Clay Mathematics Proceedings

Multiple zeta values: from numbers to motives

José Ignacio Burgos Gil and Javier Fresán

With contributions by Ulf Kühn

Contents

Preface

Multiple zeta values (MZVs for short) are real numbers of the form

$$\zeta(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 > n_2 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}, \tag{0.1}$$

where all s_i are integers greater than or equal to 1 and $s_1 \geq 2$, to ensure that the sum converges. For $\ell = 1$, these are the Riemann zeta values

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Euler proved that, when s is even, $\zeta(s)$ is a rational multiple of π^s; for example, $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. The values at odd integers are much more mysterious. Indeed, a folklore conjecture asserts that they are all "new" transcendental numbers:

Transcendence conjecture. *The numbers* $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ *are algebraically independent.*

This conjecture seems completely out of reach: at the time of writing, we do not even know whether $\zeta(3)$ is transcendental, let alone the algebraic independence with π, or whether $\zeta(5)$ is irrational!

The case $\ell = 2$ was also considered by Euler, back in his 1776 paper *Meditationes circa singulare serierum genus* ("Meditations about a singular type of series") [**Eul76**]. In an attempt to find a closed formula for $\zeta(3)$, he looked for linear relations with integer coefficients among the numbers π^3, $\pi^2 \log 2$ and $(\log 2)^3$. This led him to the discovery of remarkable identities involving double zeta values, the simplest being

$$\zeta(3) = \zeta(2, 1).$$

After more than two centuries of oblivion, multiple zeta values were independently rediscovered in the 1990s by Hoffman and Zagier. It was soon realized that these numbers appear in a wealth of different contexts, including Witten's zeta functions, deformation quantization, Vassiliev knot invariants or the theory of mixed Tate motives. Most of these topics share a physics flavour. In fact, roughly at the same time, the physicists Broadhurst and Kreimer found that a lot of Feynman amplitudes in quantum field theory can be expressed in terms of multiple zeta values. The next two decades saw extensive work by Brown, Cartier, Deligne, Drinfeld, Écalle, Goncharov, Hain, Hoffman, Kontsevich, Terasoma, Zagier, and many others. Major progress was made, but fundamental questions remain open and multiple zeta values are still nowadays an active, rapidly moving field of research

The product of two multiple zeta values is a linear combination, with integer coefficients, of multiple zeta values. For instance,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2),$$

an identity already known to Euler. The $\mathbb{Q}$-subvector space $\mathcal{Z} \subseteq \mathbb{R}$ spanned by all multiple zeta values has thus an algebra structure. Contrary to the algebra generated by Riemann zeta values, which according to the transcendence conjecture is simply a polynomial algebra in $\zeta(2), \zeta(3), \zeta(5), \ldots,$ multiple zeta values satisfy a plethora of relations which endow $\mathcal{Z}$ with a rich combinatorial structure. One can argue that the main goal of the theory is to understand all linear relations among these numbers.

To make this more precise, we attach to each $\zeta(s_1, \ldots, s_l)$ the integer $s_1 + \ldots + s_l$, which is called the *weight*. Let $\mathcal{Z}_k \subseteq \mathcal{Z}$ be the vector subspace generated by multiple zeta values of weight k, with the convention that $\mathcal{Z}_0 = \mathbb{Q}$ and $\mathcal{Z}_1 = \{0\}$. Based on a mix of numerical evidence and pure thought, Zagier conjectured that there is a direct sum decomposition

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k,$$

and that the dimension of each graded piece is given by

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k,$$

where $(d_k)_{k \geq 0}$ is a Fibonacci-like sequence of integers, defined recursively by setting $d_0 = d_2 = 1, d_1 = 0$ and $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$, so that the generating series is equal to

$$\sum_{k \geq 0} d_k t^k = \frac{1}{1 - t^2 - t^3}. \tag{0.2}$$

This would imply that $\dim_{\mathbb{Q}} \mathcal{Z}_k$ grows like a constant multiple of r^k, where $r = 1.3247...$ is the real root of $x^3 - x - 1$, which is much smaller than the number 2^{k-2} of multiple zeta values of weight k.

Plan. The goal of these notes is to give a reasonably self-contained proof of the following results towards Zagier's conjecture:

Theorem A (Deligne-Goncharov [**DG05**], Terasoma [**Ter02**]). The following inequality holds:

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k.$$

Theorem B (Brown, [**Bro12**]). Each multiple zeta value can be written as a $\mathbb{Q}$-linear combination of multiple zeta values with only $2s$ and $3s$ as exponents, *i.e.* the following family generates the $\mathbb{Q}$-vector space $\mathcal{Z}$:

$$\{\zeta(s_1, \ldots, s_\ell) \mid s_i \in \{2, 3\}\}. \tag{0.3}$$

In fact, Hoffman conjectured that (0.3) forms a *basis* of $\mathcal{Z}$. By a simple counting argument, this would imply the equality (0.2). Theorem B addresses the "algebraic" part of this conjecture, which suffices to deduce Theorem A. It is also worth mentioning that, taking these results for granted, the algebraic independence of $\pi, \zeta(3), \zeta(5), \ldots$ is a consequence of Zagier's conjecture. In a sense, we have "linearized" the transcendence conjecture.

On the negative side, let us emphasize that, despite the progress made thus far, we still do not know a single k for which $\dim_{\mathbb{Q}} \mathcal{Z}_k > 1$!

Surprisingly enough, the proofs of these easy-to-state theorems use the machinery of motives. Kontsevich noticed that multiple zeta values of weight k admit a representation as iterated integrals

$$\zeta(s_1, \ldots, s_\ell) = \int_{\Delta^k} \omega_0(t_1) \cdots \omega_0(t_{s_1-1})\omega_1(t_{s_1})\omega_0(t_{s_1+1}) \cdots \omega_1(t_k), \qquad (0.4)$$

where $\omega_0(t) = \frac{dt}{t}$ and $\omega_1(t) = \frac{dt}{1-t}$ are differential forms on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the integration domain $\Delta^k \subseteq [0, 1]^k$ is the simplex

$$\{1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 0\}.$$

This exhibits multiple zeta values as periods of algebraic varieties. In the words of Deligne, "whereas the notion of infinite sum is unfamiliar (*étrangère*) to algebraic geometry, the study of integrals of algebraic quantities is one of its sources." Thanks to the identity (0.4), "algebraic geometry, and more precisely the theory of mixed Tate motives, is useful for the study of multiple zeta values" [**Del13**, p.3].

Usually, the philosophy of motives represents a powerful tool to predict all algebraic relations between periods. However, when it comes to proving them, one is confronted with the problem that even the first step in this program—getting a category of motives with all the desired properties—remains conjectural. In contrast, for mixed Tate motives over a number field, there is an unconditional theory which relies ultimately on Borel's deep results about the K-theory of number fields. This gives good control over the group governing the symmetries of multiple zeta values. Using this group, one can construct a pro-algebraic variety, together with an action of $\mathbb{G}_m$, in such a way that the Hilbert-Poincaré series of its graded algebra of functions $\mathcal{H}$ coincides with (0.2). The *raison d'être* of this construction is the existence of a surjective map $\mathcal{H} \to \mathcal{Z}$ compatible with the weight; we shall refer to elements of $\mathcal{H}$ as "motivic multiple zeta values". This immediately implies Theorem A. To prove Theorem B, one exploits the motivic coaction, a new structure of $\mathcal{H}$, invisible at the level of numbers, which allows one to get relations amongst motivic multiple zeta values in a systematic way. A variant of the Grothendieck period conjecture asserts that the algebras $\mathcal{H}$ and $\mathcal{Z}$ are isomorphic, from which Zagier's conjecture would follow.

Outline. Let us now give a more detailed description of the contents of each chapter. The word cloud on the next page should also give a quick idea of the main concepts involved.

Chapter 1 lays out what could be called the "minimal theory" of multiple zeta values. We first define them as infinite series and prove that the product of two multiple zeta values is a linear combination of multiple zeta values by decomposing the indexation domain. This so-called *stuffle* product makes $\mathcal{Z}$ into a $\mathbb{Q}$-algebra, conjecturally graded by the weight. We

discuss Zagier's conjecture for the dimension of the graded pieces, as well as refinements due to Hoffmann, and Broadhurst and Kreimer. That progress has been made towards these conjectures relies very much on the existence of the integral representation (0.4). We prove that the decomposition of the product of two simplices yields a new algebra structure on $\mathcal{Z}$, the *shuffle* product. Comparing the stuffle and the shuffle product, one gets many relations amongst multiple zeta values but not all of them. As we explain in the last section of the chapter, to conjecturally describe the full algebraic structure, one needs to introduce a regularization process which assigns a finite value to the divergent series $\zeta(1, s_2, \ldots, s_\ell)$.

The goal of Chapter 2 is to show that multiple zeta values are periods. To begin with, we briefly recall the definition of the singular cohomology of a differential manifold and de Rham's theorem, according to which it can be computed using analytic differential forms. Grothendieck's breakthrough was to realize that, if we are dealing with algebraic varieties, algebraic differential forms suffice. This gives rise to algebraic de Rham cohomology and the period isomorphism. After introducing these concepts, we give a first interpretation, due to Goncharov and Manin, of multiple zeta values as periods of the moduli spaces $\overline{M}_{0,n}$ of stable genus zero curves. We then move to mixed Hodge structures (a first approximation to the notion of motive), discuss a number of examples and compute the extension groups of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. We end the chapter with a discussion of the problem of finding a geometric construction of these extensions, as well as a potential application to irrationality proofs.

Chapter 3 introduces iterated integrals, a second way to interpret multiple zeta values as periods. We first present the basic definitions and tackle the question of which iterated integrals are homotopy invariant. We then recall the notions of affine group schemes and Hopf and Lie algebras, which will be extensively used in the sequel. We define the pro-unipotent completion of a group and we construct it, under some finiteness assumptions, following work of Quillen. One of the main results of the chapter is Chen's π_1-de Rham theorem, which roughly says that functions on the pro-unipotent completion of the fundamental group of a differential manifold M are given by homotopy invariant iterated integrals. A consequence, due to Hain, is that when M underlies an algebraic variety, this pro-unipotent completion carries a mixed Hodge structure. The general formalism being settled, we specialize everything to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Multiple zeta values are iterated integrals along the straight path from 0 to 1. Since the endpoints do not belong to the space, this forces us to work with tangential base points. The last section examines in detail all the structures carried by the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, including Goncharov's coproduct.

In Chapter 4, we study the category of mixed Tate motives over $\mathbb{Z}$. The first two sections contain reminders of the Tannakian formalism, triangulated categories and t-structures. We then sketch the construction of Voevodsky's triangulated category of mixed motives over a field k. It is unknown how to extract an abelian category with good properties from it. However, it was observed by Levine that, when k is a number field, Borel's results on K-theory enable one to extract an abelian category of mixed Tate motives over k, which is moreover Tannakian. Even for $k = \mathbb{Q}$, this category is too large for the purposes of studying multiple zeta values. To remedy this, one defines the subcategory of mixed Tate motives over $\mathbb{Z}$. We determine the structure of its Tannaka group and show, following Deligne and Goncharov, that it contains a pro-object whose Hodge realization is the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Finally, in Chapter 5 we pull everything together to prove the main results. In the first section, we construct the graded algebra $\mathcal{H}$ of motivic multiple zeta values and a surjective map $\mathcal{H} \to \mathcal{Z}$ compatible with the grading. Using the structure of the Tannaka group of the category of mixed Tate motives over $\mathbb{Z}$, we derive Theorem A. We then present the proof of Theorem B, following closely Brown's original paper.

Warning. Before continuing, we should warn the reader that the literature contains two competing conventions for multiple zeta values, sometimes in the same paper! Other authors, including Brown, define $\zeta(s_1, \ldots, s_\ell)$, for $s_i \geq 1$ and $s_\ell \geq 2$, as the sum

$$\sum_{1 \leq n_1 < n_2 < \cdots < n_\ell} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$

In fact, one needs to fix conventions for the order of composition of paths, the definition of iterated integrals, and the expression of multiple zeta values as iterated integrals. Things get simpler if they are compatible. We have chosen those conventions for which the monodromy of a local system is a group morphism.

Prerequisites. The difficulty of the exposition increases as the notes progress. In Chapter 1, besides a couple of digressions, the emphasis is mainly on combinatorial aspects and very little background is required. From Chapter 2 on, we assume some familiarity with algebraic varieties and cohomology of sheaves, at the level of any introductory book. Chapter 3 contains a crash course on algebraic groups and Lie and Hopf algebras, which will play an important role in the sequel. However, we do not treat topics such as Lie algebra cohomology or Galois cohomology which will only appear in some proofs of the following chapter. Finally, in Chapter 4 we freely use basic notions from category theory and homological algebra, for example abelian categories. We have done our best to present all the materials in the most clear and accessible way, but occasionally we were unable to prevent the text from being sketchy. Unfortunately, Borel's theorem about the K-theory of number fields is used as a black box.

Acknowledgments. These notes are the outgrowth of a series of lectures given by the first author at the CMI summer school *Periods and Motives: Feynman Amplitudes in the 21st Century*, and by the second author at the MPIM Bonn and the University of Freiburg. We would like to thank all the participants. Parts of the paper were written while the authors were enjoying the hospitality of the ÉNS Paris, the ICMAT Madrid, the MPIM Bonn, the University of Kyoto, and the ETH Zürich.

The first author wants to thank the Clay Mathematics Institute for making the summer school possible. Moreover, he wants to acknowledge support from the MINECO grants MTM2013-42135-P and MTM2016-79400-P during the preparation of these notes.

The second author was first introduced to the subject at the master course *Multizêtas et groupe fondamental* given by Francis Brown in Jussieu during the spring of 2012. Later that summer he participated at the Alpbach workshop *Multiple zeta values*, organized by Joseph Ayoub and Sergey Gorchinsky. Finally, he wants to acknowledge support by the SNSF grants 200021-150099 and 200020-162928 during the preparation of these notes.

We would like to thank the following people who generously reviewed parts of these notes and answered our questions: Giuseppe Ancona, Joseph Ayoub, Olivier Benoist, Spencer Bloch, Francis Brown, Ishai Dan-Cohen, Frédéric Déglise, Brad Drew, Clément Dupont, Kurusch Ebrahimi-Fard, Riccardo Ferrario, Herbert Gangl, Claire Glanois, Ryotaro Harada, Annette Huber, Peter Jossen, Masanobu Kaneko, Shane Kelly, Marco Maculan, Nils Matthes, Simon Pepin Lehalleur, Corentin Perret, Jérôme Poineau, Alexander Saad, Michel Waldschmidt, Jörg Wildeshaus and Don Zagier.

We warmly thank the referees for their very careful and friendly reading of previous versions of the manuscript.

1. Classical theory of multiple zeta values (by J. I. Burgos Gil, J. Fresán and U. Kühn)

In this chapter, we introduce multiple zeta values and begin to study their basic properties. These are the real numbers

$$\zeta(s_1, \dots, s_\ell) = \sum_{n_1 > \dots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_2^{s_2}}$$

associated to tuples of integers $\boldsymbol{s} = (s_1, \dots, s_l)$ such that $s_i \geq 1$ and $s_1 \geq 2$. The quantity $s_1 + \dots + s_\ell$ is called the weight and ℓ is referred to as the length. Of great importance is that multiple zeta values cannot only be written as infinite series, as above, but also as integrals. This gives two different ways of showing that the product of $\zeta(\boldsymbol{s})$ and $\zeta(\boldsymbol{s}')$ is a linear combination, with integral coefficients, of multiple zeta values or, in more algebraic terms, that the $\mathbb{Q}$-vector space $\mathcal{Z} \subseteq \mathbb{R}$ generated by multiple zeta values has an algebra structure. From the series representation one obtains the stuffle product, whereas the integral representation gives the shuffle product. Comparing both products yields many relations amongst multiple zeta values. However, Euler's identity $\zeta(3) = \zeta(2,1)$ cannot be obtained by this method since the product of multiple zeta values has always weight at least 4. A way to solve this problem is to introduce a regularization process which assigns a finite value to the divergent series corresponding to multi-indices with $s_1 = 1$. There will be, in fact, two kinds of regularizations, modelled on the stuffle and the shuffle product. Conjecturally, all relations amongst multiple zeta values come from comparing them.

Good references for this chapter are the survey articles [**Car02**], [**Wal12**] and [**Zud03**], as well as Chapter 3 of the book [**Zha16**].

1.1. Riemann zeta values.

The Riemann zeta function is one of the most famous objects in mathematics. It is said that it encodes all arithmetic properties of prime numbers: our task is to extract them.

Definition 1.1. The *Riemann zeta function* is defined, on the half-plane of complex numbers s with $\mathrm{Re}(s) > 1$, by the absolute convergent series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \tag{1.2}$$

and extended to a meromorphic function on the whole complex plane with a single pole at $s = 1$.

The Riemann zeta function still keeps many mysteries. The most impenetrable of them is undoubtedly the Riemann hypothesis (the conjecture that all the non-trivial zeros of $\zeta(s)$ lie in the line $\mathrm{Re}(s) = 1/2$), which has many far-reaching consequences in number theory.

The aim of these notes is to glimpse at other aspects of this function, namely, what numbers do we get when evaluating ζ at integers? In fact, the story began 120 years before Riemann's paper[1], with Euler's solution to the so-called *Basel problem*, that is, the computation of the value

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Indeed, Euler showed much more:

THEOREM 1.3 (Euler, 1735). *The values of the zeta function at even positive integers are given by*

$$\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}. \tag{1.4}$$

Here B_{2k} are rational numbers, called *Bernoulli numbers* and defined by the power series identity

$$\frac{t}{e^t-1} = 1 + \sum_{k\geq 1} B_k \frac{t^k}{k!}. \tag{1.5}$$

REMARK 1.6. Note that the function

$$f(t) = \frac{t}{e^t-1} + \frac{1}{2}t = \frac{t(1+e^t)}{2(e^t-1)}$$

is even, *i.e.* satisfies $f(t) = f(-t)$. It follows that $B_1 = -\frac{1}{2}$ and $B_k = 0$ for all odd integers $k \geq 3$. The first Bernoulli numbers are easily computed:

k	2	4	6	8	10	12
B_k	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$

PROOF OF THEOREM 1.3. The key ingredient is an identity for the cotangent function, also due to Euler (see Exercise 1.19). For $x \in \mathbb{C} \setminus \mathbb{Z}$,

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{n\geq 1}\frac{2x}{x^2-n^2}. \tag{1.7}$$

Expanding the quotient inside the summation sign as a geometric series and interchanging the order of summation, we obtain

$$\pi\cot(\pi x) = \frac{1}{x} - 2\sum_{k\geq 1}\zeta(2k)x^{2k-1}. \tag{1.8}$$

Besides, we have

$$\frac{1}{e^t-1} = \frac{e^{-\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \quad \text{and} \quad -\frac{1}{e^{-t}-1} = \frac{e^{\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}},$$

[1] For the prehistory of the Riemann zeta function we refer the reader to Weil's beautiful account [**Wei89**].

from which the identity

$$\frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}=\frac{2}{t}+2\sum_{k\geq 1}\frac{B_{2k}t^{2k-1}}{(2k)!}$$

follows, using (1.5) and the vanishing of B_k for odd $k \geq 3$. Therefore,

$$\pi\cot(\pi x)=\pi i\frac{e^{\frac{2\pi i x}{2}}+e^{-\frac{2\pi i x}{2}}}{e^{\frac{2\pi i x}{2}}-e^{-\frac{2\pi i x}{2}}}=\frac{1}{x}+\sum_{k\geq 1}\frac{(2\pi i)^{2k}B_{2k}}{(2k)!}x^{2k-1}, \tag{1.9}$$

and we conclude by identifying the coefficients in (1.8) and (1.9). □

Remarks 1.10.

(1) Euler's formula (1.4) implies the equality

$$\mathbb{Q}[\zeta(2),\zeta(4),\dots]=\mathbb{Q}[\pi^2]$$

of subrings of the real numbers.

(2) Thanks to the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.11}$$

where Γ is the gamma function, we deduce the values of the Riemann zeta function at negative integers:

$$\zeta(-k)=-\frac{B_{k+1}}{k+1}$$

for all $k \geq 1$. In particular, $\zeta(-2k)=0$ for all $k \geq 1$; these are the "trivial zeros". One can also compute $\zeta(0)=-\frac{1}{2}$.

1.1.1. *Odd values.* By contrast, despite the many efforts of the mathematical community, nobody has been able to give closed formulas for the values of the Riemann zeta function at $s=3,5,7,\dots$ in terms of previously known numbers like π. This led to the following conjecture:

Conjecture 1.12 (Transcendence conjecture). *The numbers*

$$\pi,\zeta(3),\zeta(5),\dots$$

are algebraically independent, that is, for each integer $k \geq 0$ and each non-zero polynomial $P \in \mathbb{Z}[x_0,\dots,x_k]$, we have $P(\pi,\zeta(3),\dots,\zeta(2k+1)) \neq 0$.

This conjecture seems completely out of reach of the current techniques in transcendence theory. The transcendence of π was proved by Lindemann in 1882 [**Lin82**]. It follows from Euler's formula (1.4) that the numbers $\zeta(2k)$ are all transcendental. But we do not even know whether $\zeta(3)$ is transcendental —not to speak of the algebraic independence with π— or whether $\zeta(5)$ is irrational. The few known results, as the moment of writing, are summarized below. The Bourbaki seminar [**Fis04**] is an excellent survey.

- *Apéry* [**Apé79, vdP79**] proved that $\zeta(3)$ is irrational. Different proofs are now available [**Beu79**], but none of them seems to generalize to other odd values.
- *Rivoal* [**Riv00**] and *Ball* and *Rivoal* [**BR01**] proved that, if n is an odd integer ≥ 3, then
$$\dim_{\mathbb{Q}}\langle 1, \zeta(3), \zeta(5), \dots, \zeta(n)\rangle \geq \frac{1}{3}\log(n).$$
In particular, infinitely many $\zeta(2k+1)$ are irrational.
- *Zudilin* [**Zud01**] proved that at least one out of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

Remark 1.13. Recently, Brown has suggested in [**Bro16**] a common geometric framework for these irrationality proofs. The approach is based on the study of periods of the moduli spaces $M_{0,n}$ of curves of genus zero with n marked points (see paragraph 2.8.3).

Digression 1.14. Despite their "simplicity", special values of the Riemann zeta function are linked to much interesting mathematics. For instance, K-groups and regulators provide an explanation of why the values at even integers are easier to understand.

Let F be a number field and $\mathcal{O}_F$ its ring of integers. The *Dedekind zeta function* of F is defined, for $\mathrm{Re}(s) > 1$, by the convergent series
$$\zeta_F(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s},$$
where $\mathfrak{a}$ runs through all non-zero ideals of $\mathcal{O}_F$ and $N(\mathfrak{a})$ denotes the absolute norm. In particular, $\zeta_{\mathbb{Q}}$ agrees with the Riemann zeta function (1.2).

The Dedekind zeta function extends to a meromorphic function on the complex plane, with a simple pole at $s = 1$. Its residue is given by the celebrated *class number formula*
$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{w_F\sqrt{|d_F|}},$$
where r_1 (resp. $2r_2$) denotes the number of real (resp. complex) embeddings of F, h_F is the class number, w_F is the number of roots of unity contained in F, and d_F stands for the discriminant.

The remaining term R_F is defined using the *Dirichlet regulator* map
$$\begin{aligned} \rho\colon \mathcal{O}_F^\times &\longrightarrow \mathbb{R}^{r_1+r_2} \\ u &\longmapsto (\log ||u||_v)_v. \end{aligned} \tag{1.15}$$
Here v runs over all archimedean places of F and we write
$$||u||_v = \begin{cases} |\sigma(u)| & \text{if } v = \sigma \text{ is a real place,} \\ |\sigma(u)|^2 & \text{if } v = \{\sigma, \overline{\sigma}\} \text{ is a complex place.} \end{cases}$$

The product formula $\prod_v ||u||_v = 1$ implies that ρ lands in the hyperplane of points whose coordinates sum to zero. In fact, Dirichlet showed that the image of ρ is a lattice in $\mathbb{R}^{r_1+r_2-1}$, that is, a subgroup of the form $\mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_{r_1+r_2-1}$ for linearly independent vectors $v_1, \ldots, v_{r_1+r_2-1}$. By definition, its *covolume* is the Lebesgue measure of the set

$$\{x_1v_1 + \cdots + x_{r_1+r_2-1}v_{r_1+r_2-1} \mid x_i \in \mathbb{R},\ 0 \leq x_i < 1\}.$$

The covolume of the lattice $\rho(\mathcal{O}_F^\times)$ is a real number R_F, abusively called Dirichlet regulator as well.

Borel generalized this picture to other values of the Dedekind zeta function. The role of the units $\mathcal{O}_F^\times$ is replaced by the higher K-groups $K_n(\mathcal{O}_F)$, certain finitely generated abelian groups which carry a lot of information about the "hidden" arithmetic of F. Borel computed the rank of these groups and defined, for each $n \geq 2$, a map from $K_{2n-1}(\mathcal{O}_F)$ to a suitable finite-dimensional real vector space, the *Borel regulator* map, whose image is again a lattice. Its covolume is a real number R_n, also called Borel regulator. Letting $\zeta_F^*(1-n)$ denote the first non-vanishing coefficient in the Taylor expansion of the Dedekind zeta function at $s = 1-n$, he proved that there exists a rational number q_n such that

$$\zeta_F^*(1-n) = q_n R_n.$$

The Dedekind zeta function satisfies a functional equation similar to (1.11). Using it, it follows that $\zeta_F(n)$ is, up to some easy factor involving the square root of the discriminant and powers of π, a rational multiple of R_n.

When $F = \mathbb{Q}$, the K-group $K_{2n-1}(\mathbb{Z})$ has rank one if $n \geq 3$ is odd, and zero otherwise (see Section 4.4 below). Therefore, $R_n = 1$ for even n. Thus, $\zeta(n)$ is given by a rational number times a power of π for even n, while it involves the "mysterious" Borel regulator for odd n. This result will play a pivotal role in the motivic approach to multiple zeta values. For more details, we refer the reader to the original papers [**Bor74**] and [**Bor77**], the monograph [**BG02**] or the short survey [**Sou10**].

1.1.2. *Double zeta values.* In order to investigate possible relations among zeta values, Euler looked at the algebraic structure of these numbers. If we multiply two Riemann zeta values, we obtain a new kind of interesting sum:

$$\begin{aligned}
\zeta(s_1)\cdot\zeta(s_2) &= \left(\sum_{n_1\geq 1}\frac{1}{n_1^{s_1}}\right)\cdot\left(\sum_{n_2\geq 1}\frac{1}{n_2^{s_2}}\right) \\
&= \sum_{n_1,n_2\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}} \\
&= \sum_{n_1>n_2\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_2>n_1\geq 1}\frac{1}{n_2^{s_2}n_1^{s_1}} + \sum_{n=n_1=n_2\geq 1}\frac{1}{n^{s_1+s_2}}.
\end{aligned} \tag{1.16}$$

The first two terms in the last line are called *double zeta values* and admit the various representations

$$\begin{aligned}\zeta(s_1, s_2) &= \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} \\ &= \sum_{n \geq 2} \frac{1}{n^{s_1}} \left(1 + \frac{1}{2^{s_2}} + \cdots + \frac{1}{(n-1)^{s_2}}\right) \\ &= \sum_{m,n \geq 1} \frac{1}{(n+m)^{s_1} n^{s_2}}.\end{aligned}$$

With this notation, equation (1.16) can be rewritten as

$$\underbrace{\zeta(s_1) \cdot \zeta(s_2)}_{\text{product of zeta values}} = \underbrace{\zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)}_{\text{sum of zeta and double zeta values}}. \tag{1.17}$$

This identity already appears in Euler's work [**Eul76**, p.144] under the name of "prima methodus".

