 \otimes y \wedge z \mapsto (1-x) \wedge x \wedge y \wedge z\,.$$

It was conjectured in [G1], [G2] that

$$H^i\big(\mathcal{B}(F;4)^\bullet\big)_{\mathbf{Q}} = \mathrm{gr}_4^\gamma\, K_{8-i}(F)_{\mathbf{Q}}\,. \qquad (*)$$

It follows from the results of [G2]–[G4] and standard conjectures on mixed motives that one should have a weight 4 motivic complex of the following shape:

$$\mathcal{G}(F;4)^\bullet : \qquad G_4(F) \xrightarrow{\overline{\delta^{(1)}}} \mathcal{B}_3(F) \otimes F^\times \xrightarrow{\delta^{(2)}} \mathcal{B}_2(F) \otimes \bigwedge^2 F^\times \xrightarrow{\delta^{(3)}} \bigwedge^4 F^\times$$

where $G_4(F)$ is a certain abelian group *containing* the group $\mathcal{B}_4(F)$ and is generated by certain configurations of 8 points in $\mathbf{P}^3(F)$ (for details, cf. [G4], s.4-5).

**Key question.** Is it true that $\delta^{(1)}\mathcal{B}_4(F)_{\mathbf{Q}} \stackrel{?}{=} \overline{\delta^{(1)}} G_4(F)_{\mathbf{Q}}$, i.e. do the degenerate configurations as above already generate the image of all configurations?

In this note we present the results of computer experiments for $F = \mathbf{Q}$ supporting a positive answer to the latter question.

**2. The evidence.** Notice that for $F = \mathbf{Q}$ we have $\mathrm{gr}_4^\gamma\, K_{8-i}(\mathbf{Q})_{\mathbf{Q}} = 0$ modulo torsion for all $i$. Since we also expect $H^i(\mathcal{G}(\mathbf{Q},4)^\bullet) = \mathrm{gr}_4^\gamma\, K_{8-i}(\mathbf{Q})$, any $\alpha \in \ker \delta^{(2)}$

should belong to $\overline{\delta^{(1)}}G_4(\mathbf{Q})$. Therefore, if $\mathcal{B}_4(\mathbf{Q})_\mathbf{Q} = G_4(\mathbf{Q})_\mathbf{Q}$ we should be able to write any $\alpha \in \ker \delta^{(2)}$ as the boundary of an element in $\mathcal{B}_4(\mathbf{Q})$.

In the following we consider two types of non-trivial elements in $\ker \delta^{(2)}$.

**a) The first type** consists of elements of the form

$$\{1\}_3 \otimes q \ , \qquad q \in \mathbf{Q}^\times$$

(note that $\{1\}_2 = 0$ in $\mathcal{B}_2(\mathbf{Q})$). We show that they are indeed bounded by elements in $\mathcal{B}_4(\mathbf{Q})$, if the primes which have non-trivial valuation at $q$ are $\leq 59$.

We can restrict ourselves to the case of a prime number $p$, and we have been able to check the above for $p \leq 59$ (we only give examples for $p \leq 11$, though). We can proceed by induction, showing first that $\{1\}_3 \otimes p_\nu \in \mathrm{Image}(\delta^{(1)})$ for $p_\nu$ in some small set $\mathcal{P}$, and then it suffices to show that $\{1\}_3 \otimes p \in \mathrm{Image}(\delta^{(1)})$ modulo the subgroup generated by $\{1\}_3 \otimes p_\nu$ where $p_\nu \in \mathcal{P}$. For $11 \leq p \leq 59$ it was enough to take $\mathcal{P} \subset \{2,3,5,7\}$. We always compute modulo torsion in $\mathbf{Q}^\times$.