Example 1.18. One has $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$, hence $\zeta(2,2) = \frac{\pi^4}{120}$ by Euler's formula (1.4). Similarly, $\zeta(2k, 2k)$ is a rational multiple of π^{4k}.

As we have seen, products of two Riemann zeta values are linear combinations of zeta and double zeta values. To handle products of more factors, multiple zeta values of higher length are needed. These new numbers satisfy many linear relations with rational coefficients, and one can argue that the main goal of the theory is to fully understand them.

⋆ ⋆ ⋆

Exercise 1.19. Prove that the logarithmic derivative of Euler's product expansion for the sine function

$$\frac{\sin \pi z}{\pi z} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2}\right)$$

yields the identity

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n \geq 1} \frac{2x}{x^2 - n^2} \qquad (x \in \mathbb{C} \backslash \mathbb{Z}),$$

and deduce formula (1.8) in the proof of Theorem 1.3.

Exercise 1.20. Prove that the Taylor expansion of the logarithm of the gamma function at $z = 0$ is given by

$$\log \Gamma(1-z) = \gamma z + \sum_{n \geq 2} \zeta(n) \frac{z^n}{n},$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right).$$

EXERCISE 1.21 (Tornheim sums). Given three integers $a, b, c \geq 0$, consider the series

$$S(a,b,c) = \sum_{m,n\geq 1}\frac{1}{m^a n^b (m+n)^c},$$

which is sometimes called *Tornheim sum*, in reference to [**Tor50**].

(a) Prove that $S(a,b,c)$ converges if and only if $a+c>1$, $b+c>1$, and $a+b+c>2$.

(b) Show that the following Pascal triangle-like recurrence holds

$$S(a,b,c) = S(a-1,b,c+1) + S(a,b-1,c+1).$$

(c) Deduce that $S(a,b,c)$ is a linear combination, with integral coefficients, of double zeta values, *e.g.* $S(1,1,1) = 2\zeta(2,1)$.

(d) Prove by direct computation that

$$S(1,1,1) = \zeta(2,1) + \zeta(3)$$

and deduce Euler's identity $\zeta(3) = \zeta(2,1)$.

[Hint: use the equality $\frac{1}{mn(m+n)} = \frac{1}{m^2}(\frac{1}{n} - \frac{1}{m+n})$ to transform the sum over n into a telescoping series].

1.2. Definition of multiple zeta values. We now introduce multiple zeta values, the main character of these notes. In doing so, it will be convenient to use the following terminology:

DEFINITION 1.22. A multi-index

$$\boldsymbol{s} = (s_1,\ldots,s_\ell) \in \mathbb{Z}^\ell$$

is called *positive* if $s_i \geq 1$ for all $i = 1,\ldots,\ell$ and *admissible* if it is positive and, in addition, satisfies $s_1 \geq 2$. By convention, the empty multi-index ($\ell = 0$) will also be considered to be admissible.

LEMMA 1.23. *Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_\ell)$ be an admissible multi-index. Then the following series converges:*

$$\zeta(\boldsymbol{s}) = \zeta(s_1,s_2,\ldots,s_\ell) = \sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_\ell^{s_\ell}}.$$

PROOF. Since $\zeta(\emptyset) = 1$, we may assume that the multi-index $\boldsymbol{s}$ is nonempty. In view of the inequality

$$\zeta(\boldsymbol{s}) \leq \zeta(2,\underbrace{1,\ldots,1}_{\ell-1}),$$

it suffices to show that $\zeta(2,1,\dots,1)$ converges. Using the estimate

$$\sum_{k=1}^{n}\frac{1}{k}\leq 1+\log(n),$$

which is obtained by comparison with the integral $\int_1^n \frac{dx}{x}$, one gets:

$$\begin{aligned}\zeta(2,1,\dots,1) &= \sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{1}{n_1^2 n_2\cdots n_\ell}\\ &\leq \sum_{n\geq 1}\frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{\ell-1}\\ &\leq \sum_{n\geq 1}\frac{(1+\log(n))^{\ell-1}}{n^2}. \end{aligned} \tag{1.24}$$

The last series converges, as can be seen as follows: since

$$\lim_{n\to+\infty}\frac{\log(1+\log(n))}{\log(n)}=0,$$

there exists an integer n_0 such that $(1+\log(n))^{\ell-1}<\sqrt{n}$ for all $n\geq n_0$. The tail of the series (1.24) is thus bounded by the convergent series $\sum_{n\geq n_0} n^{-3/2}$. □

Definition 1.25. The *multiple zeta value* associated to an admissible multi-index $\boldsymbol{s}=(s_1,\dots,s_\ell)$ is the real number

$$\zeta(\boldsymbol{s})=\sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_\ell^{s_\ell}}.$$

The *weight* of $\zeta(\boldsymbol{s})$ is the sum of the exponents $s_1+\cdots+s_\ell$, and ℓ is called its *length*[2]. We write:

$$\mathrm{wt}(\zeta(\boldsymbol{s}))=\mathrm{wt}(\boldsymbol{s})=s_1+\cdots+s_\ell, \tag{1.26}$$

$$\ell(\zeta(\boldsymbol{s}))=\ell(\boldsymbol{s})=\ell. \tag{1.27}$$

We shall adopt the convention that $\zeta(\emptyset)=1$, so $\mathrm{wt}(1)=\ell(1)=0$.

Remark 1.28. Strictly speaking, only the weight and the length of $\boldsymbol{s}$ are well defined, since we may have $\zeta(\boldsymbol{s})=\zeta(\boldsymbol{s}')$ for different multi-indices. Conjecturally, when such an equality holds $\boldsymbol{s}$ and $\boldsymbol{s}'$ have the same weight, hence the notation (1.26) makes sense. By contrast, the length is only well defined at the level of multi-indices, as Euler's relation $\zeta(2,1)=\zeta(3)$ already shows that the same value can be represented by multi-indices of different lengths (see Exercise 1.21 or Corollary 1.57 below).

[2]Also called *depth* in the literature.

EXAMPLE 1.29. Let $2^{\{n\}}$ be the admissible multi-index of length n whose entries are all equal to 2. We compute the value of $\zeta(2^{\{n\}})$ using the method of *generating series* and Euler's product expansion

$$\frac{\sin \pi x}{\pi x} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2}\right). \tag{1.30}$$

Plugging the definition of $\zeta(2^{\{n\}})$ into the power series below we get:

$$\begin{aligned}
\sum_{n \geq 0} \zeta(2^{\{n\}})(-x^2)^n &= \sum_{n \geq 0} \sum_{m_1 > \cdots > m_n \geq 1} \left(-\frac{x^2}{m_1^2}\right) \cdots \left(-\frac{x^2}{m_n^2}\right) \\
&= \prod_{m \geq 1} \left(1 - \frac{x^2}{m^2}\right) \\
&= \sum_{n \geq 0} (-1)^n \frac{\pi^{2n}}{(2n+1)!} x^{2n}.
\end{aligned}$$

The second equality above comes from the elementary observation that, in the development of the infinite product, the terms of degree $2n$ correspond bijectively to choices of n integers $m_1 > m_2 > \cdots > m_n \geq 1$. The third equality is the combination of (1.30) and the power series expansion of the sine function. Now, identification of the coefficients yields

$$\zeta(2^{\{n\}}) = \frac{\pi^{2n}}{(2n+1)!}. \tag{1.31}$$

Note that this agrees with the result $\zeta(2,2) = \frac{\pi^4}{120}$ from Example 1.18.

1.2.1. *The algebra of multiple zeta values.*

DEFINITION 1.32. We will write $\mathcal{Z}$ for the $\mathbb{Q}$-subvector space of $\mathbb{R}$ generated by all multiple zeta values

$$\mathcal{Z} = \langle 1, \zeta(2), \zeta(3), \zeta(2,1), \zeta(4), \ldots \rangle_{\mathbb{Q}}.$$

Given integers $k, \ell \geq 0$, we also consider the subvector spaces of $\mathcal{Z}$:

$$\begin{aligned}
\mathcal{Z}_k &= \langle \zeta(\boldsymbol{s}) \mid \mathrm{wt}(\boldsymbol{s}) = k \rangle_{\mathbb{Q}}, \\
F_\ell \mathcal{Z} &= \langle \zeta(\boldsymbol{s}) \mid l(\boldsymbol{s}) \leq \ell \rangle_{\mathbb{Q}}, \\
F_\ell \mathcal{Z}_k &= \langle \zeta(\boldsymbol{s}) \mid \mathrm{wt}(\boldsymbol{s}) = k,\ \ell(\boldsymbol{s}) \leq \ell \rangle_{\mathbb{Q}}.
\end{aligned}$$

In particular, $\mathcal{Z}_0 = \mathbb{Q}$ and $\mathcal{Z}_1 = \{0\}$.

REMARK 1.33. The subspaces $F_\ell \mathcal{Z}$ define an increasing filtration of $\mathcal{Z}$:

$$\mathbb{Q} = F_0 \mathcal{Z} \subseteq F_1 \mathcal{Z} \subseteq F_2 \mathcal{Z} \subseteq \ldots.$$

There is an obvious inclusion $F_\ell \mathcal{Z}_k \subseteq F_\ell \mathcal{Z} \cap \mathcal{Z}_k$. This is actually expected to be an equality, but not known so far.

Equation (1.17) is the first indication that the $\mathbb{Q}$-vector space $\mathcal{Z}$ has the richer structure of an *algebra*. Recall that this simply means that $\mathcal{Z}$ is equipped with a bilinear "multiplication" $\mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$.

THEOREM 1.34. *The multiplication of real numbers induces an algebra structure on $\mathcal{Z}$ which is compatible with the weight and the length filtration in that, for all non-negative integers ℓ_1, ℓ_2, k_1 and k_2, one has:*

$$F_{\ell_1}\mathcal{Z}_{k_1} \cdot F_{\ell_2}\mathcal{Z}_{k_2} \subseteq F_{\ell_1+\ell_2}\mathcal{Z}_{k_1+k_2}.$$

The theorem affirms, in particular, that every product of multiple zeta values can be written as a linear combination of MZVs.

COROLLARY 1.35. *Every polynomial relation between Riemann zeta values $\zeta(k)$ gives rise to a linear relation between multiple zeta values.*

Thus, finding algebraic relations among zeta values amounts to finding linear relations among multiple zeta values; this is a first interpretation of what we meant by "*linearizing* Conjecture 1.12" in the preface .

1.2.2. *Proof of Theorem 1.34.* The result will directly follow from lemmas 1.40 and 1.41 below. Before stating them, we need to introduce the *stuffle* of two multi-indices.

CONSTRUCTION 1.36. Given positive multi-indices

$$\boldsymbol{s} = (s_1, s_2, \dots, s_\ell), \qquad \boldsymbol{s}' = (s'_1, s'_2, \dots, s'_{\ell'}),$$

consider the set of all $2 \times \ell''$-matrices, for integers $\ell'' = \max(\ell, \ell'), \dots, \ell + \ell'$, satisfying the following properties:

(1) the entries of the first row are the numbers s_i, $1 \leq i \leq \ell$, in this order, plus some interlaced zeros;
(2) the entries of the second row are the numbers s'_i, $1 \leq i \leq \ell'$, in this order, plus some interlaced zeros;
(3) no column has two zeros.

Each such matrix defines a new positive multi-index $\boldsymbol{s}'' = (s''_1, \dots, s''_{\ell''})$ by adding the two entries of each column.

An equivalent construction will be given in Exercise 1.47.

EXAMPLE 1.37. For the multi-indices $\boldsymbol{s} = (2, 1, 1)$ and $\boldsymbol{s}' = (2, 3)$, two possible choices of such a matrix are

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix},$$

from which we get the multi-index $\boldsymbol{s}'' = (2, 2, 4, 1)$, and

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix},$$

which gives $\boldsymbol{s}'' = (4, 1, 4)$. Observe that the length of $\boldsymbol{s}''$ varies.

DEFINITION 1.38. Let $\boldsymbol{s}$, $\boldsymbol{s}'$ and $\boldsymbol{s}''$ be positive multi-indices. The *stuffle multiplicity* $\mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')$ is the number of times that the multi-index $\boldsymbol{s}''$ appears in the previous construction.

By definition, the stuffle multiplicity is a non-negative integer.

EXAMPLE 1.39. In the easy case $\boldsymbol{s} = (2)$ and $\boldsymbol{s}' = (2)$, all possible matrices are

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

from which one gets multi-indices $(4), (2,2)$ and $(2,2)$. Hence

$$\mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'') = \begin{cases} 1 & \boldsymbol{s}'' = (4) \\ 2 & \boldsymbol{s}'' = (2,2) \\ 0 & \text{otherwise} \end{cases}.$$

From conditions (1)–(3) above, we immediately deduce the following properties of the stuffle multiplicity:

LEMMA 1.40. *Let $\boldsymbol{s}$, $\boldsymbol{s}'$ and $\boldsymbol{s}''$ be three positive multi-indices such that $\mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'') > 0$. Then the following holds:*

(1) $\mathrm{wt}(\boldsymbol{s}'') = \mathrm{wt}(\boldsymbol{s}) + \mathrm{wt}(\boldsymbol{s}')$*;*
(2) $\ell(\boldsymbol{s}'') \leq \ell(\boldsymbol{s}) + \ell(\boldsymbol{s}')$*;*
(3) if $\boldsymbol{s}$ and $\boldsymbol{s}'$ are admissible, then so is $\boldsymbol{s}''$.

The main reason to introduce the stuffle index is the following result which, together with the previous lemma, implies Theorem 1.34.

LEMMA 1.41. *Let $\boldsymbol{s} = (s_1, s_2, \dots, s_\ell)$ and $\boldsymbol{s}' = (s'_1, s'_2, \dots, s'_{\ell'})$ be admissible multi-indices. Then*

$$\zeta(\boldsymbol{s}) \cdot \zeta(\boldsymbol{s}') = \sum_{\boldsymbol{s}''} \mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')\zeta(\boldsymbol{s}'').$$

PROOF. Multiplying the series

$$\zeta(\boldsymbol{s}) = \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} \quad \text{and} \quad \zeta(\boldsymbol{s}') = \sum_{m_1 > \cdots > m_{\ell'} \geq 1} \frac{1}{m_1^{s'_1} \cdots m_{\ell'}^{s'_{\ell'}}},$$

one gets

$$\zeta(\boldsymbol{s})\zeta(\boldsymbol{s}') = \sum_{\substack{n_1 > \cdots > n_\ell \geq 1 \\ m_1 > \cdots > m_{\ell'} \geq 1}} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell} m_1^{s'_1} \cdots m_{\ell'}^{s'_{\ell'}}}. \tag{1.42}$$

We now decompose the sum (1.42) according to the possible orderings of the terms of the sequence $n_1, \dots n_\ell, m_1, \dots m_{\ell'}$. For instance, if $\ell = \ell' = 1$, we distinguish the three cases $n_1 > m_1$, $n_1 = m_1$ and $n_1 < m_1$, hence:

$$\sum_{\substack{n_1 \geq 1 \\ m_1 \geq 1}} \frac{1}{n_1^{s_1} m_1^{s'_1}} = \sum_{n_1 > m_1 \geq 1} \frac{1}{n_1^{s_1} m_1^{s'_1}} + \sum_{n_1 \geq 1} \frac{1}{n_1^{s_1 + s'_1}} + \sum_{m_1 > n_1 \geq 1} \frac{1}{n_1^{s_1} m_1^{s'_1}}.$$

By construction, the number of times a given sum

$$\zeta(\boldsymbol{s}'') = \sum_{k_1 > \cdots > k_{\ell''} \geq 1} \frac{1}{k_1^{s_1''} \cdots k_{\ell''}^{s_{\ell''}''}}$$

appears in this process is precisely the stuffle multiplicity $\mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')$. $\square$

Example 1.43. Let a, b, c be integers such that $a, c \geq 2$ and $b \geq 1$. We decompose the product $\zeta(a,b)\zeta(c)$:

$$\begin{aligned}
\zeta(a,b)\zeta(c) &= \sum_{\substack{n_1 > n_2 \geq 1 \\ m \geq 1}} \frac{1}{n_1^a n_2^b m^c} \\
&= \sum_{m > n_1 > n_2 \geq 1} \frac{1}{m^c n_1^a n_2^b} + \sum_{m = n_1 > n_2 \geq 1} \frac{1}{n_1^{a+c} n_2^b} + \sum_{n_1 > m > n_2 \geq 1} \frac{1}{n_1^a m^c n_2^b} \\
&\qquad + \sum_{n_1 > m = n_2 \geq 1} \frac{1}{n_1^a n_2^{b+c}} + \sum_{n_1 > n_2 > m \geq 1} \frac{1}{n_1^a n_2^b m^c} \\
&= \zeta(c,a,b) + \zeta(a+c,b) + \zeta(a,c,b) + \zeta(a,b+c) + \zeta(a,b,c).
\end{aligned}$$

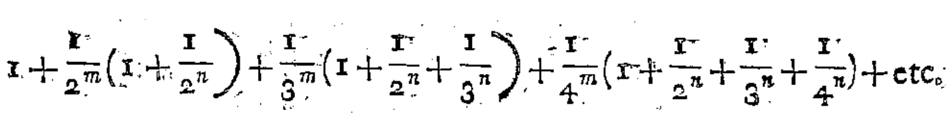
More examples will be presented in the next sections.

⋆ ⋆ ⋆

Exercise 1.44. It would have been possible, as Euler did in length two (see Figure 1 below), to define multiple zeta values as

$$\zeta^\star(s_1, s_2, \ldots, s_\ell) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_\ell \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$

Find the relation between $\zeta(s_1, s_2, \ldots, s_\ell)$ and $\zeta^\star(s_1, s_2, \ldots, s_\ell)$.



$$1 + \frac{1}{2^m}\left(1 + \frac{1}{2^n}\right) + \frac{1}{3^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right) + \frac{1}{4^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right) + \text{etc.}$$

Figure 1. Euler's definition of double zeta values in [Eul76].

Exercise 1.45. Given an integer $s \geq 2$, let $s^{\{n\}}$ be the length n multi-index $(s, \ldots, s)$.

(a) Adapt the argument from Example 1.29 to prove that

$$\sum_{n \geq 0} \zeta(s^{\{n\}}) x^n = \exp\left(\sum_{k \geq 1} (-1)^{k-1} \frac{\zeta(sk)}{k} x^k \right).$$

(b) Deduce that $\zeta(s^{\{n\}})$ belongs to the ring $\mathbb{Q}[\zeta(s), \zeta(2s), \zeta(3s), \dots]$. More precisely, consider an infinite collection of weighted variables $(x_k)_{k\geq 1}$, where x_k is given weight sk. Then, for each $n \geq 1$, there exists a polynomial with rational coefficients $P_n(x_1, \dots, x_n)$, homogeneous of weight sn, such that

$$\zeta(s^{\{n\}}) = P_n(\zeta(s), \zeta(2s), \dots, \zeta(ns)).$$

Combined with this, Euler's formula (1.4) implies that, for even s, the multiple zeta value $\zeta(s^{\{n\}})$ is a rational multiple of π^{ns}.

(c) Some explicit formulas:

$$\zeta(4^{\{n\}}) = \frac{(2\pi)^{4n}}{2^{2n-1}(4n+2)!}, \qquad \zeta(6^{\{n\}}) = \frac{6(2\pi)^{6n}}{(6n+3)!},$$

$$\zeta(8^{\{n\}}) = \frac{(2\pi)^{8n}}{2^{2n-2}(8n+4)!}\left[(\sqrt{2}+1)^{4n+2} + (\sqrt{2}-1)^{4n+2}\right].$$

Note that the last factor is rational despite its appearance.

Exercise 1.46. Use the stuffle product to prove that, for each pair of integers $n, k \geq 1$, the following holds:

$$\zeta(2k+1)\zeta(2^{\{n-k\}}) = \sum_{i=0}^{n-k} \zeta(2^{\{i\}}, 2k+1, 2^{\{n-k-i\}}) + \sum_{i=0}^{n-k-1} \zeta(2^{\{i\}}, 2k+3, 2^{\{n-k-1-i\}}).$$

Exercise 1.47. Let $\mathrm{st}(\ell, \ell'; r)$ denote the set of surjective maps

$$\sigma\colon \{1, 2, \dots, \ell+\ell'\} \longrightarrow \{1, 2, \dots, \ell+\ell'-r\}$$

satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(\ell)$ and $\sigma(\ell+1) < \cdots < \sigma(\ell+\ell')$.

(a) Determine the cardinality of $\mathrm{st}(\ell, \ell'; r)$ and show how to get from σ a matrix like the ones in Construction 1.36.

(b) Prove the identity

$$\sum_{\boldsymbol{s}''} \mathrm{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')\zeta(\boldsymbol{s}'') = \sum_{r=0}^{\min(\ell(\boldsymbol{s}), \ell(\boldsymbol{s}'))} \sum_{\sigma \in \mathrm{st}(\ell, \ell'; r)} \zeta(s''(\sigma)_1, \dots, s''(\sigma)_{\ell+\ell'-r}),$$

where $s''(\sigma)$ is the multi-index with

$$s''(\sigma)_k = \begin{cases} s_i, & \text{if } \sigma^{-1}(k) = \{i\},\ i \leq \ell, \\ s'_j, & \text{if } \sigma^{-1}(k) = \{\ell+j\}, \\ s_i + s'_j, & \text{if } \sigma^{-1}(k) = \{i, \ell+j\}. \end{cases}$$

1.3. Relations among double zeta values. We now undertake the task of finding linear relations among multiple zeta values by elementary methods. Historically, one of the first techniques consisted of reordering multiple sums by means of a partial fraction decomposition. In what follows, we show how this yields linear relations among double zeta values.

1.3.1. *Partial fraction expansions.* For a, b integers with $b \geq 0$, we shall use the standard convention for binomial numbers:

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}. \tag{1.48}$$

In particular, $\binom{a}{0} = 1$ for all a and, if $b > a \geq 0$, then $\binom{a}{b} = 0$.

LEMMA 1.49. *Let $i, j \geq 1$ be integers. The following equality of rational functions holds:*

$$\frac{1}{x^i y^j} = \sum_{r=1}^{i+j-1} \left[\frac{\binom{r-1}{i-1}}{(x+y)^r y^{i+j-r}} + \frac{\binom{r-1}{j-1}}{(x+y)^r x^{i+j-r}} \right]. \tag{1.50}$$

PROOF. We proceed by induction on i and j. The proof in the case $i = j = 1$ is a simple check. Assume that (1.50) holds for a given pair (i, j). Derivating with respect to x, we find that $\frac{1}{x^{i+1} y^j}$ is equal to

$$\begin{aligned}
\frac{1}{i} \sum_{r=1}^{i+j-1} & \left[\frac{r\binom{r-1}{i-1}}{(x+y)^{r+1} y^{i+j-r}} + \frac{r\binom{r-1}{j-1}}{(x+y)^{r+1} x^{i+j-r}} + \frac{(i+j-r)\binom{r-1}{j-1}}{(x+y)^r x^{i+j+1-r}} \right] \\
&= \frac{1}{i} \sum_{r=2}^{i+j} \left[\frac{(r-1)\binom{r-2}{i-1}}{(x+y)^r y^{i+1+j-r}} + \frac{(r-1)\binom{r-2}{j-1}}{(x+y)^r x^{i+1+j-r}} \right] \\
&\qquad + \frac{1}{i} \sum_{r=1}^{i+j-1} \frac{(i+j-r)\binom{r-1}{j-1}}{(x+y)^r x^{i+j+1-r}}.
\end{aligned}$$

Thanks to the identities

$$(r-1)\binom{r-2}{i-1} = i\binom{r-1}{i}, \quad (r-1)\binom{r-2}{j-1} = (r-j)\binom{r-1}{j-1}$$

and the convention (1.48), the previous expression becomes

$$\sum_{r=1}^{i+j} \left[\frac{\binom{r-1}{i}}{(x+y)^r y^{i+1+j-r}} + \frac{\binom{r-1}{j-1}}{(x+y)^r x^{i+1+j-r}} \right],$$

which agrees with the right-hand side of (1.50) for $(i+1, j)$. The induction step from (i, j) to $(i, j+1)$ is completely symmetric. □

COROLLARY 1.51. *Let $p, q \geq 1$ be integers. For any non-zero complex number a, the following equality of rational functions holds:*

$$\frac{1}{u^p (u-a)^q} = (-1)^q \sum_{k=0}^{p-1} \frac{\binom{q+k-1}{q-1}}{u^{p-k} a^{q+k}} + \sum_{k=0}^{q-1} (-1)^k \frac{\binom{p+k-1}{p-1}}{a^{p+k} (u-a)^{q-k}}. \tag{1.52}$$

PROOF. Take $y = u$ and $x = a - u$ in (1.50). To transform the obtained expression into (1.52), one notes that the binomial number $\binom{r-1}{q-1}$ vanishes unless $q \leq r \leq p+q-1$, hence r can be written as $r = q+k$ for $k = 0, \ldots, p-1$. The same holds for $\binom{r-1}{p-1}$. $\square$

1.3.2. *Applications.* A straightforward consequence of the partial fraction decomposition of Lemma 1.49 is the *shuffle relation*

$$\zeta(j)\zeta(k-j) = \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k-r) \tag{1.53}$$

for any $k \geq 4$ and $2 \leq j \leq k-2$. Replacing the product in the left-hand side of (1.53) by the stuffle formula (1.17) we get the linear identity

$$\begin{aligned}\zeta(j, k-j)+&\zeta(k-j, j) + \zeta(k)\\ &= \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k-r),\end{aligned} \tag{1.54}$$

which is called a *double shuffle relation*. The reason for these names will be apparent in Section 1.5.

A more sophisticated application of partial fraction decompositions gives the following result, essentially what Euler calls "tertia methodus" in [**Eul76**]. We refer the reader to [**Har17**] for a nice exposition of his techniques.

THEOREM 1.55 (Euler, 1776). *Given integers $p \geq 2$ and $q \geq 1$, the following equality holds:*

$$\begin{aligned}\zeta(p, q) = &\sum_{k=0}^{q-2} (-1)^k \binom{p+k-1}{p-1} \zeta(q-k)\zeta(p+k)\\ &+ (-1)^q \sum_{k=0}^{p-2} \binom{q+k-1}{q-1} \zeta(p-k, q+k)\\ &+ (-1)^{q-1} \binom{p+q-2}{p-1} [\zeta(p+q) + \zeta(p+q-1, 1)].\end{aligned}$$

REMARK 1.56. The assumptions $p \geq 2$ and $q \geq 1$ ensure that all the terms in the formula are convergent series. Euler also allowed the case $p = 1$. Then the sum contains divergent terms such as $\zeta(1)$ or $\zeta(1, 1)$ that one needs to regularize, see [**Har17**] for a rigorous treatment of Euler's method.

Making $q = 1$ we immediately get:

COROLLARY 1.57 (Euler's sum formula). *If $s \geq 3$, then*

$$\zeta(s) = \sum_{j=1}^{s-2} \zeta(s-j, j). \tag{1.58}$$

In particular, $\zeta(3) = \zeta(2, 1)$.

PROOF OF THEOREM 1.55. We follow [**Nie65**, III, §18, p. 48]. Let us first observe that

$$\zeta(p,q)=\sum_{n>m\geq 1}\frac{1}{n^p m^q}=\sum_{n\geq 2}\left(\sum_{a=1}^{n-1}\frac{1}{n^p(n-a)^q}\right). \tag{1.59}$$

Applying the partial fraction expansion of Corollary 1.51 to each summand in the right-hand side and separating the terms coming from $k=p-1$ and $k=q-1$, gives:

$$\begin{aligned}\sum_{a=1}^{n-1}\frac{1}{n^p(n-a)^q}=&(-1)^q\sum_{k=0}^{p-2}\sum_{a=1}^{n-1}\frac{\binom{q+k-1}{q-1}}{n^{p-k}a^{q+k}}\\ &+\sum_{k=0}^{q-2}\sum_{a=1}^{n-1}(-1)^k\frac{\binom{p+k-1}{p-1}}{a^{p+k}(n-a)^{q-k}}\\ &+(-1)^q\binom{q+p-2}{p-1}\sum_{a=1}^{n-1}\left[\frac{1}{na^{p+q-1}}-\frac{1}{a^{p+q-1}(n-a)}\right].\end{aligned}$$

The sum over n of the first two terms in the above expression converges, whereas the sum of each individual summand of the third term diverges. We will show later that the sum over n of the third term is also convergent.

Applying equation (1.59) to the first term we obtain

$$\sum_{n\geq 2}(-1)^q\sum_{k=0}^{p-2}\sum_{a=1}^{n-1}\frac{\binom{q+k-1}{q-1}}{n^{p-k}a^{q+k}}=(-1)^q\sum_{k=0}^{p-2}\binom{q+k-1}{q-1}\zeta(p-k,q+k).$$

We next observe that

$$\zeta(p)\zeta(q)=\sum_{n\geq 2}\sum_{a=1}^{n-1}\frac{1}{(n-a)^p a^q},$$

which implies that the sum over n of the second term is equal to

$$\sum_{n\geq 2}\sum_{k=0}^{q-2}\sum_{a=1}^{n-1}\frac{(-1)^k\binom{p+k-1}{p-1}}{a^{p+k}(n-a)^{q-k}}=\sum_{k=0}^{q-2}(-1)^k\binom{p+k-1}{p-1}\zeta(q-k)\zeta(p+k).$$

For the last term we use the identity

$$\begin{aligned}&\sum_{a=1}^{n-1}\frac{1}{a^{p+q-1}(n-a)}=\\ &\qquad\sum_{a=1}^{a<\frac{n}{2}}\frac{1}{(n-a)a^{p+q-1}}+\sum_{a>\frac{n}{2}}^{n-1}\frac{1}{a^{p+q-1}(n-a)}+\begin{cases}\frac{1}{\left(\frac{n}{2}\right)^{q+p}}, & n \text{ even},\\ 0, & n \text{ odd}.\end{cases}\end{aligned}$$

We note that

$$\sum_{\substack{n\geq 2\\ n \text{ even}}} \frac{1}{\left(\frac{n}{2}\right)^{q+p}} = \zeta(p+q)$$

and

$$\sum_{n\geq 2}\sum_{a>\frac{n}{2}}^{n-1} \frac{1}{a^{p+q-1}(n-a)} = \zeta(p+q-1,1).$$

We finally estimate the remaining term. For $N>2$, one has:

$$\sum_{n=2}^{N}\left(\sum_{a=1}^{n-1}\frac{1}{n(n-a)^{p+q-1}} - \sum_{a=1}^{a<\frac{n}{2}}\frac{1}{(n-a)a^{p+q-1}}\right) \\ = \sum_{n>\frac{N+1}{2}}^{N}\ \sum_{a=N-n+1}^{n-1}\frac{1}{na^{p+q-1}}.$$

Using that $p+q-1\geq 2$, one sees that the last term converges to zero as N goes to ∞. The theorem follows from summing up all the computations. $\square$

Corollary 1.60 (Nielsen). *If $n\geq 2$, the following equalities hold:*

$$\sum_{r=1}^{n-1}\zeta(2r,2n-2r) = \frac{3}{4}\zeta(2n),$$
$$\sum_{r=1}^{n-1}\zeta(2r+1,2n-2r-1) = \frac{1}{4}\zeta(2n).$$

Proof. We follow [**Nie65**, III, §19, p. 49]. We shall use the following identity, which follows from the decomposition (1.17) of the product of two zeta values and Euler's sum formula (1.58):

$$\sum_{r=2}^{p-1}\zeta(r)\zeta(p-r+1) = p\zeta(p+1) - 2\zeta(p,1). \tag{1.61}$$

Theorem 1.55 for $p=2$ and $q=2n-2$ yields the equality

$$(2n-2)\left[\zeta(2n)+\zeta(2n-1,1)\right] = \sum_{k=0}^{2n-4}(-1)^k(k+1)\zeta(k+2)\zeta(2n-k-2).$$

Note that the term $\zeta(k+2)\zeta(2n-k-2)$ is invariant under the substitution $k\mapsto 2n-k-4$ and that it appears with multiplicity $(-1)^k(2n-2)$ in the sum in the right-hand side. Therefore,

$$\begin{aligned}
2\left[\zeta(2n)+\zeta(2n-1,1)\right] &= \sum_{k=0}^{2n-4}(-1)^k\zeta(k+2)\zeta(2n-k-2)\\
&= \sum_{r=1}^{n-1}\zeta(2r)\zeta(2n-2r)-\sum_{r=1}^{n-2}\zeta(2r+1)\zeta(2n-2r-1). \qquad (1.62)
\end{aligned}$$

Summing and subtracting equations (1.62) and (1.61) for $p = 2n-1$ yields the recursion formulas

$$\begin{aligned}
\sum_{r=1}^{n-1}\zeta(2r)\zeta(2n-2r) &= \frac{2n+1}{2}\zeta(2n), \quad n\geq 2,\\
\sum_{r=1}^{n-2}\zeta(2r+1)\zeta(2n-2r-1) &= \frac{2n-3}{2}\zeta(2n)-2\zeta(2n-1,1), \quad n\geq 3.
\end{aligned}$$

The statement is proved by replacing the products of zeta values in the left hand sides by (1.17). □

Remark 1.63. The previous corollary was rediscovered by Gangl, Kaneko and Zagier, see [**GKZ06**, Thm. 1] and Exercise 1.67.

1.3.3. *Relations in low weight.* We now show how to use the above results to get linear relations among multiple zeta values of low weight.

Corollary 1.64. *The following relations hold in $\mathcal{Z}$:*

(1) *in weight 3:*
$$\zeta(3)=\zeta(2,1).$$
(2) *in weight 4:*
$$\begin{aligned}\zeta(4)&=4\zeta(3,1),\\ \zeta(2,2)&=3\zeta(3,1).\end{aligned}$$
(3) *in weight 5:*
$$\begin{aligned}\zeta(5)&=-4\zeta(4,1)+2\zeta(2,3),\\ \zeta(3,2)&=-5\zeta(4,1)+\zeta(2,3).\end{aligned}$$
(4) *in weight 6:*
$$\begin{aligned}\zeta(6)&=4\zeta(5,1)+4\zeta(3,3),\\ \zeta(2,4)&=\frac{13}{3}\zeta(5,1)+\frac{7}{3}\zeta(3,3),\\ \zeta(4,2)&=-\frac{4}{3}\zeta(5,1)+\frac{2}{3}\zeta(3,3).\end{aligned}$$

PROOF. All the relations follow from Theorem 1.55 together with the decomposition (1.17). We have already seen that the equality $\zeta(3) = \zeta(2,1)$ is the first instance of Euler's sum formula.

Let us now derive the two relations in weight 4. On the one hand, Theorem 1.55 for $p = q = 2$ gives $\zeta(2)^2 = 2\zeta(4) + 2\zeta(3,1)$. Combining this with the identity $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$, we obtain

$$\zeta(4) + 2\zeta(3,1) = 2\zeta(2,2).$$

On the other hand, by Euler's sum formula, $\zeta(4) = \zeta(3,1) + \zeta(2,2)$, hence $\zeta(4) = 4\zeta(3,1)$ and $\zeta(2,2) = 3\zeta(3,1)$.

The remaining identities are left as an exercise. □

1.3.4. *An upper bound for the dimension of $F_2\mathcal{Z}_k$.* Putting together all the identities of this section, one gets upper bounds for the dimension of the $\mathbb{Q}$-vector space generated by zeta and double zeta values of a given weight. However, as we will see in the next section, these bounds are not expected to be optimal in general (see Remark 1.94).

PROPOSITION 1.65. *If $k \geq 4$, then the $\mathbb{Q}$-vector space of zeta and double zeta values of weight k satisfies*

$$\dim_{\mathbb{Q}} F_2\mathcal{Z}_k \leq \left\lceil \frac{k-2}{2} \right\rceil.$$

PROOF. The space $F_2\mathcal{Z}_k$ is generated by the $k-1$ elements $\zeta(k)$ and $\zeta(j, k-j)$ for $j = 2, \ldots, k-1$. Recall from Corollary 1.57 that they satisfy Euler's sum formula

$$\zeta(2, k-2) + \cdots + \zeta(k-1, 1) - \zeta(k) = 0,$$

as well as the double shuffle relations (1.54)

$$\begin{aligned}\zeta(j, k-j) &+ \zeta(k-j, j) + \zeta(k) \\ &= \sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1}\right] \zeta(r, k-r), \quad j = 2, \ldots, k-2.\end{aligned}$$

Since the latter are symmetric with respect to $j \mapsto k-j$, it suffices to consider the equations for $j \leq k-j$, that is $j \leq \lfloor \frac{k}{2} \rfloor$.

One gets one equation from Euler's sum formula and $\lfloor \frac{k}{2} \rfloor - 1$ equations from the double shuffle relations. We claim that these $\lfloor \frac{k}{2} \rfloor$ equations are linearly independent. As $k - 1 - \lfloor \frac{k}{2} \rfloor = \lceil \frac{k-2}{2} \rceil$, this implies the statement. Indeed, by the convention (1.48), the double shuffle relations take the form

$$\sum_{r=j+1}^{k-1} a_r \zeta(r, k-r) - \zeta(k) = 0, \quad j = 2, \ldots, k-2,$$

with a_r positive integers. The matrix of relations is thus upper triangular with non-zero entries in the diagonal, hence invertible. □

⋆ ⋆ ⋆

Exercise 1.66. Derive the remaining relations of Corollary 1.64.

Exercise 1.67 (Gangl-Kaneko-Zagier). Define the generating function of double zeta values of weight k as

$$T_k(X,Y) = \sum_{\substack{r+s=k \\ r,s\geq 1}} \zeta(r,s) X^{r-1} Y^{s-1} \, .$$

(a) Use the double shuffle relation (1.54) to show that the following functional equation holds for all integers $k \geq 3$:

$$\begin{aligned} &T_k(X+Y,Y) + T_k(X+Y,X) \\ &\quad = T_k(X,Y) + T_k(Y,X) + \zeta(k)\frac{X^{k-1}-Y^{k-1}}{X-Y} \, . \end{aligned}$$

(b) Give an alternative proof of Corollary 1.60 using the above functional equation for $(X,Y) = (1,0)$ and $(1,-1)$.

1.4. The Zagier and the Broadhurst-Kreimer conjectures. As we have seen in the previous section, there are many linear relations between multiple zeta values. In order to elucidate the structure of the algebra $\mathcal{Z}$, one can start by performing numerical experiments.

1.4.1. *Numerical experiments.* The first step is to use clever techniques to accelerate the convergence of the infinite series defining multiple zeta values. With these techniques, one can compute them with very high precision (for instance 800 significant digits) in reasonable time[3]. Then we can apply lattice algorithms such as the LLL algorithm or, more efficiently, the PSLQ algorithm to find linear relations with integral coefficients among the computed multiple zeta values. At a given precision, we will find many spurious relations (as we are only working with rational approximations), but we can easily distinguish between true relations and spurious ones. The true relations should have small coefficients compared to the inverse of the used precision. Moreover, the true relations will survive after doubling the precision, say from 100 to 200 significant digits.

After extensive experimentation by many mathematicians, no non-trivial linear relations between multiple zeta values of different weight have been found: all known relations are homogeneous. Moreover, we can write a table with the "experimental" dimension of each vector space $\mathcal{Z}_k$. Below, k is the weight, $d_k^{\exp}$ is the apparent dimension of $\mathcal{Z}_k$ given by the experiments and 2^{k-2} is the number of admissible multi-indices of weight k, that is, the dimension $\mathcal{Z}_k$ would have had if there were no $\mathbb{Q}$-linear relations at all.

[3]See *e.g.* §4 of [Bro96] for a description of such techniques, as well as [BBV10] for the state of the art some years ago.

k	2	3	4	5	6	7	8	9	10	11	12	13
2^{k-2}	1	2	4	8	16	32	64	128	256	512	1024	2048
d_k^{exp}	1	1	1	2	2	3	4	5	7	9	12	16

TABLE 1.1. Experimental dimension

Of course, the experiments are not conclusive. There may exist linear relations with "big" coefficients that we have not yet found; then the dimension of $\mathcal{Z}_k$ would be smaller than d_k^{exp}. In fact, there is not even a single k for which the inequality $\dim_{\mathbb{Q}} \mathcal{Z}_k > 1$ is known.

Many of the relations obtained experimentally can be proved theoretically. For instance, Euler's sum formula (1.58) gives

$$\zeta(3) = \zeta(2,1),$$

the expected relation in weight 3. In weight 4, there are four admissible multi-indices but $d_4^{\mathrm{exp}} = 1$; we thus need to find three independent relations. Indeed, according to Corollary 1.64 and Example 1.132 below, we have

$$\zeta(3,1) = \frac{1}{4}\zeta(4), \quad \zeta(2,2) = \frac{3}{4}\zeta(4), \quad \zeta(2,1,1) = \zeta(4).$$

In weight 5, we expect six relations. In fact, by Corollary 1.64 and Exercise 1.138 below, we have the linear relations

$$\begin{aligned} \zeta(5) &= \frac{4}{5}\zeta(3,2) + \frac{6}{5}\zeta(2,3), & \zeta(4,1) &= -\frac{1}{5}\zeta(3,2) + \frac{1}{5}\zeta(2,3), \\ \zeta(5) &= \zeta(2,1,1,1), & \zeta(4,1) &= \zeta(3,1,1), \\ \zeta(2,1,2) &= \zeta(2,3), & \zeta(2,2,1) &= \zeta(3,2). \end{aligned} \tag{1.68}$$

However, given the lack of a theoretical proof, it is conceivable that experimental relations survive up to the number of significant digits that we have used but fail with higher precision.

1.4.2. *Does the weight define a grading?* The fact that all known relations between multiple zeta values are homogeneous led to the following:

CONJECTURE 1.69. *The subspaces $\mathcal{Z}_k \subseteq \mathcal{Z}$ are in direct sum:*

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k.$$

Together with the fact that $\mathcal{Z}_{k_1} \cdot \mathcal{Z}_{k_2} \subseteq \mathcal{Z}_{k_1+k_2}$ (Theorem 1.34), the conjecture would be reformulated below as the statement that the weight defines a *grading* on the $\mathbb{Q}$-algebra $\mathcal{Z}$.

REMARK 1.70. Assuming Conjecture 1.69, we immediately deduce that all multiple zeta values of positive weight are transcendental numbers. Indeed, let $\boldsymbol{s}$ be an admissible multi-index of weight $w > 0$. If $\zeta(\boldsymbol{s})$ were algebraic, it would satisfy a polynomial equation of the form $\sum_{k=0}^{d} a_k \zeta(\boldsymbol{s})^k = 0$, where the a_k are rational numbers. But then one would have

$$a_d\zeta(\boldsymbol{s})^d \in \mathcal{Z}_{wd} \cap \bigoplus_{d'<d} \mathcal{Z}_{wd'},$$

hence $a_d = 0$ since subspaces of different weights intersect only at 0.

1.4.3. *Zagier's conjecture.* In order to give the conjectural dimension of the $\mathbb{Q}$-vector spaces $\mathcal{Z}_k$, we need to introduce a Fibonacci-like sequence of integers. Set $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and, for $k \geq 3$,

$$d_k = d_{k-2} + d_{k-3}.$$

These numbers fit together into the generating series

$$\sum_{k\geq 0} d_k t^k = \frac{1}{1-t^2-t^3}.$$

There is an overwhelming amount of numerical evidence for the following conjecture, stated by Zagier in [**Zag94**, p. 509] "after many discussions with Drinfel'd, Kontsevich and Goncharov".

Conjecture 1.71 (Zagier). *The equality* $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ *holds.*

Hoffman proposed the following refinement of Zagier's conjecture, in which not only the dimension of $\mathcal{Z}_k$ but also a particular $\mathbb{Q}$-basis is postulated [**Hof97**, Conj. C, p. 493]:

Conjecture 1.72 (Hoffman). *For each weight k, multiple zeta values $\zeta(s_1, \ldots, s_\ell)$ with $s_i \in \{2, 3\}$ form a $\mathbb{Q}$-basis of $\mathcal{Z}_k$.*

This would imply the following representations of the spaces $\mathcal{Z}_k$:

$$\begin{aligned}
\mathcal{Z}_2 &= \langle \zeta(2) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_3 &= \langle \zeta(3) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_4 &= \langle \zeta(2,2) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_5 &= \langle \zeta(2,3), \zeta(3,2) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_6 &= \langle \zeta(2,2,2), \zeta(3,3) \rangle_{\mathbb{Q}} \\
\mathcal{Z}_7 &= \langle \zeta(2,2,3), \zeta(2,3,2), \zeta(3,2,2) \rangle_{\mathbb{Q}}
\end{aligned}$$

Remarks 1.73.

(1) The previous discussion shows that $\mathcal{Z}_5$ is generated by $\zeta(2,3)$ and $\zeta(3,2)$. Thus, the first step towards the conjecture would be to prove that these numbers are $\mathbb{Q}$-linearly independent.

(2) Having the right number of elements does not mean finding a basis. For instance, one could have thought that the elements

$$\zeta(2n_1+1, \ldots, 2n_r+1)\zeta(2)^k,$$

for $r \geq 0, k \geq 0, n_i \geq 1$, form a basis of $\mathcal{Z}$, since their number in a given weight agrees with the conjectural dimension (see Exercice 1.99). However, Gangl, Kaneko and Zagier [**GKZ06**, p.74] discovered the relation

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12),$$

which disproves such an expectation.

1.4.4. *Algebra generators of multiple zeta values.* In what follows, by a $\mathbb{Q}$-algebra (without any further qualifier) we will mean an associative commutative algebra with unit.

Definition 1.74. A *graded* $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra A, together with a direct sum decomposition (called *grading*)

$$A = \bigoplus_{k \in \mathbb{Z}} A_k$$

into $\mathbb{Q}$-vector spaces A_k such that $A_k \cdot A_{k'} \subseteq A_{k+k'}$. Note that the unit of the algebra then belongs necessarily to A_0, hence there is a map $\eta \colon \mathbb{Q} \to A_0$. A graded $\mathbb{Q}$-algebra is said to be *connected* if $A_k = 0$ for all $k < 0$ and η is an isomorphism. Moreover, A is said to be *free* if it is isomorphic to a polynomial algebra $\mathbb{Q}[X_1, \ldots, X_n, \ldots]$ with X_i homogenous of some degree.

Definition 1.75. Assume that all A_k are finite-dimensional. Then the *Hilbert-Poincaré series* of A is defined as

$$H_A(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Q}} A_k t^k.$$

If A is connected, then its Hilbert-Poincaré series has only positive degrees and the constant coefficient is equal to 1.

Lemma 1.76. *Let A be a connected graded free $\mathbb{Q}$-algebra, and let D_k denote the number of generators in degree k. Then*

$$H_A(t) = \prod_{k \geq 1} (1 - t^k)^{-D_k}. \tag{1.77}$$

Proof. Let $X_{1,1}, \ldots, X_{1,D_1}, \ldots, X_{\ell,1}, \ldots, X_{\ell,D_\ell}, \ldots$ be a set of homogenous generators of A, with $X_{i,j}$ of degree $i \geq 1$. It suffices to observe that the coefficient of t^k in the power series expansion of the product (1.77) agrees with the number of monomials of degree k in the variables $X_{i,j}$, and hence with the dimension of A_k since we are dealing with a free algebra. □

We now explain how to compute the number of algebra generators in terms of the logarithm of the Hilbert-Poincaré series. Let us keep the assumption that A is connected, and write

$$\log H_A(t) = \sum_{n \geq 1} c_n t^n. \tag{1.78}$$

Recall that the Möbius function μ takes the value 1 (resp. -1) on square-free integers with an even (resp. odd) number of prime factors, and 0 on non-square-free integers. In particular, $\mu(1) = 1$. The *Möbius inversion formula* is the statement that, if two sequences of complex numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are related by the equality $a_n = \sum_{d|n} b_d$ for all $n \geq 1$, then

$$b_n = \sum_{d|n} \mu(d) a_{n/d}.$$

LEMMA 1.79. *Let A be a connected graded free $\mathbb{Q}$-algebra, let D_k denote the number of generators in degree k and let c_k be the coefficients of $\log H_A(t)$ as in (1.78). Then the following equality holds:*

$$D_k = \sum_{d|k} \frac{\mu(d)}{d} c_{k/d}. \tag{1.80}$$

PROOF. Taking the logarithm of the identity (1.77) and using the formal power series expansion $-\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}$, one gets

$$\log H_A(t) = -\sum_{k\geq 1} D_k \log(1-t^k) = \sum_{k\geq 1} D_k \sum_{d\geq 1} \frac{t^{kd}}{d} = \sum_{n\geq 1}\left(\sum_{d|n} \frac{D_{n/d}}{d}\right) t^n.$$

Comparison of coefficients then yields

$$c_n = \sum_{d|n} \frac{D_{n/d}}{d},$$

and the equality (1.80) follows from Möbius inversion. □

Let us specialize the above discussion to the algebra $\mathcal{Z}$ of multiple zeta values. According to Zagier's conjecture, its Hilbert-Poincaré series is

$$H_{\mathcal{Z}}(t) = \frac{1}{1-t^2-t^3}.$$

CONJECTURE 1.81. *$\mathcal{Z}$ is a graded free algebra.*

Assuming this and Zagier's conjecture, we would like to compute the number D_k of algebra generators in weight k. For this, we define a sequence of integers $(P_d)_{d\geq 1}$ by the equality

$$\sum_{d\geq 1} P_d t^d = \sum_{d\geq 1} d c_d t^d = t\frac{d}{dt} \log H_{\mathcal{Z}}(t) = \frac{2t^2+3t^3}{1-t^2-t^3}.$$

Then $P_1 = 0$, $P_2 = 2$, $P_3 = 3$ and $P_d = P_{d-2} + P_{d-3}$ for all $d \geq 4$. Therefore, Lemma 1.79 gives

$$D_k = \frac{1}{k}\sum_{d|k} \mu(k/d) P_d.$$

The first values of P_k and D_k are given in Table 1.2.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
P_k	0	2	3	2	5	5	7	10	12	17	22	29	39
D_k	0	1	1	0	1	0	1	1	1	1	2	2	3

TABLE 1.2. Conjectural values of D_k for the algebra $\mathcal{Z}$

Recall that Hoffman's conjecture 1.72 predicts that multiple zeta values with exponents equal to 2 and 3 form a graded $\mathbb{Q}$-basis of $\mathcal{Z}$. It is only natural to try to extract from these elements a set of algebra generators; this is done through the theory of Lyndon words.

DEFINITION 1.82. Let X be the alphabet $\{a, b\}$ and equip the set X^* of words in X with the lexicographic order for which $a < b$. A *Lyndon word* is a non-empty word $w \in X^*$ such that, for each non-trivial decomposition $w = uv$, the inequality $w < v$ holds.

For example, ab is a Lyndon word because $ab < b$, but none of the words aa, ba, bb is Lyndon.

CONJECTURE 1.83. *$\mathcal{Z}$ is the free $\mathbb{Q}$-algebra generated by Lyndon words on the alphabet $\{2, 3\}$ with the order $2 < 3$.*

Assuming that the conjecture holds, the algebra generators in weights up to 13 are listed in Table 1.3.

weight	generators	weight	generators
2	$\zeta(2)$	8	$\zeta(2,3,3)$
3	$\zeta(3)$	9	$\zeta(2,2,2,3)$
4	$\emptyset$	10	$\zeta(2,2,3,3)$
5	$\zeta(2,3)$	11	$\zeta(2,2,2,2,3), \zeta(2,3,3,3)$
6	$\emptyset$	12	$\zeta(2,2,2,3,3), \zeta(2,2,3,2,3)$
7	$\zeta(2,2,3)$	13	$\zeta(2,2,2,2,2,3), \zeta(2,2,3,3,3)$, $\zeta(2,3,2,3,3)$

TABLE 1.3. First Lyndon words on the alphabet $\{2, 3\}$

1.4.5. *The Broadhurst-Kreimer conjecture.* So far we have only taken into account the weight of multiple zeta values. To add the length to the picture, the first difficulty one needs to face is that the length is only expected to induce a filtration and not a grading, as it is already evident from the existence of relations such as $\zeta(3) = \zeta(2, 1)$.

DEFINITION 1.84.

(1) A *filtered* $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra A, together with an increasing collection of vector subspaces

$$\cdots \subseteq F_{\ell-1}A \subseteq F_\ell A \subseteq F_{\ell+1}A \subseteq \ldots$$

indexed by $\ell \in \mathbb{Z}$ and such that $F_\ell A \cdot F_{\ell'} A \subseteq F_{\ell+\ell'} A$. The filtration is called *separated* if $\bigcap_\ell F_\ell A = 0$, and *exhaustive* if $\bigcup_\ell F_\ell A = A$.

(2) Given a filtered algebra $(A, F_\bullet)$, the *associated graded algebra* is

$$\operatorname{Gr}^F A = \bigoplus_{\ell \in \mathbb{Z}} F_\ell A / F_{\ell-1} A.$$

Note that the compatibility of the product and the filtration guarantees that $\operatorname{Gr}^F A$ inherits an algebra structure.

(3) A *filtered graded* $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra A with a filtration $F_\bullet A$ and a grading $A = \bigoplus_{k \in \mathbb{Z}} A_k$ which are compatible in the sense that

$$F_\ell A = \bigoplus_{k \in \mathbb{Z}} F_\ell A_k.$$

Given such an algebra, we set

$$A_{k,\ell} = \operatorname{Gr}^F_\ell A_k = F_\ell A_k / F_{\ell-1} A_k$$

and form the associated *bigraded* algebra $\bigoplus_{k,\ell \in \mathbb{Z}} A_{k,\ell}$.