**b) The second type** of evidence comes from the following considerations (also suggested by Goncharov): in [G1] was given, for any field $F$, an element $\kappa(x,y)$ (given below, s.5.) in the difference kernel of the following map

$$\Big(\mathcal{B}_3(F) \otimes F^\times\Big) \oplus \bigwedge^2 \mathcal{B}_2(F) \quad \stackrel{\delta^{(2)} \oplus \delta^{(0)}}{\longrightarrow} \quad \mathcal{B}_2(F) \otimes \bigwedge^2 F^\times ,$$

where $\delta^{(0)}$ is defined on generators as $\{x\}_2 \wedge \{y\}_2 \mapsto \{y\}_2 \wedge (1-x) \wedge x - \{x\}_2 \wedge (1-y) \wedge y$. For any linear combination $\xi = \sum_i n_i[x_i] \in \mathbf{Z}[F^\times]$ we can form $\kappa(\xi, y)$ (extended linearly), and if $\xi$ corresponds to a functional equation of the dilogarithm, it is zero in $\mathcal{B}_2(F)$, and the projection of $\kappa(\xi,y)$ to the first summand $\mathcal{B}_3(F) \otimes F^\times$ lies in the kernel of $\delta^{(2)}$ (the projection to the second giving zero) and should be the boundary of an element in $\mathcal{B}_4(F)$.

We will construct those "bounding" elements for infinitely many functional equations $\xi$ (in one variable). We have performed similar calculations for certain $\mathbf{Q}-$specializations of the five term relation which always could be bounded. We have not been able to obtain the corresponding result for the general five term relation, which should enable one to define $\mathcal{B}_4(F)$ explicitly.

**3. How to compute.** We take finitely many elements $[x_i]$ in $\mathbf{Z}[\mathbf{Q}^\times]$ and compute their image under the map $\beta_4 : \mathbf{Z}[\mathbf{Q}^\times] \longrightarrow \mathrm{Sym}^2(\mathbf{Q}^\times) \otimes \bigwedge^2 \mathbf{Q}^\times$ given on generators by $[x] \mapsto x \otimes x \otimes x \wedge (1-x)$. The images actually lie in a finitely generated subgroup, and the computer finds elements in $\ker \beta_4$ by looking for linear dependences among the images. We expect $\delta^{(1)}(\xi)$ to lie in $\{1\}_3 \otimes \mathbf{Q}^\times$ as indicated above, and this is exactly what we find.

For each valuation $v : \mathbf{Q}^\times \to \mathbf{Z}$ let $\iota_v \delta^{(1)}(\xi)$ denote the following homomorphism:

$$\begin{aligned} \iota_v : \ \mathcal{B}_3(\mathbf{Q}) \otimes \mathbf{Q}^\times &\longrightarrow \mathcal{B}_3(\mathbf{Q}) \\ \{x\}_3 \otimes y &\mapsto \iota_v(y) \cdot \{x\}_3 . \end{aligned}$$

We have checked numerically that $\iota_v \delta^{(1)}(\xi)$ for each valuation $v : \mathbf{Q}^\times \to \mathbf{Z}$ is mapped under the trilogarithm map to a rational multiple of $\zeta(3)$ (the denominator of this rational multiple divides 12). This numerical check could be made rigorous by deriving $\iota_v \delta^{(1)}(\xi)$ from specializations of functional equations.

If, for $p$ as above, this rational multiple is non–zero (this is proved(!) numerically), we are done.

**4. Examples (first type).** 1) It is an easy exercise to show that $\{1\}_3 \otimes 2 \in \mathrm{Image}(\delta^{(1)})$ (use $\{x\}_3 = \{1/x\}_3$ and $\{x\}_3 + \{1-x\}_3 + \{1-1/x\}_3 = \{1\}_3$ to deduce $\delta^{(1)}([2]) = \frac{7}{8}\{1\}_3 \otimes 2$).

2) For $p = 3$ we consider the following linear combination:

$$\xi = 4[1/3] + [1/4] + 8[2/3] + 2[3/4]\,.$$

We compute

$$\begin{aligned}\delta^{(1)}(\xi) &= 4(\{1/3\}_3 \otimes 3^{-1}) + \{1/4\}_3 \otimes 4^{-1} \\ &\quad + 8(\{2/3\}_3 \otimes 2/3) + 2(\{3/4\}_3 \otimes 3/4) \\ &= \big(-2\{1/4\}_3 + 8\{2/3\}_3 - 4\{3/4\}_3\big) \otimes 2 + \big(-4\{1/3\}_3 - 8\{2/3\}_3 + 2\{3/4\}_3\big) \otimes 3 \\ &= \frac{7}{3}\{1\}_3 \otimes 2 - \frac{26}{3}\{1\}_3 \otimes 3\,.\end{aligned}$$

An even simpler element is

$$\eta = 2[3] - [-3]\,,$$

for which

$$\delta^{(1)}(\eta) = \frac{13}{6}\{1\}_3 \otimes 3\,.$$

3) For $p > 3$ the preimages found become more and more complicated. We list examples for $p = 5, 7, 11$. The data for the larger $p$ can be viewed on the webpage http://www.exp-math.uni-essen.de/~herbert/

$$\begin{aligned}\delta^{(1)}\big(&-9[1/3] - 3[1/4] + 1[1/6] + 1[2/3] + 2[2/5] - 4[3/4] - 14[3/5] + 1[3/8] \\ &- 10[4/5] + 2[4/9] + 2[5/6] + 13[5/8] + 6[5/9] - 6[8/9] + 8[9/10] + 3[15/16] \\ &- 2[16/25] - 3[24/25] - 1[25/27] + 1[80/81] - 1[125/128]\ \big) \\ &= -\frac{259}{3}\{1\}_3 \otimes 2 + \frac{124}{3}\{1\}_3 \otimes 5\,,\end{aligned}$$

$$\begin{aligned}\delta^{(1)}\big(\ &4[1/3] - 10[1/4] + 86[1/7] - 22[1/8] + 4[1/9] - 5[1/28] - 4[1/49] + 2[1/64] \\ &- 2[2/9] - 6[3/4] + 70[3/7] + 25[4/7] + 20[6/7] + 24[7/8] + 4[7/9] + 2[7/16] \\ &+ 2[8/9] + 15[9/16] - 2[27/28] + 2[32/81] + 2[48/49] + 1[49/81]\ \big) \\ &= \frac{63}{4}\{1\}_3 \otimes 2 + \frac{208}{3}\{1\}_3 \otimes 3 - 114\{1\}_3 \otimes 7\,,\end{aligned}$$

$$
\begin{aligned}
\delta^{(1)}\big(\ & 12[1/3] + 2[1/4] - 12[1/5] + 2[1/6] + 1[1/9] + 17[1/11] - 6[1/12] - 4[1/16] \\
& + 4[1/25] + 5[1/45] - 1[1/81] - 1[1/100] - 3[1/121] - 1[1/243] - 2[2/3] + 26[2/11] \\
& + 8[2/27] - 4[3/4] - 2[3/5] + 1[3/8] + 30[3/11] - 5[3/25] + 1[3/128] + 4[4/5] \\
& + 10[4/15] + 4[5/6] - 1[5/8] - 4[5/9] + 30[5/11] + 4[5/16] - 4[5/27] + 3[5/32] \\
& + 43[6/11] + 7[8/9] + 38[8/11] + 14[8/33] + 6[9/10] + 41[9/11] - 11[9/20] + 2[9/25] \\
& - 2[9/64] + 48[10/11] - 19[11/12] - 44[11/15] - 23[11/16] - 41[11/20] - 4[11/27] \\
& - 16[11/36] - 7[11/75] - 20[15/16] - 10[16/25] + 5[16/27] - 30[22/25] - 35[22/27] \\
& - 21[24/25] - 8[25/27] + 28[25/33] - 2[25/36] + 2[25/121] + 1[25/729] + 8[27/32] \\
& - 9[32/33] + 2[32/275] - 6[40/121] - 33[44/45] + 14[54/55] + 7[55/64] - 2[64/75] \\
& - 5[81/121] - 7[96/121] - 9[99/100] + 1[120/121] + 14[121/125] - 4[125/128] \\
& + 1[135/256] + 6[242/243] + 1[704/729]\ \big) \\
& \quad = \frac{3269}{12}\{1\}_3 \otimes 2 + \frac{910}{3}\{1\}_3 \otimes 3 + \frac{868}{3}\{1\}_3 \otimes 5 - \frac{1330}{3}\{1\}_3 \otimes 11.
\end{aligned}
$$