Returning to the algebra of multiple zeta values, we see that the length defines a separated and exhaustive filtration

$$F_\ell \mathcal{Z} = \langle \zeta(\boldsymbol{s}) \mid \ell(\boldsymbol{s}) \leq \ell \rangle_{\mathbb{Q}}.$$

Assuming Conjecture 1.69, $\mathcal{Z}$ is hence a filtered graded algebra.

The associated bigraded algebra is not free, since $\zeta(2) \cdot \zeta(2) = \frac{5}{2}\zeta(4)$ implies that $\zeta(2)^2$ vanishes in $\mathcal{Z}_{4,2}$. To remedy this, we consider the quotient by the ideal generated by $\zeta(2)$:

$$\mathcal{Z}^\circ = \mathcal{Z} / \langle \zeta(2) \rangle.$$

It is a graded filtered algebra as well. Moreover, we equip $\mathbb{Q}[\zeta(2)]$ with the filtration $F_0 = \mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta(2)]$, and the grading that gives $\zeta(2)$ weight 2.

The following is a refinement of Conjecture 1.81.

Conjecture 1.85.

(1) $\operatorname{Gr}^F \mathcal{Z}^\circ$ *is a free bigraded algebra.*

(2) By the first part of the conjecture, there exists a morphism of filtered graded algebras $\mathcal{Z}^\circ \to \mathcal{Z}$ *that is a section of the quotient* $\mathcal{Z} \to \mathcal{Z}^\circ$*. Then the induced map* $\mathcal{Z}^\circ \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta(2)] \to \mathcal{Z}$ *is an isomorphism of filtered graded algebras.*

Definition 1.75 and lemmas 1.76 and 1.79 extend to bigraded algebras. In particular, if $A = \bigoplus_{k,\ell} A_{k,\ell}$ is a connected free bigraded algebra, then

$$H_A(x,y) = \sum_{k,\ell \geq 0} (\dim_{\mathbb{Q}} A_{k,\ell}) x^k y^\ell = \prod_{k,\ell \geq 1} (1 - x^k y^\ell)^{-D_{k,\ell}},$$

where $D_{k,\ell}$ is the number of generators in bidegree (k, ℓ).

Extensive numerical experiments support the following refinement of Zagier's conjecture, due to Broadhurst and Kreimer [**BK97**, §2]:

CONJECTURE 1.86 (Broadhurst-Kreimer). *Define integers $(D_{k,\ell})_{k\geq 3,\ell\geq 1}$ by the product expansion formula*

$$\prod_{k\geq 3}\prod_{\ell\geq 1}(1-x^k y^\ell)^{-D_{k,\ell}} = \frac{1}{1-O(x)y+S(x)y^2-S(x)y^4}, \tag{1.87}$$

where $O(x)$ and $S(x)$ are the formal series

$$O(x) = \frac{x^3}{1-x^2} = x^3+x^5+x^7+x^9+\dots,$$

$$S(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12}+x^{16}+x^{18}+x^{20}+x^{22}+2x^{24}+\dots.$$

Then $D_{k,\ell}$ is the number of generators of $\mathcal{Z}^\circ$ of weight k and length ℓ.

For shorthand, write $\mathrm{BK}^0(x,y)$ for the power series expansion of

$$\frac{1}{1-O(x)y+S(x)y^2-S(x)y^4}.$$

Arguing as in Lemma 1.79, the numbers $D_{k,\ell}$ are given by the formula

$$D_{k,\ell} = \sum_{d|(k,\ell)} \frac{\mu(d)}{d}\cdot \text{coefficient of } x^{\frac{k}{d}}y^{\frac{\ell}{d}} \text{ in } \log \mathrm{BK}^0(x,y), \tag{1.88}$$

where (k,ℓ) denotes the greatest common divisor of k and ℓ.

Taking Conjecture 1.85 for granted, the multiplicative formula (1.87) becomes equivalent to the following additive version, which is the one usually found in the literature:

CONJECTURE 1.89 (Broadhurst-Kreimer). *Define non-negative integers $(d_{k,\ell})_{k,\ell\geq 0}$ by the generating series*

$$\sum_{k,\ell\geq 0} d_{k,\ell}x^k y^\ell = \frac{1+E(x)y}{1-O(x)y+S(x)y^2-S(x)y^4},$$

where

$$E(x) = \frac{x^2}{1-x^2} = x^2+x^4+x^6+x^8+\dots.$$

Then $d_{k,\ell}$ coincides with the dimension of the space of multiple zeta values of (precisely) weight k and length ℓ, that is

$$d_{k,\ell} = \dim_{\mathbb{Q}} \mathcal{Z}_{k,\ell}.$$

REMARK 1.90. The series $E(x)$ counts even zeta values, while $O(x)$ counts the odd ones. More interestingly, Zagier realized that $S(x)$ agrees with the generating series

$$S(x) = \sum_{k\geq 1}(\dim_{\mathbb{Q}} S_k)x^k,$$

where S_k stands for the vector space of cuspidal modular forms of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. It is a classical result that

$$\dim_{\mathbb{Q}} S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \text{ even}, \ k \not\equiv 2 \bmod 12, \\ \lfloor \frac{k}{12} \rfloor - 1 & k \equiv 2 \bmod 12, \\ 0 & \text{otherwise} \end{cases}$$

(see *e.g.* §1.3 and §2.1 of [**Zag08**] for an elementary proof).

Let us denote by $\mathrm{BK}(x,y)$ the power series expansion of

$$\frac{1+E(x)y}{1-O(x)y+S(x)y^2-S(x)y^4}.$$

Expanding the fraction as a geometric series and collecting the terms with lower powers of y, we obtain

$$\begin{aligned}\mathrm{BK}(x,y) = 1 &+ [E(x)+O(x)]y + [(E(x)+O(x))O(x) - S(x)]y^2 \\ &+ [(O(x)^2-2S(x))O(x) + (O(x)^2-S(x))E(x)]y^3 + \dots .\end{aligned}$$

REMARK 1.91. Observe that $d_{k,1} = 1$ for all $k \geq 2$. Since $F_0\mathcal{Z} = \mathbb{Q}$, the Broadhurst-Kreimer conjecture holds in this case if and only if $\zeta(k)$ is irrational, which is only known for even k and $k=3$.

The first values of $d_{k,2}$ and $d_{k,3}$ are given in Table 1.4.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$d_{k,2}$	0	0	0	0	1	1	2	2	3	3	4	3	5	5	6	5	7
$d_{k,3}$	0	0	0	0	0	0	0	1	1	3	3	6	6	9	8	14	13

TABLE 1.4. First values of $d_{k,2}$ and $d_{k,3}$

Similarly, we derive

$$\begin{aligned}\log \mathrm{BK}^0(x,y) &= -\log(1-O(x)y+S(x)y^2-S(x)y^4) \\ &= O(x)y + \left(\frac{1}{2}O(x)^2 - S(x)\right)y^2 + \\ &\quad + \left(\frac{1}{3}O(x)^3 - O(x)S(x)\right)y^3 + \dots .\end{aligned}$$

REMARK 1.92. Note that $D_{k,\ell} = 0$ if k and ℓ have different parity. Indeed, in this case the integers d contributing to formula (1.88) are all odd, so k/d and ℓ/d have again different parity. However, it is clear from the above expression for $\log \mathrm{BK}^0(x,y)$ that only monomials in which the degree of x and the degree of y have the same parity appear.

LEMMA 1.93.
(1) If k is even, then $D_{k,2} = \lfloor \frac{k-2}{6} \rfloor$.
(2) If k is odd, then $D_{k,3} = \lfloor \frac{(k-3)^2-1}{48} \rfloor$.

PROOF. Specializing (1.88) to the case $\ell = 2$, we get

$$\begin{aligned} D_{k,2} &= \text{coeff. of } x^k y^2 - \frac{1}{2}\text{coeff. of } x^{\frac{k}{2}} y \text{ in } \log BK^0(x,y) \\ &= \text{coeff. of } x^k \text{ in } \left(\frac{1}{2}O(x)^2 - S(x)\right) - \frac{1}{2}\text{ coeff. of } x^{\frac{k}{2}} \text{ in } O(x). \end{aligned}$$

Since $O(x)^2 = \sum_{\substack{k\geq 6\\ even}} \frac{k-4}{2} x^k$, we find that

$$D_{k,2} = \begin{cases} \frac{k-4}{4} - \lfloor \frac{k}{12} \rfloor & k \equiv 0 \bmod 4 \\ \frac{k-6}{4} - \lfloor \frac{k}{12} \rfloor & k \equiv 2 \bmod 4, k \not\equiv 2 \bmod 12 \\ \frac{k-2}{4} - \lfloor \frac{k}{12} \rfloor & k \equiv 2 \bmod 12, \end{cases}$$

and it is a simple matter to check that this quantity agrees with $\lfloor \frac{k-2}{6} \rfloor$. The proof of the second assertion follows the same pattern (Exercise 1.102). □

REMARKS 1.94.

(1) The numbers $D_{k,2}$ and $D_{k,3}$ are known to be upper bounds for the number of generators of length 2 and 3, see [**Zag93**, §3] for $\ell = 2$ and [**Gon98**, Thm. 1.5] for $\ell = 3$. From this it follows that, in lengths $\ell = 1, 2, 3$, one has the inequality:

$$\dim_{\mathbb{Q}} (F_\ell \mathcal{Z}_k / F_{\ell-1}\mathcal{Z}_k) \leq d_{k,\ell}.$$

(2) In particular, for double zeta values we get

$$\dim_{\mathbb{Q}} F_2\mathcal{Z}_k - 1 \leq d_{k,2}.$$

By contrast, Proposition 1.65 yields the upper bound

$$\dim_{\mathbb{Q}} F_2\mathcal{Z}_k - 1 \leq \left\lceil \frac{k-4}{2} \right\rceil.$$

The right-hand side of this last inequality agrees with the coefficient of degree k of the power series $(E(x)+O(x))O(x)$, while $d_{k,2}$ is, by definition, the coefficient of degree k in $(E(x)+O(x))O(x) - S(x)$. Therefore, the bound of Proposition 1.65 is *not* optimal for those weights k such that S_k is non-trivial.

(3) Brown reformulated the Broadhurst-Kreimer conjecture in terms of the homology of a certain Lie algebra [**Bro13a**].

1.4.6. *Known results.* Not much is known about these conjectures, especially the last one. The goal of these notes is to explain in detail the following two results towards Zagier's and Hoffman's conjectures. In spite of their elementary formulation, this will carry us far away since the only known proofs are based on the theory of motives.

THEOREM 1.95 (Terasoma [**Ter02**], Deligne-Goncharov [**DG05**]). *The number d_k is an upper bound for the dimension of the $\mathbb{Q}$-vector space of multiple zeta values of weight k, that is*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k.$$

THEOREM 1.96 (Brown, [**Bro12**]). *Every multiple zeta value can be written as a $\mathbb{Q}$-linear combination of $\zeta(s_1, \dots, s_l)$ with $s_i \in \{2, 3\}$.*

REMARK 1.97. As we will see at the very end of the text, in paragraph 5.5.1, a corollary of these two theorems is that Zagier's conjecture implies the algebraic independence of odd zeta values (Conjecture 1.12).

⋆ ⋆ ⋆

EXERCISE 1.98. Prove that the sequence $(d_k)_{k \geq 0}$ satisfies

$$\lim_{k \to \infty} (d_k - \kappa r^k) = 0$$

where $\kappa = \frac{r+1}{2r+3}$ and r is the real root of $x^3 - x - 1$.

EXERCISE 1.99. Let δ_k denote the number of ordered tuples of integers $(s, n_1, \dots, n_r)$ such that $s \geq 0$, $r \geq 0$, $n_i \geq 1$, and

$$k = 2s + 2n_1 + 1 + \cdots + 2n_r + 1.$$

Show that $\delta_0 = 1$, $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_k = \delta_{k-2} + \delta_{k-3}$ for all $k \geq 3$. Therefore, $\delta_k = d_k$.

EXERCISE 1.100. Assume that the numbers $\zeta(2), \zeta(3), \zeta(5), \dots$ are algebraically independent, so that $\mathbb{Q}[\zeta(2), \zeta(3), \dots]$ is a free graded algebra. Apply Lemma 1.79 to compute the dimensions of the graded pieces, and compare them to the conjectural dimensions of multiple zeta values. Then find an example of a multiple zeta value which is not expected to be in the algebra generated by Riemann zeta values.

EXERCISE 1.101. Show that either Hoffman's or the Broadhurst-Kreimer conjecture implies Zagier's conjecture.

EXERCISE 1.102. Prove the equality $D_{k,3} = \lfloor \frac{(k-3)^2-1}{48} \rfloor$.

1.5. Integral representation of multiple zeta values. We have defined multiple zeta values as sums of infinite series. Using this representation, we proved that the vector space generated by these numbers forms an algebra under the stuffle product. We also derived some linear relations among multiple zeta values by means of the partial fraction method. Kontsevich found a different representation in terms of integrals. This way of writing multiple zeta values is central to the theory. From a combinatorial point of view, it yields a new structure, the *shuffle product*, from which many other linear relations are obtained in a systematic way. More importantly, from a conceptual point of view, the integral representation shows that multiple zeta values are *periods* of algebraic varieties. This will allow us to use algebro-geometric tools to study them, and paves the road for applications to a wealth of different areas such as knot theory or quantum field theory.

1.5.1. *Two examples.*

EXAMPLE 1.103. The following identity holds:

$$\zeta(2) = \int\limits_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} = \int_0^1 \left(\frac{1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \right) dt_1. \tag{1.104}$$

Indeed, for $0 \leq t_2 < 1$ we have the geometric series expansion

$$\frac{1}{1-t_2} = \sum_{n \geq 1} t_2^{n-1},$$

and thus

$$\int_0^{t_1} \frac{dt_2}{1-t_2} = \sum_{n \geq 1} \int_0^{t_1} t_2^{n-1} dt_2 = \sum_{n \geq 1} \frac{t_1^n}{n}.$$

Therefore we get

$$\int_0^1 \frac{1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} dt_1 = \int_0^1 \sum_{n \geq 1} \frac{t_1^n}{n} \frac{dt_1}{t_1} = \sum_{n \geq 1} \frac{1}{n} \int_0^1 t_1^{n-1} dt_1 = \sum_{n \geq 1} \frac{1}{n^2}.$$

EXAMPLE 1.105. The identity

$$\zeta(2,1) = \int\limits_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3}$$

holds. Indeed,

$$\begin{aligned}
\int\limits_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3} &= \int\limits_{1 \geq t_1 \geq t_2 \geq 0} \frac{1}{t_1} \sum_{n \geq 1} \frac{t_2^n}{n} \frac{dt_1\, dt_2}{1-t_2} \\
&= \int\limits_{1 \geq t_1 \geq t_2 \geq 0} \frac{1}{t_1} \sum_{n,m \geq 1} \frac{t_2^{n+m-1}}{n} dt_1\, dt_2 \\
&= \int\limits_{1 \geq t_1 \geq 0} \sum_{n,m \geq 1} \frac{t_1^{n+m}}{(n+m)n} \frac{dt_1}{t_1} \\
&= \sum_{n,m \geq 1} \frac{1}{(n+m)^2 n} \\
&= \zeta(2,1).
\end{aligned}$$

REMARK 1.106. As we will see in Section 3.7, the above integrals are particular cases of *iterated* integrals, but for the moment we will think of them just as ordinary integrals over a simplex.

1.5.2. *The integral representation.* A piece of notation is needed to describe the general integral representation of multiple zeta values.

NOTATION 1.107. Given a real number $0 \leq t \leq 1$, we define

$$\Delta^p(t) = \{(t_1, \dots, t_p) \in \mathbb{R}^p |\ t \geq t_1 \geq t_2 \geq \dots \geq t_p \geq 0\}.$$

When $t = 1$, we will simply write $\Delta^p = \Delta^p(1)$. Furthermore, consider the measures on the open interval $]0, 1[$

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}.$$

If $\boldsymbol{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$ is a positive multi-index (*i.e.* all $s_i \geq 1$), we write $r_i = s_1 + \dots + s_i$ for each $i = 1, \dots, l$. In particular, $r_1 = s_1$ and r_l is the weight of $\boldsymbol{s}$. For convenience, we also set $r_0 = 0$. Let $\omega_{\boldsymbol{s}}$ be the measure on the interior of the simplex $\Delta^{\mathrm{wt}(\boldsymbol{s})}$ given by

$$\omega_{\boldsymbol{s}} = \prod_{i=1}^{\ell} \underbrace{\omega_0(t_{r_{i-1}+1}) \cdots \omega_0(t_{r_i - 1})}_{s_i - 1 \text{ times}} \omega_1(t_{r_i}).$$

For example, one has:

$$\begin{aligned}
\omega_{(2)} &= \frac{dt_1}{t_1} \frac{dt_2}{1-t_2}, \\
\omega_{(2,2)} &= \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} \frac{dt_4}{1-t_4}, \\
\omega_{(2,1)} &= \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3}, \\
\omega_{(1,3)} &= \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3} \frac{dt_4}{1-t_4}.
\end{aligned}$$

The following result is attributed to Kontsevich:

THEOREM 1.108. *Let $\boldsymbol{s} = (s_1, \dots, s_\ell)$ be an admissible multi-index. The multiple zeta value $\zeta(\boldsymbol{s})$ can be obtained by a convergent improper integral:*

$$\zeta(\boldsymbol{s}) = \zeta(s_1, \dots, s_\ell) = \int_{\Delta^{\mathrm{wt}(\boldsymbol{s})}} \omega_{\boldsymbol{s}}. \tag{1.109}$$

In order to easily prove this theorem, we introduce the polylogarithm functions, which will also be of use later in Chapter 3.

DEFINITION 1.110. Let $\boldsymbol{s} = (s_1, \dots, s_\ell)$ be a positive multi-index and t a complex number with $|t| < 1$. We define

$$\mathrm{Li}_{\boldsymbol{s}}(t) = \sum_{n_1 > n_2 > \dots > n_\ell \geq 1} \frac{t^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_\ell^{s_\ell}}.$$

We call $\mathrm{Li}_{\boldsymbol{s}}$ the *(multiple) polylogarithm function (of one variable).*

REMARK 1.111. Similarly, one can define *multiple polylogarithms of several variables* by

$$\mathrm{Li}_{\boldsymbol{s}}(t_1,\dots,t_\ell)=\sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{t_1^{n_1}\cdots t_\ell^{n_\ell}}{n_1^{s_1}n_2^{s_2}\cdots n_\ell^{s_\ell}}$$

whenever the complex numbers t_i satisfy $|t_1|<1$ and $|t_i|\leq 1$ for $i=2,\dots,\ell$.

The following proposition is a straightforward consequence of basic results in complex analysis:

PROPOSITION 1.112. *If $\boldsymbol{s}$ is a positive multi-index, then the function $\mathrm{Li}_{\boldsymbol{s}}$ is holomorphic on the open unit disk $|t|<1$. Moreover, if $\boldsymbol{s}$ is admissible, then $\mathrm{Li}_{\boldsymbol{s}}$ can be extended continuously to the closed disk $|t|\leq 1$ and*

$$\mathrm{Li}_{\boldsymbol{s}}(1)=\zeta(\boldsymbol{s}).$$

For instance,

$$\mathrm{Li}_1(t)=\sum_{n\geq 1}\frac{t}{n}=-\log(1-t)=\int_0^t\frac{dt_1}{1-t_1}. \tag{1.113}$$

An important property of polylogarithms is that they satisfy many functional equations, the easiest being:

PROPOSITION 1.114. *The following identities hold for all $|t|<1$:*

$$\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{dt_1}{t_1}=\mathrm{Li}_{s_1+1,\dots,s_\ell}(t), \tag{1.115}$$

$$\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{dt_1}{1-t_1}=\mathrm{Li}_{1,s_1,\dots,s_\ell}(t). \tag{1.116}$$

PROOF. We first prove equation (1.115):

$$\begin{aligned}\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{dt_1}{t_1}&=\int_0^t\sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{t_1^{n_1}}{n_1^{s_1}n_2^{s_2}\dots n_l^{s_\ell}}\frac{dt_1}{t_1}\\&=\sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{t^{n_1}}{n_1^{s_1+1}n_2^{s_2}\dots n_l^{s_\ell}}\\&=\mathrm{Li}_{s_1+1,\dots,s_\ell}(t).\end{aligned}$$

Similarly, equation (1.116) follows from the manipulations

$$\begin{aligned}\int_0^t \mathrm{Li}_{s_1,\dots,s_\ell}(t_1)\frac{dt_1}{1-t_1}&=\int_0^t\sum_{n_1>n_2>\cdots>n_\ell\geq 1}\frac{t_1^{n_1}}{n_1^{s_1}n_2^{s_2}\dots n_\ell^{s_\ell}}\sum_{m\geq 0}t_1^m dt_1\\&=\sum_{n_0>n_1>\cdots>n_\ell\geq 1}\frac{t^{n_0}}{n_0 n_1^{s_1}n_2^{s_2}\dots n_\ell^{s_\ell}}\\&=\mathrm{Li}_{1,s_1,\dots,s_\ell}(t),\end{aligned}$$

where we have written $n_0=n_1+m+1>n_1$. $\square$

Now Theorem 1.108 is a particular case of the next result.

THEOREM 1.117. *If $\boldsymbol{s}$ is a positive multi-index and $0 < t < 1$ a real number, then the following identity holds:*

$$\mathrm{Li}_{\boldsymbol{s}}(t) = \int_{\Delta^{\mathrm{wt}(\boldsymbol{s})}(t)} \omega_{\boldsymbol{s}}.$$

PROOF. The proof is by induction on the weight of $\boldsymbol{s}$. If $\mathrm{wt}(\boldsymbol{s}) = 1$, then $\boldsymbol{s} = (1)$ and the statement is just formula (1.113). The inductive step follows from the functional equations in Proposition 1.114. Indeed, let $\boldsymbol{s} = (s_1, \dots, s_\ell)$ be a positive multi-index and assume that the result is true for all multi-indices of lower weight. If $s_1 > 1$, we write $\boldsymbol{s}' = (s_1 - 1, \dots, s_\ell)$. Then, by the identity (1.115) and induction,

$$\mathrm{Li}_{\boldsymbol{s}}(t) = \int_0^t \mathrm{Li}_{\boldsymbol{s}'}(t_1)\frac{dt_1}{t_1} = \int_0^t \int_{\Delta^{\mathrm{wt}(\boldsymbol{s}')}(t_1)} \omega_{\boldsymbol{s}'}\frac{dt_1}{t_1} = \int_{\Delta^{\mathrm{wt}(\boldsymbol{s})}(t)} \omega_{\boldsymbol{s}}.$$

The case $s_1 = 1$ is similar, using equation (1.116) instead. □

1.5.3. *The shuffle product.* Since multiple zeta values are integrals along simplices, certain combinatorial properties of the latter translate into relations among the former. Let us first illustrate this with an example.

EXAMPLE 1.118. We have the following equalities:

$$\begin{aligned}
\zeta(2)^2 &= \int\limits_{1\geq t_1\geq t_2\geq 0} \frac{dt_1\,dt_2}{t_1(1-t_2)} \cdot \int\limits_{1\geq u_1\geq u_2\geq 0} \frac{du_1\,du_2}{u_1(1-u_2)} \\
&= \int\limits_{\substack{1\geq t_1\geq t_2\geq 0\\ 1\geq u_1\geq u_2\geq 0}} \frac{dt_1\,dt_2\,du_1\,du_2}{t_1(1-t_2)u_1(1-u_2)} \\
&= \sum_{i=1}^{6} \int_{U_i} \frac{dt_1\,dt_2\,du_1\,du_2}{t_1(1-t_2)u_1(1-u_2)} \\
&= \zeta(3,1) + \zeta(3,1) + \zeta(2,2) + \zeta(3,1) + \zeta(3,1) + \zeta(2,2) \\
&= 4\zeta(3,1) + 2\zeta(2,2),
\end{aligned}$$

where the sets U_i, $i = 1, \dots, 6$, are defined by

$$\begin{aligned}
U_1 &= \{1 \geq t_1 \geq u_1 \geq t_2 \geq u_2 \geq 0\},\\
U_2 &= \{1 \geq t_1 \geq u_1 \geq u_2 \geq t_2 \geq 0\},\\
U_3 &= \{1 \geq t_1 \geq t_2 \geq u_1 \geq u_2 \geq 0\},\\
U_4 &= \{1 \geq u_1 \geq t_1 \geq u_2 \geq t_2 \geq 0\},\\
U_5 &= \{1 \geq u_1 \geq t_1 \geq t_2 \geq u_2 \geq 0\},\\
U_6 &= \{1 \geq u_1 \geq u_2 \geq t_1 \geq t_2 \geq 0\}.
\end{aligned}$$

The third equality comes from the decomposition

$$\{(t_1, t_2, u_1, u_2) \mid 1 \geq t_1 \geq t_2 \geq 0,\ 1 \geq u_1 \geq u_2 \geq 0\} = \bigcup_{i=1}^{6} U_i,$$

and the fourth one from Theorem 1.108.

REMARK 1.119. This expression of $\zeta(2)^2$ as linear combination of double zeta values is different from the one obtained by means of the series representation in Example 1.18. Combining both, we recover one of the relations which was proved in Corollary 1.64 using the method of partial fraction expansions, namely:

$$\zeta(4) = 4\zeta(3,1)$$

To generalize the previous example, we consider shuffles:

DEFINITION 1.120. A permutation σ of the set $\{1, 2, \ldots, r+s\}$ is called a *shuffle of type* (r, s) if the following two conditions are satisfied:

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r)$$
$$\sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s).$$

We denote the set of all shuffles of type (r, s) by $ш(r, s)$.

REMARK 1.121. By definition, a shuffle is a permutation that respects the ordering of two distinguished subsets. The name comes from the way gamblers shuffle a deck of cards in western saloons.

EXAMPLE 1.122. The set of shuffles of type $(2, 2)$ consists of

$$\mathrm{Id}, (23), (243), (123), (1243), (13)(24).$$

Shuffles allow us to decompose a product of two simplices into a union of simplices, and therefore to express a product of integrals over simplices as a linear combination of integrals.

PROPOSITION 1.123. *Let $r, s \geq 0$ be integers, $0 < t < 1$ a real number and $\mu_i \in \{\omega_0, \omega_1\}$ for $i = 1, \ldots, r+s$. Then*

$$\int\limits_{\Delta^r(t)} \mu_1(t_1) \cdots \mu_r(t_r) \int\limits_{\Delta^s(t)} \mu_{r+1}(t_{r+1}) \cdots \mu_{r+s}(t_{r+s})$$
$$= \sum_{\sigma \in ш(r,s)} \int\limits_{\Delta^{r+s}(t)} \mu_{\sigma^{-1}(1)}(t_1) \cdots \mu_{\sigma^{-1}(r+s)}(t_{r+s}).$$

PROOF. Using the decomposition

$$\Delta^r(t) \times \Delta^s(t)$$
$$= \bigcup_{\sigma \in ш(r,s)} \{(t_1, \ldots, t_{r+s}) \mid t \geq t_{\sigma^{-1}(1)} \geq \cdots \geq t_{\sigma^{-1}(r+s)} \geq 0\},$$

together with the fact that the intersection of two simplices on the right-hand side is a set of measure zero we obtain

$$\begin{aligned}\int_{\Delta^r(t)} \mu_1(t_1)\cdots\mu_r(t_r) &\int_{\Delta^s(t)} \mu_{r+1}(t_{r+1})\cdots\mu_{r+s}(t_{r+s}) \\ &= \int_{\Delta^r(t)\times\Delta^s(t)} \mu_1(t_1)\cdots\mu_{r+s}(t_{r+s}) \\ &= \sum_{\sigma\in ш(r,s)} \int_{t\geq t_{\sigma^{-1}(1)}\geq\cdots\geq t_{\sigma^{-1}(r+s)}\geq 0} \mu_1(t_1)\cdots\mu_{r+s}(t_{r+s}) \\ &= \sum_{\sigma\in ш(r,s)} \int_{\Delta^{r+s}(t)} \mu_{\sigma^{-1}(1)}(t_1)\cdots\mu_{\sigma^{-1}(r+s)}(t_{r+s}),\end{aligned}$$

where, in the last equality we have made the change of variables $t_i = t_{\sigma^{-1}(i)}$ to put the set $t \geq t_{\sigma^{-1}(1)} \geq \cdots \geq t_{\sigma^{-1}(r+s)} \geq 0$ as $\Delta^{r+s}(t)$. □

1.5.4. *Multi-indices and binary sequences.* To easily exploit the preceding proposition to derive relations among polylogarithms, and in particular among multiple zeta values, we need a new notation.