**5. Examples (second type).**

Let $p(t)$ be the algebraic function defined by the equation

$$
f(t, p(t)) = 0\,, \qquad \text{where}
$$

$$
f(t,x) = x^a(1-x)^b - t\,, \qquad a, b \in \mathbf{Z}\,.
$$

For the non-critical points $t \in \mathbf{C}$ let $\{p_i(t)\}_i$ be locally analytic branches of $p(t)$.

Note that the sum $\sum_i [p_i(t)]$ is well–defined globally as a divisor on $\mathbf{Z}[\mathcal{O}_C]^G$ where $\mathcal{O}_C$ denotes the ring of locally analytic functions on the curve $C$ with defining equation $f(t,x) = 0$ and $G$ is the symmetric group on $\deg(f)$ elements.

We deduce

$$
1 - p_i = \pm \frac{\prod_i p_i^{1/b}}{p_i^{a/b}}, \quad 1 - \frac{1}{p_i} = \pm \frac{\prod_i p_i^{1/b}}{p_i^{(a+b)/b}}\,.
$$

Let $c$ be defined by $a+b+c=0$ and denote $\xi = \sum_i [p_i(t)] \in \mathbf{Z}[\mathbf{C}(t)]$. Notice that $\xi \in \ker \delta_2$ (up to 2−torsion).

The element $\kappa(x,y)$ (cf. 2.b) was defined by Goncharov in [G2] via

$$
\begin{aligned}
-\kappa(x,y) = & \left( \{1-x\}_3 + \{1-y\}_3 + \left\{\frac{1-x^{-1}}{1-y^{-1}}\right\}_3 - \left\{\frac{1-x}{1-y}\right\}_3 \right) \otimes \frac{x}{y} \\
& + \{\frac{x}{y}\}_3 \otimes \frac{1-y}{1-x} - \{x\}_3 \otimes (1-y) + \{y\}_3 \otimes (1-x) \ + \ \{x\}_2 \wedge \{y\}_2\,.
\end{aligned}
$$

Now we can compute (note that for an element $\xi \in \mathcal{R}_2(F)$ the contributions from $\bigwedge^2 \mathcal{B}_2(F)$ cancel)

$$
\begin{aligned}
-\kappa(\xi,y) = &\sum_i \left( \{1-p_i\}_3 + \{1-y\}_3 + \left\{\frac{1-p_i^{-1}}{1-y^{-1}}\right\}_3 - \left\{\frac{1-p_i}{1-y}\right\}_3 \right) \otimes \frac{p_i}{y} \\
&+ \sum_i \{\frac{p_i}{y}\}_3 \otimes \frac{1-y}{1-p_i} - \sum_i \{p_i\}_3 \otimes (1-y) + \sum_i \{y\}_3 \otimes (1-p_i) \\
= &\sum_i \Bigg( \{1-p_i\}_3 + \{1-y\}_3 + \left\{\frac{1-p_i^{-1}}{1-y^{-1}}\right\}_3 - \left\{\frac{1-p_i}{1-y}\right\}_3 \\
&\qquad + \frac{a}{b}\{\frac{p_i}{y}\}_3 - \frac{a}{b}\{y\}_3 - \frac{1}{b}\sum_j \{\frac{p_j}{y}\}_3 + \frac{a+b}{b}\{y\}_3 \Bigg) \otimes p_i \\
&+ \sum_i \left( -\{1-p_i\}_3 - \{1-y\}_3 - \left\{\frac{1-p_i^{-1}}{1-y^{-1}}\right\}_3 + \left\{\frac{1-p_i}{1-y}\right\}_3 \right) \otimes y \\
&+ \sum_i \left(\{\frac{p_i}{y}\}_3 - \{p_i\}_3\right) \otimes (1-y)\,.
\end{aligned}
$$