DEFINITION 1.124. A *binary sequence* is an element $\alpha \in \{0,1\}^k$. We call k the *weight* of α, while its *length* is defined as the number of ones in the sequence. A sequence is called *positive* if it ends in one and *admissible* if it ends in one and starts with zero.

We will use the following notation to go from multi-indices to binary sequences and the other way around.

NOTATION 1.125. To each positive multi-index $\boldsymbol{s} = (s_1,\dots,s_\ell)$ we attach the positive binary sequence

$$\mathrm{bs}(\boldsymbol{s}) = (0^{\{s_1-1\}}, 1, \dots, 0^{\{s_\ell-1\}}, 1)$$

where $0^{\{s\}}$ means that the entry zero is repeated s times. By convention, the empty binary sequence is admissible of weight and length both equal to zero. Clearly, bs is a bijection between the set of positive multi-indices and the set of positive binary sequences which respects the weight and the length. Moreover, it restricts to a bijection between the subsets of admissible objects on both sides.

If $\alpha = (\varepsilon_1,\dots,\varepsilon_r)$ is a binary sequence we will set

$$\omega_\alpha = \omega_{\varepsilon_1}\dots\omega_{\varepsilon_r}.$$

In particular, if $\boldsymbol{s}$ is a positive multi-index then

$$\omega_{\boldsymbol{s}} = \omega_{\mathrm{bs}(\boldsymbol{s})}.$$

Moreover, if α is positive we will denote $\mathrm{Li}_\alpha(t) = \mathrm{Li}_{\mathrm{bs}^{-1}(\alpha)}(t)$ and if it is also admissible, we write $\zeta(\alpha) = \zeta(\mathrm{bs}^{-1}(\alpha))$.

1.5.5. *The shuffle product.*

DEFINITION 1.126. Let $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$, $\alpha' = (\varepsilon_{r+1}, \dots, \varepsilon_{r+s})$ and α'' be three binary sequences of lengths r, s and t respectively. Then the shuffle index $⧢(\alpha, \alpha'; \alpha'')$ is the number of shuffles of type (r, s) which send $\alpha\alpha'$ to α''. That is,

$$⧢(\alpha, \alpha'; \alpha'') = \#\{\sigma \in ⧢(r, s) \mid \alpha'' = (\varepsilon_{\sigma^{-1}(1)}, \dots, \varepsilon_{\sigma^{-1}(r+s)})\}.$$

Clearly, $⧢(\alpha, \alpha'; \alpha'') = 0$ unless $t = r + s$.

The next result is the analogue of Lemma 1.40 for the shuffle index; it follows directly from the definition as well.

LEMMA 1.127. *Let α, α' and α'' be three binary sequences such that $⧢(\alpha, \alpha'; \alpha'') > 0$. Then*

(1) $\mathrm{wt}(\alpha'') = \mathrm{wt}(\alpha) + \mathrm{wt}(\alpha)$;
(2) $\ell(\alpha'') = \ell(\alpha) + \ell(\alpha')$;
(3) if both α and α' are positive (resp. admissible), then so is α''.

With this notation, Proposition 1.123 translates into the following result, which is the analogue of Lemma 1.41 for shuffles.

LEMMA 1.128. *Let α and α' be positive binary sequences. Then*

$$\mathrm{Li}_\alpha(t)\,\mathrm{Li}_{\alpha'}(t) = \sum_{\alpha''} ⧢(\alpha, \alpha'; \alpha'')\,\mathrm{Li}_{\alpha''}(t).$$

Moreover, if α and α' are admissible, then

$$\zeta(\alpha) \cdot \zeta(\alpha') = \sum_{\alpha''} ⧢(\alpha, \alpha'; \alpha'')\zeta(\alpha'').$$

1.5.6. *An involution.* Another useful identity comes from exploiting the symmetry $t \mapsto 1 - t$.

PROPOSITION 1.129. *Let $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$ be an admissible binary sequence. Then*

$$\int\limits_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} \omega_{\varepsilon_1}(t_1) \cdots \omega_{\varepsilon_r}(t_r) = \int\limits_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} \tilde{\omega}_{\varepsilon_r}(t_1) \cdots \tilde{\omega}_{\varepsilon_1}(t_r),$$

where $\tilde{\omega}_0 = \omega_1$ and $\tilde{\omega}_1 = \omega_0$.

PROOF. The change of variables $s_i = 1 - t_i$ transforms the measure $\omega_0(t_i)$ into $\omega_1(s_i) = \tilde{\omega}_0(s_i)$, and $\omega_1(t_i)$ into $\omega_0(s_i) = \tilde{\omega}_1(s_i)$. Hence

$$\int\limits_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} \omega_{\varepsilon_1}(t_1) \cdots \omega_{\varepsilon_r}(t_r) = \int\limits_{0 \leq s_1 \leq \cdots \leq s_r \leq 1} \widetilde{\omega}_{\varepsilon_1}(s_1) \cdots \widetilde{\omega}_{\varepsilon_r}(s_r).$$

The sought formula follows after renaming the variables in the right as $s_i = t_{r-i}$. $\square$

DEFINITION 1.130. For a binary sequence $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$, we write

$$\tau(\alpha) = (1 - \varepsilon_r, \dots, 1 - \varepsilon_1).$$

If α is admissible, then so is $\tau(\alpha)$.

As a consequence of Proposition 1.129 and Theorem 1.108, we deduce

COROLLARY 1.131. *If α is an admissible binary sequence, then*

$$\zeta(\alpha) = \zeta(\tau(\alpha)).$$

EXAMPLE 1.132. We have:

$$\zeta(4) = \zeta((0,0,0,1)) = \zeta((0,1,1,1)) = \zeta(2,1,1).$$

⋆ ⋆ ⋆

EXERCISE 1.133. Justify the exchange of the integral and the summation sign in the computations of examples 1.103 and 1.105.

EXERCISE 1.134. Show that the number of shuffles of type (r, s) is the binomial number $\binom{r+s}{r}$.

EXERCISE 1.135. Manipulating series, show directly that

$$\zeta(3) = \int_{1 \geq t_1 \geq t_2 \geq t_3 \geq 0} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3}$$

and, more generally,

$$\zeta(s) = \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_s \geq 0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \frac{dt_s}{1 - t_s}.$$

EXERCISE 1.136. Use Lemma 1.128 to check the shuffle relation (1.53) for $\zeta(2)\zeta(3)$. Same in the general case $\zeta(i)\zeta(j)$.

EXERCISE 1.137. Find a formula for $\zeta(s)\zeta(p,q)$ with shuffles.

EXERCISE 1.138. Check the identities

$$\zeta(5) = \zeta(2,1,1,1), \qquad \zeta(4,1) = \zeta(3,1,1),$$
$$\zeta(2,1,2) = \zeta(2,3), \qquad \zeta(2,2,1) = \zeta(3,2)$$

with the help of Proposition 1.129.

1.6. Quasi-shuffle products and the Hoffman algebra. In the previous sections, we have seen two methods to express a product of multiple zeta values as a linear combination of MZVs. The first, using the series representation, is called the stuffle product, and the second, using the integral representation, is called the shuffle product. As seen in examples 1.18 and 1.118, both methods may give different linear combinations for the same product of MZVs leading to linear relations among MZVs. The stuffle product is easily expressed in terms of multi-indices as in Lemma 1.41, while the shuffle product is expressed more conveniently using binary sequences as in Lemma 1.128. We now want to put a little order to make clearer the combinatorial structure of MZVs.

1.6.1. *Alphabets and the quasi-shuffle product.*

NOTATION 1.139. Let $A = \{a_i\}_{i\in S}$ be a countable set. The elements of A will be called *letters* and A is called an *alphabet*. Let $\mathbb{Q}A$ be the $\mathbb{Q}$-vector space with A as a basis. Let $\mathbb{Q}\langle A\rangle$ be the non-commutative polynomial algebra over A, i.e. $\mathbb{Q}\langle A\rangle = \langle a_{i_1}, a_{i_2}, \dots, a_{i_n}\rangle_{\mathbb{Q}}$ is the vector space with the set of *words* in the letters of A as a basis, which is equipped with the concatenation product

$$a_{i_1}\cdots a_{i_n}\cdot a_{j_1}\cdots a_{j_m} = a_{i_1}\cdots a_{i_n}a_{j_1}\cdots a_{j_m}.$$

We say that a word $w = a_1\cdots a_n$ has *length* $\ell(w) = n$ and set $\ell(1) = 0$, as we consider 1 as the empty word.

DEFINITION 1.140. Let A be an alphabet and let

$$\Diamond : \mathbb{Q}A \times \mathbb{Q}A \to \mathbb{Q}A$$

be a commutative and associative product. We define a new product $*_\Diamond$ on $\mathbb{Q}\langle A\rangle$ recursively by setting $1 *_\Diamond w = w *_\Diamond 1 = w$ and

$$aw *_\Diamond bv = a(w *_\Diamond bv) + b(aw *_\Diamond v) + (a\Diamond b)(w *_\Diamond v),$$

for any pair $a, b \in A$ of letters, and $w, v \in \mathbb{Q}\langle A\rangle$ of words. This product is extended to $\mathbb{Q}\langle A\rangle$ by $\mathbb{Q}$-linearity and is called the *quasi-shuffle product* associated to $\Diamond$.

THEOREM 1.141 (Hoffman [**Hof00**]). *The vector space $\mathbb{Q}\langle A\rangle$ equipped with the product $*_\Diamond$ is a commutative $\mathbb{Q}$-algebra.*

PROOF. Let us check the commutativity

$$u_1 *_\Diamond u_2 = u_2 *_\Diamond u_1 \tag{1.142}$$

by induction on $\ell(u_1) + \ell(u_2)$. If one of u_1 or u_2 is the empty word, then (1.142) holds trivially. Thus let $u_1 = aw$ and $u_2 = bv$ with letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$. Then, by definition of the product $*_\Diamond$ and the induction hypothesis, we get

$$u_1 *_\Diamond u_2 - u_2 *_\Diamond u_1 = (a\Diamond b)(w *_\Diamond v) - (b\Diamond a)(v *_\Diamond w).$$

Since $\Diamond$ is assumed to be commutative, (1.142) follows from induction.

The proof of the associativity is similar and is left as an exercise. □

Let us give some examples of quasi-shuffle products.

1.6.2. *Stuffle product.* We first introduce the stuffle product. Let Y be the alphabet with letters $y_1, y_2, y_3, \dots$, together with the product

$$\Diamond_1 \colon \mathbb{Q}Y \times \mathbb{Q}Y \to \mathbb{Q}Y, \qquad y_i \Diamond_1 y_j = y_{i+j}.$$

The product $\Diamond_1$ is commutative and associative. The product $*_{\Diamond_1}$ on $\mathbb{Q}\langle Y\rangle$ will be denoted by $*$ and called the *stuffle product*. By definition, it is given by

$$y_i w * y_j v = y_i(w * y_j v) + y_j(y_i w * v) + y_{i+j}(w * v). \tag{1.143}$$

EXAMPLE 1.144. We have $y_i * y_j = y_i y_j + y_j y_i + y_{i+j}$ and

$$\begin{aligned} y_2 * y_3 y_4 &= y_2(y_3 y_4) + y_3(y_2 * y_4) + y_5(y_4) \\ &= y_2 y_3 y_4 + y_3(y_2 y_4 + y_4 y_2 + y_6) + y_5 y_4 \\ &= y_2 y_3 y_4 + y_3 y_2 y_4 + y_3 y_4 y_2 + y_3 y_6 + y_5 y_4. \end{aligned}$$

NOTATION 1.145. A positive multi-index $\boldsymbol{s} = (s_1, \dots, s_\ell)$ defines a word

$$y_{\boldsymbol{s}} = y_{s_1} \cdots y_{s_\ell}.$$

In fact, the set of positive multi-indices and the set of words in the alphabet Y are in bijection. We will use this bijection to identify both sets.

LEMMA 1.146. *The stuffle product is given by*

$$y_{\boldsymbol{s}} * y_{\boldsymbol{s}'} = \sum_{\boldsymbol{s}''} \operatorname{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'') y_{\boldsymbol{s}''}.$$

PROOF. Let $\boldsymbol{s} = (s_1, \dots)$ and $\boldsymbol{s}' = (s'_1, \dots)$ be two positive multi-indices. Thus $y_{\boldsymbol{s}} = y_{s_1} v$ and $y'_{\boldsymbol{s}} = y_{s'_1} w$. The matrices used to define the stuffle indices $\operatorname{st}(\boldsymbol{s}, \boldsymbol{s}'; \boldsymbol{s}'')$ in Definition 1.38 fall into three types.

$$\begin{pmatrix} s_1 & \cdots \\ 0 & \cdots \end{pmatrix}, \qquad \begin{pmatrix} 0 & \cdots \\ s'_1 & \cdots \end{pmatrix}, \qquad \begin{pmatrix} s_1 & \cdots \\ s'_1 & \cdots \end{pmatrix}.$$

The matrices of the first type give rise to the term $y_{s_1}(v * y_{\boldsymbol{s}'})$, the matrices of the second type to the term $y_{s'_1}(y_{\boldsymbol{s}} * w)$ and the matrices of the third type to the term $y_{s_1+s'_1}(v * w)$. □

Since the words of the alphabet Y are related to multi-indices and the product of $\mathbb{Q}\langle Y\rangle$ is the stuffle product, one may expect to have a morphism of $\mathbb{Q}$-algebras

$$\begin{aligned} (\mathbb{Q}\langle Y\rangle, *) &\longrightarrow (\mathcal{Z}, \cdot) \\ y_{s_1} \cdots y_{s_\ell} &\longmapsto \zeta(s_1, \dots, s_\ell). \end{aligned}$$

But since multiple zeta values are defined only when $s_1 > 1$ we have to restrict the source of this map. Later, in Section 1.7 we will see how to extend the evaluation map to the whole $(\mathbb{Q}\langle Y\rangle, *)$.

DEFINITION 1.147. A word $w = y_{s_1} \dots y_{s_\ell}$ is called *admissible* if $s_1 > 1$, *i.e.* if it corresponds to an admissible multi-index. We will denote by $\mathbb{Q}\langle Y\rangle^0$ the subspace of $\mathbb{Q}\langle Y\rangle$ generated by admissible words.

PROPOSITION 1.148.

(1) $(\mathbb{Q}\langle Y\rangle^0, *)$ *is a subalgebra of* $(\mathbb{Q}\langle Y\rangle, *)$.
(2) *We have a morphism of* $\mathbb{Q}$*-algebras*

$$\mathbb{Q}\langle Y\rangle^0 \to \mathcal{Z}$$

determined by the assignment

$$y_{s_1} \cdots y_{s_l} \mapsto \zeta(s_1, \dots, s_l).$$

PROOF. The first statement can be checked directly from the definition of the product $*$. Alternatively, it follows from Lemma 1.146 and part (3) of Lemma 1.40. The second statement follows from lemmas 1.146 and 1.41. $\square$

Since we have identified positive multi-indices with words in the alphabet Y, we often just write $\zeta(w)$ instead of $\zeta(s_1, \dots, s_l)$ for $w = y_{s_1} \dots y_{s_l}$, thus

$$\zeta(w * v) = \zeta(w)\zeta(v) \tag{1.149}$$

for all words $w, v \in \mathbb{Q}\langle Y\rangle^0$.

1.6.3. *Shuffle product.* We now introduce the shuffle product. Let X be the alphabet in two letters $X = \{x_0, x_1\}$, equipped with the trivial product $a\Diamond_2 b = 0$. We will denote by $\sqcup\!\sqcup$ the corresponding product $*_{\Diamond_2}$ and call it the *shuffle product*[4].

DEFINITION 1.150. We call $\mathfrak{H} = (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup)$ the *Hoffman algebra*.

PROPOSITION 1.151. *Given two words* $x_{\varepsilon_1} \dots x_{\varepsilon_r}$ *and* $x_{\varepsilon_{r+1}} \dots x_{\varepsilon_{r+s}}$ *on the alphabet* X*, their shuffle product is given by*

$$x_{\varepsilon_1} \dots x_{\varepsilon_r} \sqcup\!\sqcup\, x_{\varepsilon_{r+1}} \dots x_{\varepsilon_{r+s}} = \sum_{\sigma \in \sqcup\!\sqcup(r,s)} x_{\varepsilon_{\sigma^{-1}(1)}} \dots x_{\varepsilon_{\sigma^{-1}(p+q)}}.$$

PROOF. Exercise 1.163. $\square$

EXAMPLE 1.152. We have

$$\begin{aligned} x_0x_1 \sqcup\!\sqcup\, x_0x_1 &= 2x_0x_1x_0x_1 + 4x_0^2x_1^2 \\ x_0x_1 \sqcup\!\sqcup\, x_0^2x_1 &= x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2. \end{aligned}$$

NOTATION 1.153. There is an obvious bijection between binary sequences and words in the alphabet X: to a binary sequence $\alpha = (\varepsilon_1, \dots, \varepsilon_r)$ we associate the word $x_\alpha = x_{\varepsilon_1} \dots x_{\varepsilon_r}$. Using this bijection, we can transfer the shuffle index, as introduced in Definition 1.126, to words in the alphabet X. The resulting index will be denoted by $\sqcup\!\sqcup(u, v; w)$.

[4]This justifies the name *quasi-shuffle*.

With this notation, Proposition 1.151 can be rewritten as

$$u \sqcup\!\sqcup\, v = \sum_{w} \sqcup\!\sqcup(u, v; w) w. \tag{1.154}$$

This equation hints at the existence of an algebra morphism from $\mathfrak{H}$ to multiple zeta values. As in the case of the alphabet Y, one needs to restrict to the space where the series are convergent.

DEFINITION 1.155. A word in the alphabet X is said to be *positive* if it ends in x_1 and is said to be *admissible* if it ends in x_1 and starts in x_0.

Let $\mathfrak{H}^1$ (resp. $\mathfrak{H}^0$) be the subspace generated by positive (resp. admissible) words, so that

$$\mathfrak{H} \supset \mathfrak{H}^1 \supset \mathfrak{H}^0.$$

PROPOSITION 1.156.

(1) $(\mathfrak{H}^0, \sqcup\!\sqcup)$ *and* $(\mathfrak{H}^1, \sqcup\!\sqcup)$ *are subalgebras of* $(\mathfrak{H}, \sqcup\!\sqcup)$.

(2) There is a morphism of $\mathbb{Q}$*-algebras*

$$\zeta : \mathfrak{H}^0 \to \mathcal{Z}$$

given by the assignment

$$x_\alpha \mapsto \zeta(\alpha).$$

(Recall that the multiple zeta value corresponding to an admissible binary sequence was defined as $\zeta(\mathrm{bs}^{-1}(\alpha))$ *in Notation 1.125).*

PROOF. Exercise 1.164. □

Since we are identifying binary sequences and words in the alphabet X, we will often write $\zeta(x_\alpha)$ instead of $\zeta(\alpha)$. With this notation, Proposition 1.156 says that the following identity holds for all $w, v \in \mathfrak{H}^0$:

$$\zeta(w \sqcup\!\sqcup\, v) = \zeta(w)\zeta(v). \tag{1.157}$$

1.6.4. *Double shuffle relations.* In the same way that positive multi-indices can be translated into binary sequences, there is a map between $\mathbb{Q}\langle Y \rangle$ and $\mathfrak{H}$. This map does not preserve the product structure, the stuffle product on one side, the shuffle product on the other. We can define a second product on $\mathfrak{H}$ that is compatible with the stuffle product in $\mathbb{Q}\langle Y \rangle$.

DEFINITION 1.158. The *stuffle product* in $\mathfrak{H}$, denoted $*$, is defined inductively as follows:

$$\begin{aligned} 1 * w = w * 1 &= w && \forall\, w \in \mathfrak{H} \\ x_0^p * w = w * x_0^p &= w x_0^p && \forall\, p > 0,\ \forall\, w \in \mathfrak{H} \\ z_p w * z_q v &= z_p(w * z_q v) + z_q(z_p w * v) + z_{p+q}(w * v) && \forall\, w, v \in \mathfrak{H}, \end{aligned}$$

where $z_p = x_0^{p-1} x_1$.

PROPOSITION 1.159.

a) $(\mathfrak{H}, *)$ *is a commutative and associative* $\mathbb{Q}$*-algebra.*
b) The map

$$\begin{aligned}(\mathbb{Q}\langle Y\rangle, *) &\hookrightarrow (\mathfrak{H}, *)\\ y_i &\mapsto z_i = x_0^{i-1}x_1.\end{aligned}$$

is an algebra monomorphism.

PROOF. Exercise 1.165. □

THEOREM 1.160. *Let* $\zeta : \mathfrak{H}^0 \to \mathbb{R}$ *be as before. Then we have*

$$\zeta(w \sqcup\!\sqcup v - w * v) = 0.$$

PROOF. This follows from equations (1.149) and (1.157). □

This theorem is a source of relations among MZVs called *double shuffle* relations. Nevertheless, it is clear that they are not enough to describe all relations among MZVs. For instance, we do not obtain any relation in weight 3, while we know the Euler relation, and we can only produce one relation in weight 4, while there are at least 3 independent relations in weight 4. In order to obtain the needed relations we will need to consider products with non-admissible words. This will be done in the next section.

⋆ ⋆ ⋆

EXERCISE 1.161. Show that

$$x_0^r \sqcup\!\sqcup x_0^s = x_0 \dots x_0 \sqcup\!\sqcup x_0 \dots x_0 = \frac{(r+s)!}{r!s!}x_0^{r+s}.$$

EXERCISE 1.162. Prove that, for a letter a and words u and v, the following identity holds:

$$a \sqcup\!\sqcup uv = (a \sqcup\!\sqcup u)v + u(a \sqcup\!\sqcup v) - uav.$$

EXERCISE 1.163. Prove Proposition 1.151.

EXERCISE 1.164. Prove Proposition 1.156.

EXERCISE 1.165. Prove Proposition 1.159.

EXERCISE 1.166. Given a multi-index $\boldsymbol{s}$ and an integer $M \geq 0$, we set

$$\zeta_M(\boldsymbol{s}) = \sum_{M>m_1>m_2>\dots>m_\ell>0} \frac{1}{m_1^{s_1}\dots m_l^{s_\ell}}.$$

(a) Show that, if $\boldsymbol{s}$ is admissible, then $\lim_{M\to\infty}\zeta_M(\boldsymbol{s}) = \zeta(\boldsymbol{s})$.

(b) Recall that we identified words and multi-indexes. Prove that

$$\zeta_M\colon (\mathfrak{H}^1, *) \to \mathbb{Q}$$

is a group morphism, *i.e.* for all $w, v \in \mathfrak{H}^1$

$$\zeta_M(w * v) = \zeta_M(w)\zeta_M(v).$$

EXERCISE 1.167. Using the identification between words in the alphabet X and binary sequences, we obtain a map

$$\mathrm{Li}\,:\,\mathfrak{H}^1 \to \mathcal{C}^\infty((0,1)).$$

Prove that this map is a homomorphism, that is, for all $w, v \in \mathfrak{H}^1$,

$$\mathrm{Li}_{w \sqcup\!\sqcup v}(t) = \mathrm{Li}_w(t) \cdot \mathrm{Li}_v(t).$$

1.7. Regularization and the Ihara-Kaneko-Zagier theorem. In this section, we discuss how to extend multiple zeta values to non-admissible words and use this extension to derive relations among them. Conjecturally, all relations can be obtained in this way. We follow the paper [**IKZ06**].

1.7.1. *The stuffle algebra as a polynomial ring.*

THEOREM 1.168. *The map of* $(\mathbb{Q}\langle Y\rangle^0, *)$*-algebras*

$$\begin{array}{cccc} \varphi\colon & \mathbb{Q}\langle Y\rangle^0[T] & \longrightarrow & \mathbb{Q}\langle Y\rangle \\ & T & \longmapsto & y_1 \end{array}$$

is an isomorphism.

PROOF. We first show that φ is surjective, which amounts to saying that any $w \in \mathbb{Q}\langle Y\rangle$ can be written as a polynomial in y_1 with coefficients in $\mathbb{Q}\langle Y\rangle^0$. The bijection between the sets of multi-indices and words in the alphabet Y induces a grading by the weight wt and a filtration by the length l on the space $\mathbb{Q}\langle Y\rangle$ given by

$$\mathrm{wt}(y_{s_1}\cdots y_{s_\ell}) = s_1 + \cdots + s_\ell$$
$$\ell(y_{s_1}\cdots y_{s_\ell}) = \ell.$$

If we show that, for a fixed length ℓ and word $w \in F_\ell\mathbb{Q}\langle Y\rangle$, there are elements $v_1 \in F_\ell\mathbb{Q}\langle Y\rangle^0$ and $v_2, v_3 \in F_{\ell-1}\mathbb{Q}\langle Y\rangle$ such that

$$w = v_1 + v_2 * y_1 + v_3, \tag{1.169}$$

then the claim follows by induction on ℓ.

Any word of length ℓ can be written as

$$w = \underbrace{y_1\cdots y_1}_{m} y_{s_1}\cdots y_{s_{\ell-m}} = \{y_1\}^m y_{s_1}\cdots y_{s_{\ell-m}}$$

with $s_1 \neq 1$ and $m \geq 0$.

We next prove, by induction on m, that w can be written as in (1.169). For $m = 0$, we have $w \in \mathbb{Q}\langle Y\rangle^0$. Thus we can choose $v_1 = w$, $v_2 = v_3 = 0$. For the induction step we compute

$$\{y_1\}^{m-1} y_{s_1} \cdots y_{s_{\ell-m}} * y_1 = \\ m \cdot w + \sum_{i=1}^{\ell-m} \{y_1\}^{m-1} y_{s_1} \cdots y_{s_i} y_1 y_{s_{i+1}} \cdots y_{s_{\ell-m}} - v_3$$

with $v_3 \in F_{\ell-1}\mathbb{Q}\langle Y\rangle$. Applying the induction hypothesis with respect to m we deduce that w can be written as in (1.169). It follows that φ is surjective.