If we use functional equations for the 3–logarithm we find that $-\kappa(\xi,y)$ is equal to

$$
\begin{aligned}
&\sum_i \Bigg( \frac{1}{abc}\left\{\frac{\prod_j p_j}{y^a(1-y)^b}\right\}_3 \\
&\quad + \frac{1}{a}\sum_j \{\frac{1-p_j}{1-y}\}_3 + \frac{1}{c}\sum_j \{\frac{1-y^{-1}}{1-p_j^{-1}}\}_3 + \frac{1}{b}\sum_j \{\frac{y}{p_j}\}_3 - \frac{1}{a}\sum_j \{1-p_j\}_3 + \{\frac{y-1}{y}\}_3 \\
&\qquad - \{\frac{1-p_i}{1-y}\}_3 + \{\frac{1-y^{-1}}{1-p_i^{-1}}\}_3 - \frac{1}{b}\sum_j \{\frac{y}{p_j}\}_3 + \{1-p_i\}_3 + \{1-y\}_3 \\
&\qquad\qquad + \frac{a}{b}\{\frac{y}{p_i}\}_3 + \{y\}_3 \Bigg) \otimes p_i \\
&+ \Bigg( -\frac{1}{bc}\left\{\frac{\prod_j p_j}{y^a(1-y)^b}\right\}_3 - (a+b)\{1-y\}_3 \\
&\quad - \sum_j \{\frac{1-p_j}{1-y}\}_3 - \frac{a}{c}\sum_j \{\frac{1-y^{-1}}{1-p_j^{-1}}\}_3 - \frac{a}{b}\sum_j \{\frac{y}{p_j}\}_3 + \sum_j \{1-p_j\}_3 - a\{\frac{y-1}{y}\}_3 \\
&\quad + \sum_j \{\frac{1-p_j}{1-y}\}_3 - \sum_j \{\frac{1-y^{-1}}{1-p_j^{-1}}\}_3 - \sum_j \{1-p_j\}_3 \Bigg) \otimes y \\
&+ \Bigg( -\frac{1}{ac}\left\{\frac{\prod_j p_j}{y^a(1-y)^b}\right\}_3 \\
&\quad - \frac{b}{a}\sum_j \{\frac{1-p_j}{1-y}\}_3 - \frac{b}{c}\sum_j \{\frac{1-y^{-1}}{1-p_j^{-1}}\}_3 - \sum_j \{\frac{y}{p_j}\}_3 + \sum_j \{p_j\}_3 - b\{\frac{y-1}{y}\}_3 \\
&\qquad - \sum_j \{\frac{y}{p_j}\}_3 + \sum_j \{p_j\}_3 \Bigg) \otimes (1-y)\,.
\end{aligned}
$$

Here the functional equations for the trilogarithm are given as the first 2 lines of each of the 3 large brackets in the previous expression. But the last element is the boundary of the following one:

$$\eta_{a,b}(t,y) = \frac{1}{abc}\left\{\frac{\prod_j p_j}{y^a(1-y)^b}\right\}_4 + \frac{b}{a}\sum_j\{\frac{1-p_j}{1-y}\}_4 - \frac{b}{c}\sum_j\{\frac{1-y^{-1}}{1-p_j^{-1}}\}_4 - \frac{a}{b}\sum_j\{\frac{y}{p_j}\}_4$$
$$-\frac{b}{a}\sum_j\{1-p_j\}_4 - b\{\frac{y-1}{y}\}_4 - c\{y\}_4\,.$$