To prove the injectivity of φ, we write each non-zero $P \in \mathbb{Q}\langle Y\rangle^0[T]$ as

$$P = w_1 T^m + w_2$$

with $0 \neq w_1 \in \mathbb{Q}\langle Y\rangle^0$ and w_2 of degree less than m in the variable T. Then

$$\varphi(P) = m! y_1^m w_1 + v_2$$

where all the words in v_2 have less than m factors y_1 in the front. Thus $\varphi(P) \neq 0$ and φ is injective. □

1.7.2. *The shuffle algebra as a polynomial ring.* Mutatis mutandi, one can prove the analogous result for the shuffle product.

Theorem 1.170.

(1) *The map of* $(\mathfrak{H}^0, ⧢)$*-algebras*

$$\begin{array}{rccc} \psi_1\colon & \mathfrak{H}^0[T] & \longrightarrow & \mathfrak{H}^1 \\ & T & \longmapsto & x_1 \end{array}$$

is an isomorphism.

(2) *The map of* $(\mathfrak{H}^1, ⧢)$*-algebras*

$$\begin{array}{rccc} \psi_2\colon & \mathfrak{H}^1[U] & \longrightarrow & \mathfrak{H} \\ & U & \longmapsto & x_0 \end{array}$$

is an isomorphism.

Therefore the map of $(\mathfrak{H}^0, ⧢)$*-algebras*

$$\begin{array}{rccc} \psi\colon & \mathfrak{H}^0[T,U] & \longrightarrow & \mathfrak{H} \\ & T & \longmapsto & x_1 \\ & U & \longmapsto & x_0 \end{array}$$

is an isomorphism.

Proof. Exercise 1.203. □

1.7.3. *Regularized zeta values.* Using the previous theorems we define the stuffle and shuffle regularization maps.

Definition 1.171. The *stuffle regularization map*

$$\mathrm{reg}_*^T\colon\ \mathfrak{H}^1 = \mathbb{Q}\langle Y\rangle \ \longrightarrow\ \mathfrak{H}^0[T] = \mathbb{Q}\langle Y\rangle^0[T]$$

is defined as $\mathrm{reg}_*^T = \varphi^{-1}$, while the *shuffle regularization maps*

$$\mathrm{reg}_{ш}^T\colon \mathfrak{H}^1 \to \mathfrak{H}^0[T], \text{ and}$$
$$\mathrm{reg}_{ш}^{T,U}\colon \mathfrak{H} \to \mathfrak{H}^0[T,U]$$

as $\mathrm{reg}_{ш}^T = \psi_1^{-1}$ and $\mathrm{reg}_{ш}^{T,U} = \psi^{-1}$.

Theorems 1.168 and 1.170 allow us to extend the function ζ in a formal way.

Definition 1.172. The *stuffle regularized* zeta map, denoted ζ_*^T, is the composition

$$\mathbb{Q}\langle Y\rangle \xrightarrow{\mathrm{reg}_*^T} \mathbb{Q}\langle Y\rangle^0[T] \xrightarrow{\zeta} \mathcal{Z}[T] \subset \mathbb{R}[T].$$

We denote by ζ_* the composition of ζ_*^T with the evaluation at $T=0$.

The *shuffle regularized* zeta map, denoted by $\zeta_{ш}^T$, is the composition

$$\mathfrak{H}^1 \xrightarrow{\mathrm{reg}_{ш}^T} \mathfrak{H}^0[T] \xrightarrow{\zeta} \mathcal{Z}[T] \subset \mathbb{R}[T].$$

Similarly, we write $\zeta_{ш}^{T,U}$ for the composition

$$\mathfrak{H} \xrightarrow{\mathrm{reg}_{ш}^{T,U}} \mathfrak{H}^0[T,U] \xrightarrow{\zeta} \mathcal{Z}[T,U] \subset \mathbb{R}[T,U].$$

We denote by $\zeta_{ш}$ the composition of $\zeta_{ш}^{T,U}$ with the evaluation at $T=U=0$. We will also denote by $\zeta_{ш}$ its restriction to $\mathfrak{H}^1$.

By identifying $(\mathbb{Q}\langle Y\rangle,*)$ with $(\mathfrak{H}^1,*)$, we will also consider ζ_*^T as a map from $(\mathfrak{H}^1,*)$ to $\mathbb{R}[T]$. This map is characterized by the conditions

$$\begin{aligned} \zeta_*^T(w) &= \zeta(w) \in \mathbb{R}, \qquad \text{if } w \in \mathfrak{H}^0,\\ \zeta_*^T(x_1) &= T,\\ \zeta_*^T(v*w) &= \zeta_*^T(v)\zeta_*^T(w). \end{aligned}$$

In the same way, the map $\zeta_{ш}^T$ is characterized by the identities

$$\begin{aligned} \zeta_{ш}^T(w) &= \zeta(w) \in \mathbb{R}, \qquad \text{if } w \in \mathfrak{H}^0,\\ \zeta_{ш}^T(x_1) &= T,\\ \zeta_{ш}^T(v ш w) &= \zeta_{ш}^T(v)\zeta_{ш}^T(w). \end{aligned}$$

The maps ζ_*, $\zeta_{ш}$ and $\zeta_{ш}^{T,U}$ are determined by similar conditions. For future reference we single out the properties characterizing $\zeta_{ш}$.

PROPOSITION 1.173. *The map* $\zeta_{ш}\colon \mathfrak{H} \to \mathbb{R}$ *is the only map satisfying*

$$\zeta_{ш}(w) = \zeta(w) \in \mathbb{R}, \qquad \text{if } w \in \mathfrak{H}^0, \tag{1.174}$$

$$\zeta_{ш}(x_0) = 0,\ \zeta_{ш}(x_1) = 0, \tag{1.175}$$

$$\zeta_{ш}(v ш w) = \zeta_{ш}(v)\zeta_{ш}(w). \tag{1.176}$$

COROLLARY 1.177. *The image of* $\zeta_{ш}$ *agrees with* $\mathcal{Z}$.

PROOF. By Theorem 1.170, every element $w \in \mathfrak{H}$ can be written as a polynomial in x_0 and x_1 with coefficients in $\mathfrak{H}^0$ with respect to the shuffle product. By Proposition 1.173, we deduce $\zeta_{ш}(w) \in \mathcal{Z}$. $\square$

EXAMPLE 1.178. On the one hand, we have

$$\begin{aligned}\zeta_*^T(1,2) &= \zeta_*^T(y_1y_2)\\ &= \zeta_*^T(y_2 * y_1 - y_2y_1 - y_3)\\ &= \zeta(2)T - \zeta(2,1) - \zeta(3),\end{aligned}$$

which yields $\zeta_*(1,2) = -\zeta(2,1) - \zeta(3)$. On the other hand,

$$\begin{aligned}\zeta_{ш}^T(1,2) &= \zeta_{ш}^T(x_1x_0x_1)\\ &= \zeta_{ш}^T(x_0x_1 ш x_1 - 2x_0x_1x_1)\\ &= \zeta(2)T - 2\zeta(2,1).\end{aligned}$$

Therefore, $\zeta_{ш}(1,2) = -2\zeta(2,1)$.

1.7.4. *Comparing the shuffle and the stuffle regularizations.* As we just saw in the previous example, the regularizations $\zeta_{ш}^T(w)$ and $\zeta_*^T(w)$ are in general different from each other. In order to compare them, we introduce the formal power series

$$A(u) = e^{\gamma u}\Gamma(1+u) = \exp\left(\sum_{n\geq 2} \frac{(-1)^n}{n}\zeta(n)u^n\right),$$

where γ is the Euler-Mascheroni constant, and the second identity follows from Exercise 1.20. We write

$$A(u) = \sum_{k\geq 0} \gamma_k u^k.$$

Observe that γ_k is a linear combination, with rational coefficients, of multiple zeta values of weight k. Here are the first values:

k	0	1	2	3	4	5
γ_k	1	0	$\frac{\zeta(2)}{2}$	$-\frac{\zeta(3)}{3}$	$\frac{\zeta(2,2)}{4} + \frac{3\zeta(4)}{8}$	$-\frac{\zeta(2,3)}{6} - \frac{\zeta(3,2)}{6} - \frac{11\zeta(5)}{30}$

We define an $\mathbb{R}$-linear map $\varrho : \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$\varrho(T^n) = \frac{d^n}{du^n}\left(A(u)e^{Tu}\right)\Big|_{u=0} = n!\sum_{k=0}^{n} \gamma_k \frac{T^{n-k}}{(n-k)!}. \tag{1.179}$$

THEOREM 1.180 (Ihara-Kaneko-Zagier, [**IKZ06**]). *The following identity holds for all words* $w \in \mathfrak{H}^1$:

$$\zeta_{⧢}^T(w) = \varrho(\zeta_*^T(w)).$$

EXAMPLE 1.181. Since $\gamma_0 = 1$ and $\gamma_1 = 0$, we have $\varrho(1) = 1$ and $\varrho(T) = T$. Combining this with Example 1.178 we find

$$\begin{aligned}\varrho(\zeta_*^T(1,2)) &= \varrho(\zeta(2)T - \zeta(2,1) - \zeta(3)) \\ &= \zeta(2)T - \zeta(2,1) - \zeta(3).\end{aligned}$$

On the other hand,

$$\zeta_{⧢}^T(1,2) = \zeta(2)T - 2\zeta(2,1),$$

hence we recover Euler's relation $\zeta(2,1) = \zeta(3)$.

PROOF OF THEOREM 1.180. The idea is to view $\zeta_{⧢}^T(w) = \varrho(\zeta_*^T(w))$ as an identity of functions in T. Let $M > 0$ be an integer and $w = (y_{s_1} \cdots y_{s_\ell})$ a word in the alphabet Y. We write

$$\zeta_M(w) = \sum_{M > m_1 > m_2 > \cdots > m_\ell > 0} \frac{1}{m_1^{s_1} \cdots m_\ell^{s_\ell}}.$$

Note that, if w is admissible, then $\lim_{M\to\infty} \zeta_M(w) = \zeta(w)$. We extend ζ_M to a map $\mathbb{Q}\langle Y \rangle \to \mathbb{R}$ by linearity. Then ζ_M satisfies the stuffle relation

$$\zeta_M(w_1)\zeta_M(w_2) = \zeta_M(w_1 * w_2).$$

From the approximation of the harmonic series

$$\zeta_M(y_1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M-1} = \log M + \gamma + O\left(\frac{1}{M}\right)$$

and the representation of $\zeta_*^T(w)$ as a polynomial on $\zeta_*(1)$, it follows that there exists $j \geq 0$ such that, for M large enough, one has

$$\zeta_M(w) = \zeta_*^{\log M + \gamma}(w) + O(M^{-1}\log^j M), \tag{1.182}$$

where $\zeta_*^{\log M+\gamma}(w)$ means the evaluation at $T = \log M + \gamma$ of $\zeta_*^T(w)$.

Recall that, to each positive multi-index $\boldsymbol{s}$ we have associated a polylogarithm function. Using the identification of positive multi-indices with words in the alphabet Y and linearity, we attach, to each element $w \in \mathfrak{H}^1$ a function Li_w on the segment $(0,1)$. If $w \in \mathfrak{H}^0$, then

$$\lim_{t\to 1^-} \mathrm{Li}_w(t) = \zeta(w).$$

Moreover we have for all w, w' and $t \in (0,1)$

$$\mathrm{Li}_w(t) \cdot \mathrm{Li}_{w'}(t) = \mathrm{Li}_{w ⧢ w'}(t).$$

Since

$$\mathrm{Li}_{y_1}(t) = \log\left(\frac{1}{1-t}\right),$$

when $t \to 1^-$

$$\mathrm{Li}_w(t) = \zeta_{ш}^{\log\left(\frac{1}{1-t}\right)}(w) + O\left((1-t)\log^j\left(\frac{1}{1-t}\right)\right) \tag{1.183}$$

for some $j \geq 0$ depending on w. Here $\zeta_{ш}^{\log\left(\frac{1}{1-t}\right)}(w)$ means the evaluation at $T = \log\left(\frac{1}{1-t}\right)$ of $\zeta_{ш}^{T}(w)$.

By explicit calculations,

$$\begin{aligned}
\mathrm{Li}_w(t) &= \sum_{m_1>m_2>\cdots>m_\ell>0} \frac{t^{m_1}}{m_1^{s_1}\cdots m_\ell^{s_\ell}} \\
&= \sum_{m\geq 1}\left(\sum_{m>m_2>\cdots>m_\ell>0} \frac{1}{m^{s_1}m_2^{s_2}\cdots m_\ell^{s_\ell}}\right)t^m \\
&= \sum_{m\geq 1}(\zeta_{m+1}(w)-\zeta_m(w))t^m \\
&= (1-t)\sum_{m\geq 2}\zeta_m(w)t^{m-1}.
\end{aligned}$$

In the last equality we use that $\zeta_1(w) = 0$.

We now need Lemma 1.184 below. We apply it to the polynomials $P(T) = \zeta_*^T(w)$ and $Q(T) = \varrho(\zeta_*^T(w))$. We derive

$$\begin{aligned}
\mathrm{Li}_w(t) &= (1-t)\sum_{m\geq 2}\zeta_m(w)t^{m-1} \\
&\overset{(1.182)}{=} (1-t)\sum_{m\geq 2}\zeta_*^{\log m+\gamma}(w)t^{m-1} + (1-t)\sum_{m\geq 1} O\left(\frac{\log^j m}{m}\right)t^{m-1} \\
&\overset{1.184}{=} Q\left(\log\frac{1}{1-t}\right) + O\left((1-t)\log^j\left(\frac{1}{1-t}\right)\right).
\end{aligned}$$

Comparing with the asymptotic expansion we get the claimed identity

$$\zeta_{ш}^T(\boldsymbol{s}) = \varrho(\zeta_*^T(\boldsymbol{s})). \qquad \square$$

The next lemma is used in the proof of Theorem 1.180.

Lemma 1.184.

(1) Let $P(T) \in \mathbb{R}[T]$ and $Q(T) = \varrho(P(T))$. Then

$$\sum_{m\geq 2} P(\log(m)+\gamma)t^{m-1} = \frac{1}{1-t}Q\left(\log\frac{1}{1-t}\right) + O\left(\log^j\left(\frac{1}{1-t}\right)\right) \tag{1.185}$$

for some $j \in \mathbb{N}$, as $t \to 1^-$.

(2) As $t \to 1^-$, we have

$$\sum_{m\geq 2} \frac{\log^j m}{m} t^{m-1} = O\left(\log^{j+1}\left(\frac{1}{1-t}\right)\right). \tag{1.186}$$

Proof. Let us prove (1.186) first. Since

$$\sum_{m\geq 2} \frac{1}{m} t^{m-1} = -1 - \frac{1}{t}\log(1-t),$$

for $j = 0$ the left hand side of (1.186) is of type $O\left(\log\left(\frac{1}{1-t}\right)\right)$ as $t \to 1^-$, which proves the statement in this case. Now we proceed by induction on j. We have

$$\log^{j+1}(m) \leq c_j \sum_{n=1}^{m} \frac{\log^j n}{n}$$

for $m \geq 1$, $j \geq 0$. This follows easily from the integral

$$\int_1^m \frac{\log^j(x)}{x} dx = \frac{\log^{j+1}(m)}{j+1}.$$

Hence for $t < 1$ we obtain

$$\begin{aligned}\sum_{m\geq 1} \frac{\log^{j+1}(m)}{m} t^{m-1} &\leq c_j \sum_{m\geq 1} \frac{t^{m-1}}{m} \sum_{n=1}^{m} \frac{\log^j(n)}{n} \\ &= c_j \sum_{n\geq 1} \frac{\log^j(n)}{n} t^{n-1} \sum_{r\geq 1} \frac{t^{r-1}}{r+n-1} \\ &< c_j \left(\sum_{n\geq 1} \frac{\log^j(n)}{n} t^{n-1}\right)\left(\frac{1}{t}\log\left(\frac{1}{1-t}\right)\right).\end{aligned}$$

Now (1.186) follows by induction on j for all $j \geq 0$.

We now establish the identity (1.185). By construction, ϱ is a linear map on $\mathbb{R}[T]$ and it therefore suffices to prove (1.185) for $P(T) = (T-\gamma)^n$. Thus we put $Q(T) = \varrho((T-\gamma)^n)$. Then, by equation (1.179),

$$Q(T) = \frac{d^n}{du^n}\left(A(u)e^{(T-\gamma)u}\right)\Big|_{u=0} = \frac{d^n}{du^n}\left(\Gamma(1+u)e^{Tu}\right)\Big|_{u=0}.$$

Hence

$$\begin{aligned}\frac{1}{1-t}Q\left(\log\left(\frac{1}{1-t}\right)\right) &= \frac{d^n}{du^n}\left(\frac{\Gamma(1+u)}{(1-t)^{1+u}}\right)\bigg|_{u=0}\\ &= \frac{d^n}{du^n}\left(\sum_{m\geq 1}\frac{\Gamma(m+u)}{\Gamma(m)}t^{m-1}\right)\Bigg|_{u=0}\\ &= \sum_{m\geq 1}\frac{\Gamma^{(n)}(m)}{\Gamma(m)}t^{m-1},\end{aligned}$$

where $\Gamma^{(n)}(m)$ is the n-th derivative of the Γ function evaluated at m. Now we use that, for $m\to\infty$ and all n, we have the estimate

$$\frac{\Gamma^{(n)}(m)}{\Gamma(m)} = \log(m)^n + O\left(\frac{\log^{n-1}(m)}{m}\right). \tag{1.187}$$

Using this and (1.186) we obtain

$$\begin{aligned}\sum_{m\geq 1}\frac{\Gamma^{(n)}(m)}{\Gamma(m)}t^{m-1} &= \sum_{m\geq 1}\log^n(m)t^{m-1} + O\left(\log^n\left(\frac{1}{1-t}\right)\right)\\ &= \sum_{m\geq 1}P(\log(m)+\gamma)t^{m-1} + O\left(\log^n\left(\frac{1}{1-t}\right)\right),\end{aligned}$$

concluding the proof of the lemma. □

1.7.5. *The extended double shuffle relations.* We now introduce the extended double shuffle relations. We first recall the two commutative diagrams

$$\begin{tikzcd}(\mathfrak{H}^1,ш) \arrow[r, "\mathrm{reg}^T_{ш}"] \arrow[dr, "\zeta^T_{ш}"'] & (\mathfrak{H}^0,ш)[T] \arrow[d, "w\mapsto\zeta(w),T\mapsto T"] \\ & \mathbb{R}[T]\end{tikzcd}\quad , \quad \begin{tikzcd}(\mathfrak{H}^1,*) \arrow[r, "\mathrm{reg}^T_{*}"] \arrow[dr, "\zeta^T_{*}"'] & (\mathfrak{H}^0,*)[T] \arrow[d, "w\mapsto\zeta(w),T\mapsto T"] \\ & \mathbb{R}[T]\end{tikzcd}\quad . \tag{1.188}$$

Definition 1.189. Let $(R,\cdot)$ be a $\mathbb{Q}$-algebra and $Z_R\colon\mathfrak{H}^0\to R$ a map. We say that (R,Z_R) satisfies the *finite double shuffle relations* if Z_R is an algebra homomorphism $Z_R\colon(\mathfrak{H}^0,ш)\to(R,\cdot)$, as well as an algebra homomorphism $Z_R\colon(\mathfrak{H}^0,*)\to(R,\cdot)$, that is:

$$Z_R(w\,ш\,v) = Z_R(w)\cdot Z_R(v) = Z_R(w*v). \tag{1.190}$$

Composing Z_R with the regularization maps $\mathrm{reg}^T_{ш}$ and reg^T_{*}, we obtain extensions

$$\begin{aligned}Z^T_{R,ш}&\colon(\mathfrak{H}^1,ш)\to R[T],\\ Z^T_{R,*}&\colon(\mathfrak{H}^1,*)\to R[T].\end{aligned}$$

Since R is a $\mathbb{Q}$-algebra that receives a map from $\mathfrak{H}^0$, we can define the formal power series

$$A_R(u) = \exp\left(\sum_{n\geq 2} \frac{(-1)^n}{n} Z_R(y_n) u^n\right).$$

By analogy with ρ, we can define a linear map $\varrho_R : R[T] \to R[T]$ by

$$\varrho_R(e^{Tu}) = A_R(u)e^{Tu}. \tag{1.191}$$

Definition 1.192. Assume that (R, Z_R) satisfies the finite double shuffle relations. We say (R, Z_R) satisfies the *extended double shuffle relations*, if in addition, for all $w \in \mathfrak{H}^1$, one has

$$Z^T_{R,ш} = \varrho_R(Z^T_{R,*}(w)). \tag{1.193}$$

Combining theorems 1.160 and 1.180 we obtain the main result of this section.

Theorem 1.194. *The pair $(\mathbb{R}, \zeta)$ satisfies the extended double shuffle relations.*

In particular, since for $w_0 \in \mathfrak{H}^0$ we have that $x_1 ш w_0 - x_1 * w_0 \in \mathfrak{H}^0$ (Exercise 1.198), we deduce the Hoffman relation

$$\zeta(x_1 ш w_0 - x_1 * w_0) = 0. \tag{1.195}$$

Moreover, the following holds for all $w_0 \in \mathfrak{H}^0$

$$\zeta^{T=0}_{ш}(x_1^m * w_0) = 0.$$

1.7.6. *The universal algebra satisfying the extended double shuffle relations.* Let $\varphi\colon R \to R'$ be a morphism of $\mathbb{Q}$-algebras. If (R, Z_R) satisfies the extended double shuffle relations, then so does $(R', \varphi \circ Z_R)$. Let R_{EDS} be the universal algebra with this property. Thus R_{EDS} is a quotient of $\mathfrak{H}^0$ by certain relations and, for any (R, Z_R) satisfying the extended double shuffle relations, there exists a map $\varphi_R\colon R_{\mathrm{EDS}} \to R$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{H}^0 \arrow[r] \arrow[dr, "Z_R"'] & R_{\mathrm{EDS}} \arrow[d, "\varphi_R"] \\ & R. \end{tikzcd}$$

The following conjecture describes the structure of the algebra of multiple zeta values.

Conjecture 1.196. *The map $\varphi_{\mathbb{R}}$ is injective, that is the algebra $\mathcal{Z}$ of MZVs is isomorphic to R_{EDS}.*

Remark 1.197. The finite double shuffle relations are linear and homogeneous with respect to the weight. Moreover the extended double shuffle relations are also homogeneous (Exercise 1.201). Since the coefficients of the power series A_R are polynomials in zeta values, the extended double shuffle relations relations are polynomial in the MZVs. Since products of MZVs can be reduced to linear combinations of MZVs using either the shuffle or the stuffle product, we can reduce the extended double shuffle relations relations to linear ones. Hence all possible relations among MZVs are conjectured to be generated by homogeneous linear relations.

⋆ ⋆ ⋆

EXERCISE 1.198. Show that, if $w \in \mathfrak{H}^0$, then $x_1 ш w - x_1 * w \in \mathfrak{H}^0$.

EXERCISE 1.199. Deduce Euler's sum formula (1.58) from the Hoffman relation. [Hint: take $w = z_p$.]

EXERCISE 1.200. Show that γ_k is a polynomial in $\zeta(2), \zeta(3), \ldots$, that is homogeneous of weight k.

EXERCISE 1.201. Use Exercise 1.200 to prove that the EDS relations are homogeneous.

EXERCISE 1.202. What identities do we get from a comparison of $\zeta_*(1,1,2)$ and $\zeta_ш(1,1,2)$?

EXERCISE 1.203. Prove Theorem 1.170.

EXERCISE 1.204. Verify

$$\begin{aligned}\zeta_M(\boldsymbol{s}) &= \sum_{k=0}^{n} a_k \left(\log M + \gamma + O\left(\frac{1}{M}\right)\right)^k \\ &= \sum_{k=0}^{n} a_k (\log M + \gamma)^k + O\left(\frac{1}{M}\log^{n-1}(M)\right).\end{aligned}$$

EXERCISE 1.205. Prove (1.187).

2. Periods of mixed Hodge structures

In this chapter, we introduce the first tools from algebraic geometry that will be needed for the study of multiple zeta values. The main goal is to show that all these numbers can be obtained by integrating an algebraic differential form over a topological cycle on a variety defined over $\mathbb{Q}$; the extra structures carried by cohomology will then give non-trivial information about these numbers. With this in mind, we begin by recalling the definition of the singular cohomology of a topological space M. If M underlies a complex algebraic variety X, this cohomology can be computed using differential forms with polynomial coefficients: it is isomorphic to the algebraic de Rham cohomology. A remarkable consequence is that, when X is defined over $\mathbb{Q}$, we get two different rational structures on the same complex vector space which are not compatible. This is not bad news, quite the opposite: the comparison between them produces an interesting class of complex numbers called *periods*. Another important consequence is that the cohomology of X is equipped with two filtrations, whose interaction gives rise to a mixed Hodge structure. We explain the definition and give many examples, in particular of Hodge structures of mixed Tate type. Conjecturally, Hodge structures capture all algebraic relations between periods. As an illustration, we explain in detail how to interpret $\zeta(2)$ as a period of an algebraic variety and how this can be used to prove that it is a rational multiple of π^2.

2.1. Singular homology and cohomology. We begin by briefly recalling the definition of the singular homology and cohomology of a topological space. For each integer $n \geq 0$, let

$$\Delta^n_{\mathrm{st}} = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\, t_i \geq 0,\, i = 0, \ldots, n\}$$

be the standard simplex of dimension n. For each $i = 0, \ldots, n$, there are *face* maps $\delta^n_i \colon \Delta^{n-1}_{\mathrm{st}} \to \Delta^n_{\mathrm{st}}$ given by

$$\delta^n_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Let M be a topological space. A continuous map $\sigma : \Delta^n_{\mathrm{st}} \to M$ is called a *singular n-chain*. For each $n \geq 0$, let

$$C_n(M) = \bigoplus_{\sigma} \mathbb{Z}\sigma$$

be the free abelian group generated by singular n-chains. Elements of $C_n(M)$ are thus finite $\mathbb{Z}$-linear combinations of continuous maps $\sigma \colon \Delta^n_{\mathrm{st}} \to M$.

For each $n \geq 1$, we define a *boundary* map

$$\begin{aligned} \partial_n : C_n(M) &\longrightarrow C_{n-1}(M) \\ \sigma &\longmapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta_i^n). \end{aligned} \tag{2.1}$$

Thanks to the alternating signs, these maps satisfy $\partial_{n-1} \circ \partial_n = 0$, thus making $(C_\bullet(M), \partial_\bullet)$ into a complex (see Exercise 2.18).

Definition 2.2. The *singular homology* of M is the homology of the complex $(C_\bullet(M), \partial_\bullet)$, that is

$$H_n(M, \mathbb{Z}) = \begin{cases} C_0(M)/\mathrm{Im}(\partial_1) & n = 0, \\ \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}) & n \geq 1. \end{cases}$$

Elements in the kernel of the boundary map ∂_n are usually called (closed) *cycles* and $\mathrm{Im}(\partial_{n+1})$ is the group of *boundaries*.