**Remark.** Zagier initially gave the corresponding element for $\xi = [x]+[1/x]$, more precisely: $\kappa(\xi,y) = d_1(-A(x,y))$, where

$$A(x,y) = \frac{1}{2}\{\frac{x(1-y)^2}{y(1-x)^2}\}_4 - \frac{1}{2}\{xy\}_4 + \frac{1}{2}\{\frac{x}{y}\}_4 + 2\{\frac{1-x}{1-y}\}_4 + 2\{\frac{y(1-x)}{x(1-y)}\}_4 + 2\{\frac{x-1}{x(1-y)}\}_4$$
$$+2\{\frac{y(1-x)}{y-1}\}_4 - \{y\}_4 - 2\{1-x\}_4 + 2\{1-y\}_4 - 2\{\frac{x-1}{x}\}_4 + 2\{\frac{y-1}{y}\}_4\,.$$

**6. Corollary.** *There exists a family of functional equations in 2 variables for the 4-logarithm of the following form: let $\{p_j = p_j(t)\}$ and $\{q_k = q_k(u)\}$ be the set of roots of $C_1 : x^a(1-x)^b = t$ and $C_2 : x^d(1-x)^e = u$, respectively ($a,b,d,e \in \mathbf{Z}$, $a+b+c = 0 = d+e+f$). Denote $D = bd - ae = ce - bf$. Then $\mathcal{L}_4$ vanishes on the following element in $\mathbf{Z}[\mathcal{O}_X]$ where $X$ is the product of the two curves $C_1$ and $C_2$:*

$$\frac{1}{abc\,D}\sum_k[\frac{t}{q_k^a(1-q_k)^b}] + \frac{1}{def\,D}\sum_j[\frac{u}{p_j^d(1-p_j)^e}]$$
$$-\frac{1}{ad}\sum_{j,k}[\frac{1-q_k}{1-p_j}] - \frac{1}{cf}\sum_{j,k}[\frac{1-p_j^{-1}}{1-q_k^{-1}}] - \frac{1}{be}\sum_{j,k}[\frac{p_j}{q_k}]$$
$$+\frac{1}{f}\sum_k[\frac{q_k-1}{q_k}] - \frac{1}{c}\sum_j[\frac{p_j-1}{p_j}]\,.$$

**Proof.** Write $\kappa(\sum_j[p_j],\sum_k[q_k])$ in two different ways:

$$\sum_k \kappa(\textstyle\sum_j[p_j],q_k) = \kappa(\sum_j[p_j],\sum_k[q_k]) = \displaystyle\sum_j \kappa(p_j,\textstyle\sum_k[q_k])\,.$$

Each of the summands on the left and right hand side can be expressed as the boundary of some element in $\mathbf{Z}[\mathcal{O}_X]$, e.g. (using the anticommutativity of $\kappa$):

$$\sum_k \kappa(\textstyle\sum_j[p_j],q_k) = -\displaystyle\sum_k d_1\Big(\eta_{a,b}(t,q_k)\Big)\,,$$

and the difference of the left and right hand side is therefore expressible as $d_1\zeta$ for some $\zeta \in \mathbf{Z}[\mathcal{O}_X]$. It remains to check that $\zeta$ is (up to elements in $\ker d_1$) the one given in the theorem—we have used functional equation (4.1.6) from [Ga]:

$$bc\sum_j\{1-p_j\}_4 \,-\, ac\sum_j\{p_j\}_4 \,-\, ab\sum_j\{\frac{p_j-1}{p_j}\}_4 \,=\, 0\,,$$

to simplify the expression.

**Remark.** 1. Kummer's functional equation for the 4-logarithm can be recovered e.g. as the special case $(a,b)=(1,1)$, $(d,e)=(1,-2)$.

2. The family in the corollary is essentially the one given in [Ga], Thm. 4.4.

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