The above construction is functorial: if $f\colon M_1 \to M_2$ is a continuous map between two topological spaces, sending a singular n-chain $\sigma\colon \Delta^n_{\mathrm{st}} \to M_1$ to $f \circ \sigma\colon \Delta^n_{\mathrm{st}} \to M_2$ induces a morphism of complexes $f_*\colon C_\bullet(M_1) \to C_\bullet(M_2)$, hence a group morphism

$$f_*\colon H_n(M_1, \mathbb{Z}) \to H_n(M_2, \mathbb{Z}).$$

Example 2.3. Let $M = \mathbb{C} \setminus \{0\}$ be the punctured complex plane. Consider the singular chains

$$\begin{aligned} \sigma_0\colon \Delta^0_{\mathrm{st}} &\to M, & 1 &\mapsto 1 \\ \sigma_1\colon \Delta^1_{\mathrm{st}} &\to M, & (t, 1-t) &\mapsto \exp(2\pi i t). \end{aligned}$$

Then $H_0(M, \mathbb{Z})$ and $H_1(M, \mathbb{Z})$ are both free groups of rank one, generated by σ_0 and σ_1 respectively. All the other homology groups vanish.

Dualizing, we find the free abelian group of *singular n-cochains*

$$C^n(M) = \mathrm{Hom}_{\mathbb{Z}}(C_n(M), \mathbb{Z}),$$

as well as coboundary maps $d^n\colon C^n(M) \to C^{n+1}(M)$ which yield again a complex $(C^\bullet(M), d^\bullet)$, this time cohomological.

Definition 2.4. The *singular cohomology* of M is the cohomology of the complex $(C^\bullet(M), d^\bullet)$, that is:

$$H^n(M, \mathbb{Z}) = \begin{cases} \mathrm{Ker}(d^0) & n = 0, \\ \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) & n \geq 1. \end{cases}$$

Remarks 2.5.

(1) We have defined singular homology and cohomology with integral coefficients, but the same construction extends to other coefficient rings such as $\mathbb{Q}$ or $\mathbb{R}$. Most of the time it will be enough for our purposes to work with rational coefficients.

(2) Singular homology and cohomology are invariants defined for any topological space. When M is a differentiable manifold, instead of continuous maps $\sigma\colon \Delta^n_{\mathrm{st}} \to M$ we may use piecewise smooth maps or even smooth maps. The resulting groups are the same.

(3) Working with rational coefficients, we can identify singular cohomology with the linear dual of singular homology

$$H^n(M, \mathbb{Q}) \simeq \operatorname{Hom}(H_n(M, \mathbb{Q}), \mathbb{Q}),$$

and think of cohomology classes as linear functionals on homology. This isomorphism cannot hold for integral coefficients since the group $\operatorname{Hom}(H_n(M, \mathbb{Z}), \mathbb{Z})$ is always torsion free, while $H^n(M, \mathbb{Z})$ may have torsion (see Exercise 2.19).

In the sequel, we will mainly consider the singular cohomology of topological spaces given by the complex points of algebraic varieties over subfields of $\mathbb{C}$. It deserves a special name:

Definition 2.6. Let k be a subfield of $\mathbb{C}$ and X an algebraic variety over k. The *Betti cohomology* $H^*_B(X)$ is the singular cohomology of the space of complex points $X(\mathbb{C})$ equipped with the analytic topology:

$$H^*_{\mathrm{B}}(X) = H^*(X(\mathbb{C}), \mathbb{Q}).$$

2.1.1. *Relative homology and cohomology.* There is also a relative version of homology and cohomology: if $\iota\colon N \hookrightarrow M$ is a topological subspace, the morphism of complexes $\iota_*\colon C_\bullet(N) \to C_\bullet(M)$ is injective. Recall *e.g.* from [**Wei94**, 1.5] that its *cone* is the complex $C_\bullet(M, N) = \operatorname{cone}(i_*)$ given by

$$C_n(M, N) = C_{n-1}(N) \oplus C_n(M)$$

in degree n, together with the differential

$$\partial(a, b) = (-\partial a, -\iota_*(a) + \partial b).$$

Definition 2.7. The *relative homology* of a pair of topological spaces (M, N) is the homology of this complex:

$$H_*(M, N; \mathbb{Z}) = H_*(C_\bullet(M, N)).$$

We refer the reader to Exercise 2.20 for an alternative definition.

By construction, $C_\bullet(M, N)$ fits into a short exact sequence of complexes

$$0 \longrightarrow C_\bullet(M) \longrightarrow C_\bullet(M, N) \longrightarrow C_\bullet(N)[-1] \longrightarrow 0,$$

where the left map sends b to $(0, b)$, and the right map sends (a, b) to $-a$. Above, the shifted complex $C_\bullet(N)[-1]$ has $C_{n-1}(N)$ as degree n term, with

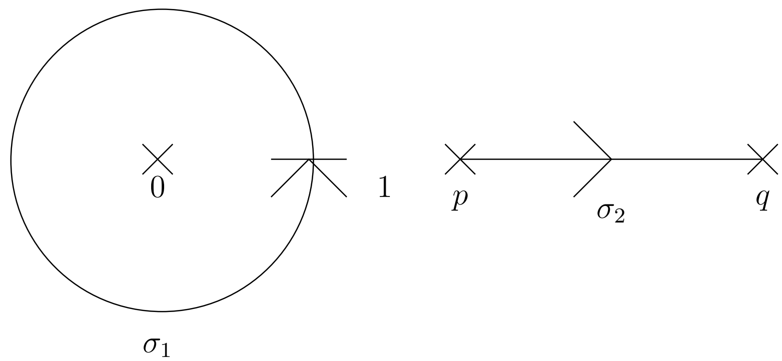


FIGURE 2. A basis of $H_1(\mathbb{C} \setminus \{0\}, \{p, q\}; \mathbb{Z})$

differential $-\partial_{n-1}$, so that the relation $H_n(C_\bullet(N)[-1]) = H_{n-1}(N, \mathbb{Z})$ holds. The associated long exact sequence then reads

$$\begin{tikzcd} \cdots \arrow[r] & H_n(M, \mathbb{Z}) \arrow[r] & H_n(M, N; \mathbb{Z}) \arrow[r] & H_{n-1}(N, \mathbb{Z}) \arrow[dll, out=0, in=180] \\ & H_{n-1}(M, \mathbb{Z}) \arrow[r] & H_{n-1}(M, N; \mathbb{Z}) \arrow[r] & \cdots, \end{tikzcd} \tag{2.8}$$

and the connecting morphisms are nothing other than the maps

$$\iota_* \colon H_*(N, \mathbb{Z}) \to H_*(M, \mathbb{Z})$$

induced by the inclusion $\iota \colon N \hookrightarrow M$.

REMARK 2.9. An element of the relative homology $H_n(M, N; \mathbb{Z})$ is represented by a pair (σ_N, σ_M) consisting of singular chains $\sigma_N \in C_{n-1}(N)$ and $\sigma_M \in C_n(M)$ such that $\partial\sigma_N = 0$ and $\partial\sigma_M = -\iota_*\sigma_N$. Since ι_* is injective, the singular chain σ_N is determined by the latter condition, which implies the former. In other words, relative homology classes are represented by chains in M whose boundary is contained in N.

EXAMPLE 2.10. Consider $M = \mathbb{C} \setminus \{0\}$ and let $N = \{p, q\} \subseteq M$ be a subspace consisting of two distinct points. Let $\sigma_2 \colon \Delta^1_{\mathrm{st}} \to M$ be any continuous map such that $\sigma_2((0, 1)) = p$ and $\sigma_2((1, 0)) = q$. Then

$$\partial\sigma_2 = p - q \in C_0(N),$$

so σ_2 defines a relative chain. It follows from the long exact sequence (2.8) that the only non-trivial relative homology group is $H_1(M, N; \mathbb{Z})$, which has a basis given by the chain σ_1 from Example 2.3 and σ_2 (see Figure 2).

In a similar way, one defines relative cohomology groups

$$H^n(M, N).$$

2.1.2. *Properties of singular homology.* Singular homology and cohomology have many useful properties such as

(1) *Homotopy invariance*: if M_1 and M_2 are topological spaces with the same homotopy type, then $H_*(M_1, \mathbb{Z}) \simeq H_*(M_2, \mathbb{Z})$.

(2) *Mayer-Vietoris*: for any two open subspaces U and V such that $M = U \cup V$, there is a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(U\cap V,\mathbb{Z}) \arrow[r] & H_n(U,\mathbb{Z})\oplus H_n(V,\mathbb{Z}) \arrow[r] & H_n(M,\mathbb{Z}) \arrow[dll, out=-10, in=170] \\ & H_{n-1}(U\cap V,\mathbb{Z}) \arrow[r] & \cdots \end{tikzcd}$$

(3) *Künneth formula*: there is a natural isomorphism

$$H_n(M_1 \times M_2, \mathbb{Q}) \simeq \bigoplus_{i+j=n} H_i(M_1, \mathbb{Q}) \otimes H_j(M_2, \mathbb{Q}).$$

Note that, for the Künneth formula to be true as stated we need rational coefficients. A more more complex formula that involve Tor groups is true with integer coefficients.

2.1.3. *Sheaf cohomology.* We have already introduced singular cohomology. To define periods, we will also need another type of cohomology called algebraic de Rham cohomology. But before we can introduce it we need to discuss general sheaf cohomology. We will give now give a brief summary sheaf cohomology. More details can be found, for instance in [**Har77**, Chapter II] or in [**KS06**].

Definition 2.11. Let M be a topological space. A *sheaf of abelian groups* F on M is an assignment that to each open subset $U \subset M$ assigns an abelian group $F(U)$ satisfying the following properties

(1) If $U \subset V$ is an inclusion of open subsets, then there is a restriction map $\rho_{U,V} \colon F(V) \to F(U)$. The notation

$$t\,|_U := \rho_{u,V}(t)$$

is frequently used.

(2) If $U \subset V \subset W$ are inclusions of open subsets, then

$$\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}.$$

(3) If $U = \bigcup_{i \in I} U_i$, with U and U_i, $i \in I$ open subsets, and $t \in F(A)$ such that $t\,|_{U_i} = 0$ for all $i \in I$ then $t = 0$.

(4) If $U = \bigcup_{i \in I} U_i$ as before and $t_i \in F(U_i)$ are such that

$$t_i\,|_{U_i \cap U_j} = t_j\,|_{U_i \cap U_j}, \quad \forall i, j \in I$$

then there exists $t \in F(U)$ such that $t\,|_{U_i} = t$.

Properties 1 and 2 are summarized stating that F is a functor from the category $\mathbf{Op}(M)$ of open subsets of M to the category $\mathbf{Ab}$ of abelian groups. Property 3 is the locality property, while 4 is the gluing property. Since all of the sheaves we will see in this notes are sheaves of abelian groups, for shorthand we well use the word sheaf as a synonym of sheaf of abelian groups.

The category of sheaves (of abelian groups) $\mathbf{ShAb}(M)$ is an abelian category. In particular all the paraphernalia of abelian categories applies to sheaves. For instance it makes sense to talk about kernels, cokernels and images, of complexes of sheaves and of the cohomology of a complex of sheaves, also the notion of exact sequence of sheaves is well defined and the usual definition of injective object in homological algebra also applies to sheaves.

Given a sheaf F and an open subset $U \subseteq M$, the elements of $F(U)$ are called *sections* of F over U. The group $F(M)$ is also denoted by $\Gamma(M, F)$ and its elements are called *global sections*. The assignment $F \mapsto \Gamma(M, F)$ gives rise to the global section functor $\Gamma\colon \mathbf{ShAb} \to \mathbf{Ab}$. This functor is not exact, but only left exact: if

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is an exact sequence of sheaves, then the sequence of abelian groups

$$0 \to \Gamma(M, F_1) \to \Gamma(M, F_2) \to \Gamma(M, F_3)$$

is exact, but the rightmost map does not need to be surjective. This observation is the starting point of the definition of sheaf cohomology. In the language of derived categories (see Section 4.2), *sheaf cohomology* is the derived functor of global sections.

The category $\mathbf{ShAb}(M)$ has enough injectives. This means that, for each sheaf F, there exists a long exact sequence

$$0 \to F \to I^0 \to I^1 \to I^2 \to \dots$$

where all the I^i are injective sheaves. Such an exact sequence is called an *injective resolution*. Taking global sections yields a complex

$$\Gamma(M, I^0) \longrightarrow \Gamma(M, I^1) \longrightarrow \Gamma(M, I^2) \to \cdots$$

and the sheaf cohomology groups of F are defined as

$$H^n(M, F) = H^n(\Gamma(M, I^\bullet)).$$

The resulting group does not depend on the choice of the injective resolution.

For theoretical purposes, injective resolutions are very nice, but it is not easy to write them down explicitly and it is useful to have more concrete ways to compute cohomology. A sheaf A is called *acyclic* if $H^i(X, A) = 0$ for all $i > 0$. An *acyclic resolution* of a sheaf F is an exact sequence

$$0 \to F \to A^0 \to A^1 \to A^2 \to \dots$$

where all the A^i are acyclic sheaves. The sheaf cohomology of F can be computed using any acyclic resolution, that is

$$H^n(M,F) = H^n(\Gamma(M,A^*)).$$

REMARK 2.12. Every sheaf F has a canonical acyclic resolution called the *Godement resolution*. It is constructed as follows: one first assigns to each open subset U of M the product

$$\mathcal{C}^0(F)(U) = \prod_{x\in U} F_x$$

of the stalks of F at all points in U. Together with the obvious restriction maps, one obtains a sheaf $\mathcal{C}^0(F)$ on M. Moreover, the natural morphism of sheaves $F \to \mathcal{C}^0(F)$ is injective. Then one defines

$$\mathcal{C}^1(F) = \mathcal{C}^0(\mathcal{C}^0(F)/F).$$

Iterating this process yields the Godement resolution

$$\mathcal{C}^*(M,F)\colon \qquad \mathcal{C}^0(F) \longrightarrow \mathcal{C}^1(F) \longrightarrow \mathcal{C}^2(F) \longrightarrow \cdots$$

An important property of the Godement resolution is that it gives a functorial way of choosing an acyclic resolution: if $F \to G$ is a morphism of sheaves, then there is a morphism of complexes $\mathcal{C}^*(M,F) \to \mathcal{C}^*(M,G)$ compatible with the composition of morphisms.

EXAMPLE 2.13. Under mild assumptions on the topological space M, singular cohomology can be identified with the sheaf cohomology of the constant sheaf. More precisely, let $\underline{\mathbb{Z}}$ be the sheaf that assigns to each open subset $U \subseteq M$ the group $\underline{\mathbb{Z}}(U) = \mathbb{Z}^{\pi_0(U)}$, where $\pi_0(U)$ stands for the number of connected components of U. Note that $\underline{\mathbb{Z}}(U)$ can be identified with the group of locally constant functions $U \to \mathbb{Z}$. Then, if M is a locally contractible topological space, then

$$H^*(M,\mathbb{Z}) = H^*(M,\underline{\mathbb{Z}}),$$

where the left-hand group is singular cohomology and the right hand group is sheaf cohomology. The same result is true for other rings of coefficients.

2.1.4. *Hypercohomology.* Consider now a complex of sheaves of abelian groups

$$F^\bullet\colon \cdots \to F^{n-1} \xrightarrow{d} F^n \xrightarrow{d} F^{n+1} \to \cdots .$$

For the cohomology of $F^\bullet$ we can mean two thinks. The first is to take the cohomology in the abelian category **ShAb**, in this case the cohomology objects will be sheaves. The second is to consider the derived functor of the functor of global sections Γ. In this second case the cohomology objects will be abelian groups. In order to distinguish between these two possibilities the second is classically called the hypercohomology of the complex.

Recall that a *resolution* of $F^\bullet$ is a complex $D^\bullet$, together with a quasi-isomorphism $F^\bullet \to D^\bullet$. If all the sheaves D^n are injective, we say that $D^\bullet$

is an injective resolution. If all the sheaves D^n are acyclic, then $D^\bullet$ is called an acyclic resolution.

DEFINITION 2.14. The *hypercohomology* of $F^\bullet$ is the cohomology of the complex of global sections of any acyclic resolution $D^\bullet$ of $F^\bullet$:

$$\mathbb{H}^n(X, F^\bullet) = H^n(\Gamma(X, D^\bullet))$$

EXAMPLE 2.15. Since Godement's canonical resolution of Remark 2.12 is functorial, we can use it to construct a resolution of any bounded below complex of sheaves. Let

$$\cdots \to F^{n-1} \xrightarrow{d} F^n \xrightarrow{d} F^{n+1} \to \ldots$$

be a complex of sheaves, with $F^n = 0$ for $n \leq n_0 \in \mathbb{Z}$. For each n, let $C^*(M, F^n)$ be Godement's canonical resolution. By the functoriality of Godement's resolution, there are commuting maps

$$d^{\mathrm{hor}} \colon C^m(M, F^n) \to C^m(M, F^{n+1}), \quad d^{\mathrm{ver}} \colon C^m(M, F^n) \to C^{m+1}(M, F^n).$$

The total complex of $C^*(M, F^\bullet)$ is the complex $\mathrm{Tot}^\bullet(C(M, F))$ with

$$\mathrm{Tot}^n(C(M, F)) = \bigoplus_{p+q=n} C^q(M, F^p)$$

and differential d given, for $x \in C^q(M, F^p)$ by

$$dx = d^{\mathrm{hor}} x + (-1)^p d^{\mathrm{ver}} x.$$

There is sheaf quasi-isomorphism $F^\bullet \to \mathrm{Tot}^\bullet(C(M, F))$ that makes $\mathrm{Tot}^\bullet(C(M, F))$ an acyclic resolution of $F^\bullet$. Thus

$$\mathbb{H}^n(X, F^\bullet) = H^n(\Gamma(X, \mathrm{Tot}^\bullet(C(M, F)))).$$

Spectral sequences will also be needed at some point. The reader can find an introduction to spectral sequences, for instance, in the book [**BT82**]. In particular, the hypercohomology of a complex of sheaves comes always equipped with a spectral sequence

$$E_1^{p,q} = H^q(X, F^p) \Longrightarrow \mathbb{H}^{p+q}(X, F^\bullet). \tag{2.16}$$

EXAMPLE 2.17. As we have seen in Example 2.13, singular cohomology can be writen as sheaf cohomology. In the same spirit, relative cohomology as in 2.1.1 can be writen as the hypercohomology of a complex of sheaves. Let X be a topological space and $\iota \colon Y \to X$ a closed immersion. The sheaf $\iota_* \underline{\mathbb{Z}}_Y$ is defined as

$$\iota_* \underline{\mathbb{Z}}_Y(U) = \underline{\mathbb{Z}}_Y(U \cap Y).$$

There is a morphism of sheaves $\underline{\mathbb{Z}}_X \to \iota_* \underline{\mathbb{Z}}_Y$. If X and Y are both locally contractible, then

$$H^*(Y, \mathbb{Z}) = H^*(X, \iota_* \underline{\mathbb{Z}}_Y), \qquad H^*(X, Y, \mathbb{Z}) = \mathbb{H}^*(X, \underline{\mathbb{Z}}_X \to \iota_* \underline{\mathbb{Z}}_Y).$$

⋆ ⋆ ⋆

EXERCISE 2.18. Prove that the boundary (2.1) satisfies $\partial_{n-1} \circ \partial_n = 0$.

EXERCISE 2.19. Prove that there is a natural short exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}(M,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^n(M,\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_n(M,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Whenever $H_{n-1}(M,\mathbb{Z})$ is torsion-free, the Ext group vanishes and we get an isomorphism between $H^n(M,\mathbb{Z})$ and the linear dual of $H_n(M,\mathbb{Z})$.

EXERCISE 2.20 (An alternative definition of relative homology). We keep the notations from paragraph 2.1.1. Given a topological space M and a subspace N, show that the boundary maps ∂_n yield a complex

$$\cdots \longrightarrow \frac{C_n(M)}{C_n(N)} \longrightarrow \frac{C_{n-1}(M)}{C_{n-1}(N)} \longrightarrow \cdots \tag{2.21}$$

which is quasi-isomorphic to $C_\bullet(M,N)$. Therefore, one can also define the relative homology of the pair (M,N) as the homology of (2.21).

EXERCISE 2.22. Combine the Mayer-Vietoris exact sequence with Example 2.3 to compute the homology of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

2.2. Algebraic de Rham cohomology.

2.2. Algebraic de Rham cohomology. Inspired by ideas of Atiyah and Hodge, Grothendieck introduced the de Rham cohomology of algebraic varieties over fields of characteristic zero in the paper [**Gro66**], written shortly after Hironaka's proof of resolution of singularities. In this section, we explain the definition and give some elementary examples.

2.2.1. *Motivation: de Rham's theorem.* Before going into Grothendieck's construction, we shall give a quick review of the more familiar objects in differential geometry. The reader is encouraged to consult [**BT82**] for a very nice exposition of the subject.

Let M be a differentiable manifold of dimension n. Recall that a differential p-form can be written in local coordinates as

$$\omega = \sum_{1\le i_1<i_2<\cdots i_p\le n} f_{i_1,\dots,i_p}(x_1,\dots,x_n)dx_{i_1}\wedge\cdots\wedge dx_{i_p}, \tag{2.23}$$

where $f_{i_1,\dots,i_p}(x_1,\dots,x_n)$ are $\mathcal{C}^\infty$-functions. Let $E^p(M)$ denote the real vector space of differential p-forms and

$$E(M) = \bigoplus_{p=0}^{n} E^p(M).$$

The *exterior derivative* $d\colon E(M)\to E(M)$ is the unique $\mathbb{R}$-linear map which sends p-forms to $(p+1)$-forms and satisfies the axioms:

(a) If f is a smooth function, df is the differential of f.
(b) $d^2 = 0$.
(c) If α is a p-form, then $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta$.

We thus get a complex

$$0 \to E^0(M) \xrightarrow{d} E^1(M) \xrightarrow{d} \cdots \xrightarrow{d} E^n(M) \to 0,$$

whose cohomology $H^j_{\mathrm{dR}}(M)$ is called *the de Rham cohomology* of M.

A classical theorem of de Rham asserts that the singular cohomology $H^*(M, \mathbb{R})$ can be computed using differential forms.

THEOREM 2.24 (de Rham). *Let $0 \leq j \leq n$. The map*

$$H^j_{\mathrm{dR}}(M, \mathbb{R}) \longrightarrow H^j(M, \mathbb{R})$$

which sends the class of a differential form ω to the integration functional $\int \omega \colon H_j(M, \mathbb{R}) \to \mathbb{R}$ is an isomorphism.

Remarkably enough, when M is the underlying topological space of a complex algebraic variety, it suffices to consider differential forms with polynomial coefficients. In this way, one obtains a purely algebraic definition of cohomology, as we now explain.

2.2.2. *Kähler differentials.* We first recall the notion of Kähler differentials, the algebraic substitute for the differential forms (2.23). Let k be a field of characteristic zero and A a finitely generated reduced[5] k-algebra, so that $X = \mathrm{Spec}(A)$ is an affine algebraic variety over k.

DEFINITION 2.25. A k-linear *derivation* on A is an A-module M, together with a k-linear morphism $D \colon A \to M$ satisfying the Leibniz rule

$$D(ab) = aD(b) + bD(a) \tag{2.26}$$

for all $a, b \in A$. Note that (2.26) implies that $Dr = 0$ for all $r \in k$, that is, elements of k are "constants".

DEFINITION 2.27. The module of *Kähler differentials* $\Omega^1_{A/k}$ is the quotient of the free A-module generated by symbols da, for $a \in A$, by the submodule spanned by the following elements for all $r \in k$ and all $a, b \in A$:

$$dr, \qquad d(a+b) - da - db, \qquad d(ab) - adb - bda.$$

By construction, the map $d \colon A \to \Omega^1_{A/k}$ sending a to da is a k-linear derivation. It is actually the *universal* one, in the sense that, given any k-linear derivation $D \colon A \to M$, there exists a unique morphism of A-modules $\varphi \colon \Omega^1_{A/k} \to M$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & \Omega^1_{A/k} \arrow[d, "\varphi"] \\ & M. \end{tikzcd} \tag{2.28}$$

[5] Recall that reduced means that there are no non-zero nilpotent elements in A.

EXAMPLE 2.29. Let $A = k[x_1, \dots, x_n]$. Then $\Omega^1_{A/k}$ is the free A-module generated by $dx_1, \dots, dx_n$. Indeed, let $D\colon A \to M$ be any k-linear derivation. It follows from the Leibniz rule (2.26) that

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i),$$

where $\partial f / \partial x_i$ stands for the partial derivative in the usual sense. Thus, D is determined by the images of the x_i. More generally, if

$$A = k[x_1, \dots, x_n]/(f_1, \dots, f_m),$$

then the module of Kähler differentials $\Omega^1_{A/k}$ has generators $dx_1, \dots, dx_n$ and relations $df_j = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i$ for $j = 1, \dots, m$.

2.2.3. *Algebraic de Rham cohomology of affine varieties.*

PROPOSITION 2.30. *If $X = \operatorname{Spec}(A)$ is smooth of dimension n, then the module of Kähler differentials $\Omega^1_{A/k}$ is locally free of rank n.*

We refer to Exercise 2.57 for an example of why the smoothness condition is necessary. For each integer $p \geq 0$, let

$$\Omega^p_{A/k} = \Lambda^p \Omega^1_{A/k}$$

be the p-th exterior power. In particular, $\Omega^0_{A/k} = A$ and $\Omega^p_{A/k} = 0$ for $p > n$. The derivation d extends canonically to a complex

$$A \longrightarrow \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k} \longrightarrow \cdots \longrightarrow \Omega^n_{A/k}.$$

DEFINITION 2.31. The *algebraic de Rham cohomology* of $X = \operatorname{Spec}(A)$ is the cohomology of this complex

$$H^*_{\mathrm{dR}}(X) = H^*(\Omega^\bullet_{A/k}).$$

EXAMPLE 2.32. Consider the affine variety $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, the algebraic analogue of Example 2.3. The de Rham complex reads

$$\begin{aligned} k[t, t^{-1}] &\xrightarrow{d} k[t, t^{-1}]dt \\ t^m &\longmapsto mt^{m-1}dt. \end{aligned}$$

The cohomology is thus given by the kernel and cokernel of d. Since all $t^m dt$ are in the image of d except for $m = -1$, one finds:

$$H^i_{\mathrm{dR}}(\mathbb{G}_m) = \begin{cases} k & i = 0, \\ k\frac{dt}{t} & i = 1, \\ 0 & \text{else.} \end{cases}$$

EXAMPLE 2.33. Let $a, b \in k$ be such that $4a^3 + 27b^2 \neq 0$. Then the polynomial $f(x) = x^3 + ax + b$ has no double roots, hence the equation $y^2 = f(x)$ defines an affine elliptic curve $X \subseteq \mathbb{A}_k^2$. In the language of algebraic geometry, $X = \operatorname{Spec}(A)$, where A is the ring

$$A = k[x, y]/(y^2 - x^3 - ax - b).$$

Again, $H^0_{\mathrm{dR}}(X) = k$ and $H^i_{\mathrm{dR}}(X)$ vanishes for $i \geq 2$. The only interesting cohomology is $H^1_{\mathrm{dR}}(X) = \operatorname{Coker}(d\colon A \to \Omega^1_A)$.

Since f and f' are coprime, there exist polynomials $P, Q \in k[x]$ such that $Pf + Qf' = 1$. We consider the differential

$$\omega = Pydx + 2Qdy \in \Omega^1_A.$$

Using that, in Ω^1_A, the identity $2ydy = f'(x)dx$ holds, one finds

$$dx = y\omega, \quad dy = \frac{1}{2} f'(x)\omega. \tag{2.34}$$

Thus, any element of Ω^1_A can be uniquely written as $(R + Sy)\omega$ for polynomials $R, S \in k[x]$. By (2.34), all differentials of the form $Sy\omega$ are exact, so we only need to decide when $R\omega$ is exact. For this, compute

$$d(Ty) = T'ydx + Tdy = (T'f + \frac{1}{2}Tf')\omega$$

for $T \in k[x]$. Choosing T with leading term $\frac{2}{3+2m}x^m$ for $m \geq 0$, one gets $d(Ty) = (x^{m+2} + \cdots)\omega$, from which it follows that the image of the differential d consists of elements $(R + Sy)\omega$ with $R, S \in k[x]$, R of degree at least two and S arbitrary. We deduce that

$$H^1_{\mathrm{dR}}(X) = \langle \omega, x\omega \rangle_k.$$

Let us now turn to the situation where X is any smooth variety over k, not necessarily affine. Glueing the differential forms on affine open subsets, we get a sheaf on X.

PROPOSITION 2.35. *There exists a unique coherent sheaf $\Omega^1_{X/k}$ on X whose restriction to every affine open subset U of X is the $\mathcal{O}_X$-module associated to $\Omega^1_{\mathcal{O}_X(U)/k}$.*

Recall that the Kähler differentials Ω^1_X form a locally free sheaf of rank n, equipped with the universal k-derivation $d\colon \mathcal{O}_X \to \Omega^1_X$. Let Ω^i_X denote the i-th exterior power of Ω^1_X. Then d extends to maps $d^i\colon \Omega^i_X \to \Omega^{i+1}_X$ satisfying $d^{i+1} \circ d^i = 0$. We denote by $(\Omega^\bullet_X, d)$ the resulting *de Rham complex* of locally free sheaves:

$$\Omega^\bullet_X\colon \quad \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \tag{2.36}$$

2.2.4. *Algebraic de Rham cohomology.*

DEFINITION 2.37. Let X be a smooth projective variety over a field k of characteristic zero. The *algebraic de Rham cohomology* of X is the hypercohomology of the de Rham complex:

$$H^*_{\mathrm{dR}}(X) = \mathbb{H}^*(X, \Omega^\bullet_X).$$

REMARK 2.38. When X is affine, there is no need to use hypercohomology. In this case, we let $\Omega^p(X)$ denote the space of global p-differentials on X. Then $(\Omega^\bullet(X), d)$ is a complex of k-vector spaces called the *global de Rham complex*. The algebraic de Rham cohomology agrees with the cohomology of this complex:

$$H^*_{\mathrm{dR}}(X) = H^*(\mathcal{O}(X) \to \Omega^1(X) \to \Omega^2(X) \to \cdots). \tag{2.39}$$

In general, when X is not affine, the cohomology of the global de Rham complex does not coincide with the algebraic de Rham cohomology. For example, $\Omega^p(X)$ vanishes for $p > n$, hence so does the right-hand side of (2.39), while a variety will in general have non-trivial cohomology $H^*_{\mathrm{dR}}(X)$ up to degree $2n$. Most of the varieties of these notes will be affine, so we will often be able to use the global de Rham complex.

2.2.5. *Relative de Rham cohomology.* There is also a relative version of algebraic de Rham cohomology. For simplicity, we explain the construction only in the affine case. Let X be a smooth affine variety over k, and consider a smooth closed subscheme $\iota : Z \hookrightarrow X$, which is hence automatically affine. There is a restriction morphism of complexes $\iota^* \colon \Omega^\bullet(X) \to \Omega^\bullet(Z)$. Note that, in contrast to the situation for relative singular homology, the map ι^* is far from being injective. Let $\Omega^\bullet(X, Z)$ denote the complex

$$\Omega^n(X, Z) = \Omega^n(X) \oplus \Omega^{n-1}(Z),$$

together with the differential

$$d(\alpha, \beta) = (d\alpha, \iota^*(\alpha) - d\beta).$$

REMARK 2.40. It is instructive to compare this complex to the one used to define relative homology in Section 2.1.1. Mimicking the construction of the cone for cochain complexes, we obtain:

$$\mathrm{cone}(\iota^*)^n = \Omega^{n+1}(X) \oplus \Omega^n(Z),$$

with differential

$$d(\alpha, \beta) = (-d\alpha, -\iota^*(\alpha) + d\beta).$$

Therefore, recalling that the shift $[-1]$ changes the sign of the differential, we see that $\Omega^\bullet(X, Z)$ coincides with the complex $\mathrm{cone}(\iota^*)[-1]$. This last complex is also called the *simple* of ι^*. The use of the simple or of the cone of a morphism of complexes depends on whether we want that the degree in the obtained complex agrees with the degree in the source complex or in the target complex.

Definition 2.41. For X a smooth affine variety and $Z \hookrightarrow X$ a smooth closed subscheme, the *relative de Rham cohomology* of the pair (X, Z) is the cohomology of the complex $\Omega^\bullet(X, Z)$:

$$H^*_{\mathrm{dR}}(X, Z) = H^*(\Omega^\bullet(X, Z)).$$

Again, a relative de Rham class is represented by a pair of differential forms (α, β) such that α is closed and the restriction of α to Z is equal to $d\beta$. However, in general, α is not determined by β.

Example 2.42. Consider $X = \operatorname{Spec} k[t, t^{-1}]$ and let $Z = \{p, q\}$ be the closed subscheme of X defined by two rational points. Then $\Omega^\bullet(Z)$ is concentrated in degree zero, $\Omega^0(Z) = k \oplus k$ and the map

$$\iota^* \colon \Omega^0(X) = k[t, t^{-1}] \to \Omega^0(Z) = k \oplus k$$

is given by evaluating functions at p and q, that is, $\iota^*(f) = (f(p), f(q))$. Therefore, the complex $\Omega^\bullet(X, Z)$ reads

$$\begin{aligned} d \colon k[t, t^{-1}] &\longrightarrow k[t, t^{-1}]dt \oplus k \oplus k \\ f &\longmapsto (f'(t)dt, f(p), f(q)). \end{aligned} \tag{2.43}$$

The differential d is injective and has image

$$\operatorname{Im}(d) = \langle (0, 1, 1), (nt^{n-1}dt, p^n, q^n) \mid n \in \mathbb{Z} \setminus \{0\} \rangle_k,$$

from which it follows easily that $H^1_{\mathrm{dR}}(X, Z)$ is the k-vector space generated by the relative differential forms

$$\omega_1 = (0, 1, 0), \quad \omega_2 = (dt/t, 0, 0).$$

Remark 2.44. The de Rham cohomology of affine smooth varieties vanishes above the dimension. If $n = \dim X$, and $Z \subsetneq X$ is a closed smooth subscheme of smaller dimension, then a useful part of the long exact sequence of relative cohomology is

$$\cdots \to H^{n-1}_{\mathrm{dR}}(Z) \to H^n_{\mathrm{dR}}(X, Z) \to H^n_{\mathrm{dR}}(X) \to 0. \tag{2.45}$$

2.2.6. *The case of normal crossings divisors.* In the sequel, we will also need to use relative de Rham cohomology in the case where Z is not smooth, but a simple normal crossings divisor. Using some homological algebra, the above definition extends to this setting.

Definition 2.46. A divisor D on a smooth algebraic variety X has *simple normal crossings* if all the irreducible components are smooth and, for each $p \in X$, there exists a local equation of D of the form $x_1 \cdots x_r$ for independent local parameters $x_i \in \mathcal{O}_{X,p}$ and $r \leq \dim X$.

It follows from the definition that the intersection of m distinct irreducible components of a simple normal crossings divisor D is a smooth subvariety of codimension m in X.

Construction 2.47. Let X be a smooth irreducible affine variety over k and D a simple normal crossings divisor, with irreducible components $D_0, \ldots, D_r$. For simplicity, we assume that all the D_i are defined over k as well. For each subset $I \subseteq \{0, \ldots, r\}$, we set

$$D_I = \bigcap_{i \in I} D_i.$$

We define $D^0 = X$ and, for $p = 1, \ldots, r+1$,

$$D^p = \coprod_{|I|=p} D_I.$$

Then there is a double complex of k-vector spaces

$$K^{p,q} = \Omega^q(D^p),$$

where the vertical differentials d^{ver} are $(-1)^p d$, and the horizontal differentials d^{hor} are linear combinations, with ± 1 coefficients, of restriction maps. More precisely, $d^{\mathrm{hor}} \colon K^{p,q} \to K^{p+1,q}$ is given by

$$\bigoplus_{\substack{|I|=p \\ |J|=p+1 \\ I \subset J}} \varepsilon(I,J) d_{IJ},$$

where $d_{IJ} \colon \Omega^q(D_I) \to \Omega^q(D_J)$ denotes the restriction map and the sign $\varepsilon(I,J)$ is defined as follows: if $J = \{j_0, \ldots, j_p\}$ with $j_0 < \ldots < j_p$, and $I = \{j_0, \ldots, \widehat{j_\ell}, \ldots, j_p\}$, then $\varepsilon(I,J) = (-1)^\ell$.

Note that, thanks to the factor $(-1)^p$ in the definition of d^{ver}, the vertical and the horizontal differentials anticommute. Let $\Omega^\bullet(X,D)$ denote the total complex associated to $K^{p,q}$, that is

$$\left(\Omega^\bullet(X,D) = \bigoplus_{p+q=\bullet} K^{p,q}, \ \partial = d^{\mathrm{hor}} + d^{\mathrm{ver}} \right).$$

Definition 2.48. The *relative de Rham cohomology* $H^*_{\mathrm{dR}}(X,D)$ is the cohomology of the complex $\Omega^\bullet(X,D)$.

As for any total complex associated to a double complex, the cohomology can be computed by means of the spectral sequence

$$E_1^{p,q} = H^q(\Omega^\bullet(D^p)) \Longrightarrow H^{p+q}_{\mathrm{dR}}(X,D). \tag{2.49}$$

Let $n = \dim X$. By definition, a class in the top degree cohomology $H^n(X,D)$ is represented by a tuple

$$(\omega_0, \ldots, \omega_n) \in \bigoplus_{p=0}^{n} \Omega^{n-p}(D^p).$$

What is more, one can always choose $\omega_p = 0$ for $p = 1, \ldots, n$, so that all classes in $H^n(X,D)$ are indeed represented by some $\omega \in \Omega^n(X)$. The key point is that the restriction maps $\Omega^{n-p-1}(D^p) \to \Omega^{n-p-1}(D^{p+1})$ are all surjective [**HMS17**, Lemma 3.3.20]. We will see in the example below how to use this to find a representative; the general case is analogous.

EXAMPLE 2.50. Let $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ and let $D \subset X$ be a triangle. After an affine transformation, we may assume without loss of generality that D is the union of the lines

$$D_0 = \{x = 0\}, \qquad D_1 = \{y = 0\}, \qquad D_2 = \{x + y = 1\}.$$

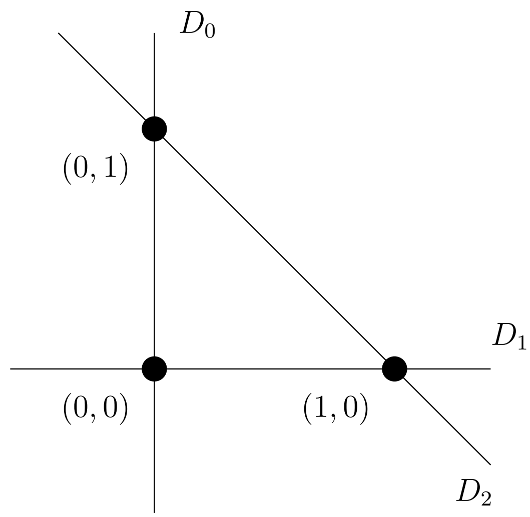


FIGURE 3. The triangle D

In this case, the double complex is equal to

$$(\Omega^\bullet(\mathbb{A}^2), d) \longrightarrow \bigoplus_{i=0}^{2} (\Omega^\bullet(D_i), -d) \longrightarrow \bigoplus_{0 \le i < j \le 2} (\Omega^\bullet(D_i \cap D_j), d) \longrightarrow 0.$$

To make all the above terms and maps explicit, we write $D_0 = \operatorname{Spec} k[y]$, $D_1 = \operatorname{Spec} k[x]$ and we parametrize $D_2 = \operatorname{Spec} k[x, y]/(x + y - 1)$ by the coordinate $z = x$. Then one gets:

$$\begin{tikzcd}
k[x,y]dx \wedge dy & & \\
k[x,y]dx \oplus k[x,y]dy \arrow[u, "d"] \arrow[r, "c"] & k[x]dx \oplus k[y]dy \oplus k[z]dz & \\
k[x,y] \arrow[u, "d"] \arrow[r, "a"] & k[x] \oplus k[y] \oplus k[z] \arrow[u, "-d"] \arrow[r, "b"] & k \oplus k \oplus k,
\end{tikzcd}$$

where the maps a and b are given by

$$\begin{aligned} a\colon f(x,y) &\longmapsto (f(0,x), f(y,0), f(z,1-z)), \\ b\colon (f(x), g(y), h(z)) &\longmapsto (g(0)-f(0), h(0)-f(1), h(1)-g(1)), \end{aligned}$$

and c is induced from a in the obvious way.

Therefore, the spectral sequence (2.49) reads

$$\begin{tikzcd} 0 & & \\ 0 & 0 & \\ k \arrow[r] & k \oplus k \oplus k \arrow[r] & k \oplus k \oplus k \end{tikzcd}$$

where the first map sends a to (a,a,a) and the second one is given by $(a,b,c) \mapsto (b-a, c-a, c-b)$. Since the only non-vanishing cohomology of the bottom complex is in degree two, generated by $(1,0,0)$, the second page of the spectral sequence is reduced to $E_2^{2,0} = k$. It follows that $H^i_{\mathrm{dR}}(\mathbb{A}^2, D)$ vanishes for $i \neq 2$ and is one-dimensional for $i = 2$.

To produce a differential $\omega \in \Omega^2(\mathbb{A}^2)$ representing the cohomology class, we follow the "zig-zag" method, which consists of

- finding $\omega_1 \in k[x] \oplus k[y] \oplus k[z]$ such that $b(\omega_1) = (1,0,0)$,
- applying d^{ver} to get $\omega_2 = -d\omega_1$ one row upper,
- choosing $\omega_3 \in k[x,y]dx \oplus k[x,y]dy$ such that $c(\omega_3) = -\omega_2$.

Then, setting $\omega = -d\omega_3$, one has

$$\partial(\omega_1 + \omega_3) = b(\omega_1) - d\omega_1 + c(\omega_3) + d\omega_3 = (1,0,0) - \omega,$$

so ω and $(1,0,0)$ are cohomologous.

$$\begin{tikzcd} \omega & & \\ \omega_3 \arrow[u] & \omega_2 \arrow[l] & \\ & \omega_1 \arrow[u] & (1,0,0) \arrow[l] \end{tikzcd}$$

FIGURE 4. The zig-zag method

It is straightforward to check that one can take

$$\omega_1 = (y-1, 0, 0), \quad \omega_2 = (-dy, 0, 0), \quad \omega_3 = (1-x)dy - ydx.$$

This yields the differential form $\omega = 2dx \wedge dy$, which defines a relative cohomology class since it has top degree. In conclusion:

$$H^i_{\mathrm{dR}}(X, D) = \begin{cases} \langle dx \wedge dy \rangle_k & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

⋆ ⋆ ⋆

Exercise 2.51. Prove that the axioms (a)-(c) of the definition of the exterior derivative imply that, in local coordinates,

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Exercise 2.52. Let k be a field of characteristic zero. Show that $H^0_{\mathrm{dR}}(\mathbb{A}^n_k) = k$ and that all the other cohomology groups vanish.

Exercise 2.53. In Example 2.33 we saw that a basis of the de Rham cohomology of an affine elliptic curve $X \subseteq \mathbb{A}^2_k$ is given by the classes of the differentials ω and $x\omega$. Let $\overline{X} \subseteq \mathbb{P}^2_k$ be the projective completion of X, that is, the smooth projective curve obtained by adjoining to X the point at infinity $O = [0 : 1 : 0]$. Prove that ω extends to a holomorphic differential on $\overline{X}$, whereas $x\omega$ has a double pole at O.

Exercise 2.54. We have defined de Rham cohomology for varieties over a field of characteristic zero. Show by means of an example that the same definition gives pathological results in positive characteristic (for instance, the cohomology of $\mathbb{A}^1$ has infinite dimension).

Exercise 2.55. Show that the differential ω_1 from Example 2.42 is cohomologous to $(\frac{1}{q-p}dt, 0, 0)$. Deduce that $(dt/t, 0, 0)$ and $(dt, 0, 0)$ form another basis of the relative cohomology group $H^1_{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0, \infty\}, \{p, q\})$ and compare it to the previous one.

Exercise 2.56. Let A be a k-algebra and $A \otimes_k A \to A$ the multiplication map which sends an element $\sum_i a_i \otimes b_i$ to $\sum a_i b_i$. Set

$$I = \mathrm{Ker}(A \otimes_k A \to A).$$

The goal of the exercise is to prove that $\Omega^1_{A/k} \simeq I/I^2$ as A-modules:

(a) Show that the map $a \mapsto 1 \otimes a - a \otimes 1$ induces a k-linear derivation $A \to I/I^2$, hence, by the universal property (2.28), a morphism of A-modules $\varphi \colon \Omega^1_{A/k} \to I/I^2$.

(b) Consider the ring $R = A \oplus \Omega^1_{A/k}$, where A acts on $\Omega^1_{A/k}$ through the A-module structure and the product of two elements of $\Omega^1_{A/k}$ is always zero, together with the map

Prove that the module of Kähler differentials is equal to $\Omega^1_{A/k} = I/I^2$ and the universal k-derivation is given by $a \mapsto 1 \otimes a - a \otimes 1$.

Exercise 2.57 (Kähler differentials are not locally free for singular varieties). Set $A = k[x,y]/(xy)$ and $X = \operatorname{Spec}(A)$. By Example 2.29, the module of Kähler differentials $\Omega^1_{X/k}$ has generators dx and dy, and one relation $xdy = -ydx$. Let $\omega = xdy$.

(a) Show that $k[\omega]$ is a *torsion* submodule of $\Omega^1_{X/k}$ which sits into an exact sequence $0 \to k[\omega] \to \Omega^1_{X/k} \to k[x]dx \oplus k[y]dy$.

(b) Prove that $\Omega^2_X = \langle dx \wedge dy \rangle_k$.

2.3. Periods. In this paragraph, we introduce a class of complex numbers called periods. They will form a countable subring of $\mathbb{C}$ halfway between algebraic and transcendental numbers: although they tend to be transcendental, they share with algebraic numbers the property that they contain, in some sense, "a finite amount of information". Moreover, this information has geometric nature. From the modern point of view, periods appear when comparing de Rham and Betti cohomology of algebraic varieties over number fields. We refer to [**HMS17**] for a detailed exposition of the subject.

2.3.1. *Naive periods.* The following elementary definition was first written down by Kontsevich and Zagier [**KZ01**]:

Definition 2.58. A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals

$$\int_\sigma f(x_1, \dots, x_n) dx_1 \cdots dx_n, \tag{2.59}$$

where f is a rational function with rational coefficients and $\sigma \subseteq \mathbb{R}^n$ is a subset defined by finite unions and intersections of domains of the form $\{g(x_1, \dots, x_n) \geq 0\}$ with g a rational function with rational coefficients.

One may replace "rational function" by "algebraic function" and "rational coefficients" by "algebraic coefficients" in the above definition, and still obtain the same class of numbers. Standard examples of naive periods include the following:

- All algebraic numbers (see Exercise 2.74).
- The number $\pi = \int_{x^2+y^2\leq 1} dxdy$.
- Logarithms of rational numbers $\log(q) = \int_1^q \frac{dx}{x}$.
- Elliptic integrals $\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$.

2.3.2. *The comparison isomorphism.* Let k be a subfield of $\mathbb{C}$ and X a smooth algebraic variety over k. As we have seen, the singular cohomology of $X(\mathbb{C})$ is a graded $\mathbb{Q}$-vector space and the de Rham cohomology of X is a graded k-vector space. Both are related by Grothendieck's comparison isomorphism.

THEOREM 2.60 (Grothendieck, [Gro66]). *Let X be a smooth variety over a subfield k of $\mathbb{C}$. Then there is a canonical isomorphism*

$$\mathrm{comp}_{\mathrm{B,dR}} \colon H^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^i_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.61}$$

When X is an affine variety, all classes in de Rham cohomology are represented by differential forms. Then the comparison isomorphism is induced by the pairing

$$\begin{array}{ccc} H^i_{\mathrm{dR}}(X) \otimes H_i(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & \mathbb{C} \\ \omega \otimes \sigma & \longmapsto & \int_\sigma \omega. \end{array} \tag{2.62}$$

The fact that (2.62) depends only on the classes of ω and σ, and is thus well defined, follows from Stokes' theorem.

REMARK 2.63. Later on, we will also need the inverse of the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$, which will be written as

$$\mathrm{comp}_{\mathrm{dR,B}} \colon H^i_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C}.$$

IDEA OF THE PROOF. The strategy to prove the comparison isomorphism is to relate Betti cohomology to an analytic version of de Rham cohomology. Indeed, by "analytification", the algebraic de Rham complex (2.62) becomes the analytic de Rham complex

$$\Omega^\bullet_{X^{\mathrm{an}}} \colon \quad \mathcal{O}^{\mathrm{an}}_X \xrightarrow{d} \Omega^1_{X^{\mathrm{an}}} \xrightarrow{d} \Omega^2_{X^{\mathrm{an}}} \xrightarrow{d} \cdots$$

of the analytic complex manifold $X^{\mathrm{an}}_{\mathbb{C}}$ associated to the base change $X \times_k \mathbb{C}$. The hypercohomology of $\Omega^\bullet_{X^{\mathrm{an}}}$ defines the analytic de Rham cohomology groups $H^i_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}})$ and, again by analytification, we get a canonical morphism of complex vector spaces:

$$H^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \longrightarrow H^i_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}). \tag{2.64}$$

Besides, according to the Poincaré lemma, the complex $\Omega^\bullet_{X^{\mathrm{an}}}$ is a resolution of the constant sheaf $\mathbb{C}_{X^{\mathrm{an}}}$. Since singular cohomology is isomorphic to sheaf cohomology with values in the constant sheaf, we obtain a canonical isomorphism

$$H^i_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}).$$

The proof is thus reduced to show that (2.64) is an isomorphism. If we assume X to be proper, this is a straightforward consequence of Serre's GAGA theorem, together with the existence of spectral sequences relating algebraic (resp. analytic) de Rham cohomology to the sheaf cohomology $H^q(X, \Omega^p_X)$ (resp. $H^q(X^{\mathrm{an}}_{\mathbb{C}}, \Omega^p_{X^{\mathrm{an}}})$). The proof of the general case is more difficult. □

REMARK 2.65. The theorem does not hold if the smoothness assumption is removed. For instance, if X is the affine plane curve defined by the equation $x^5 + y^5 + x^2y^2 = 0$, one can show that $\dim H^1_{\mathrm{dR}}(X) > \dim H^1_B(X)$ [AK11, Example 4.4]. However, the theorem remains true for singular X with the "correct" definition of de Rham cohomology [HMS17].

There is also a relative version of the comparison isomorphism:

THEOREM 2.66. *Let k be a subfield of $\mathbb{C}$, X a smooth variety and $Z \subseteq X$ either a smooth closed subvariety or a normal crossings divisor, with everything defined over k. Then there is a canonical isomorphism*

$$H^i_{\mathrm{dR}}(X,Z) \otimes_k \mathbb{C} \xrightarrow{\sim} H^i_B(X,Z) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.67}$$

REMARK 2.68. Recall that if X is affine and $\iota\colon Z \hookrightarrow X$ is a smooth closed subvariety, relative cohomology classes are represented by pairs (ω_X, ω_Z) and (σ_X, σ_Z) satisfying

$$\partial \sigma_X = -\iota_* \sigma_Z, \quad \iota^* \omega_X = d\omega_Z, \quad d\omega_X = 0.$$

Then the period pairing is given by:

$$\begin{array}{ccc} H^i_{\mathrm{dR}}(X,Z) \otimes H_i^B(X,Z) & \longrightarrow & \mathbb{C} \\ (\omega_X, \omega_Z) \otimes (\sigma_X, \sigma_Z) & \longmapsto & \int_{\sigma_X} \omega_X + \int_{\sigma_Z} \omega_Z. \end{array}$$

2.3.3. *Cohomological periods.* The comparison isomorphism does not respect the rational structures, as it is already clear from the following basic example. In particular, in the case where $k = \mathbb{Q}$, the vector spaces $H^i_{\mathrm{dR}}(X)$ and $H^i_B(X)$ are isomorphic (they have the same dimension), but there is no canonical isomorphism between them!

EXAMPLE 2.69. Let $X = \mathbb{G}_m = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$, so the complex points are $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. We know from examples 2.3 and 2.32 that

$$H^1_{\mathrm{dR}}(X) = \mathbb{Q}\tfrac{dt}{t}, \qquad H_1(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}\sigma,$$

where σ is the counterclockwise oriented unit circle. Then the comparison isomorphism is given by multiplication by:

$$\int_\sigma \frac{dt}{t} = 2\pi i.$$

The fact that the comparison isomorphism does not respect the rational structures gives rise to the *periods*.

DEFINITION 2.70. Let $k \subset \mathbb{C}$ be a number field. Let X be a smooth variety and $Z \subseteq X$ a normal crossings divisor, both defined over k. We call a *period* of the pair (X, Z) any coefficient of a matrix of the isomorphism (2.67) with respect to rational bases of both sides.

It is shown in [**HMS17**, 11.2] that the naive and the cohomological definitions of periods yield the same subring of $\mathbb{C}$. However, starting from an integral representation as in (2.59) it is in general not easy to find the pair (X, Z), as we will see when discussing the case of $\zeta(2)$.

2.3.4. *Examples.*

EXAMPLE 2.71. All algebraic numbers are periods. Indeed, let k be a number field and consider the zero-dimensional variety $X = \mathrm{Spec}(k)$, which we regard as defined over $\mathbb{Q}$. Then $H^0_{\mathrm{dR}}(X)$ is canonically identified with the $\mathbb{Q}$-vector space k. By its very definition, $X(\mathbb{C})$ is the set of complex embeddings of k, hence $H^0_B(X) = \mathbb{Q}^{\mathrm{Hom}(k,\mathbb{C})}$. If we choose a basis $a_1, \ldots, a_n$ of k over $\mathbb{Q}$ and $\sigma_1, \ldots, \sigma_n$ denote the complex embeddings of k, the period matrix is $(\sigma_i(a_j))_{i,j}$.

EXAMPLE 2.72. Let $X = \mathbb{G}_{m,\mathbb{Q}}$ and $Z = \{1, q\}$ for $q \in \mathbb{Q} \setminus \{0, 1\}$. In Example 2.10 we obtained generators σ_1 and σ_2 of $H_1^{\mathrm{B}}(X, Z)$ and in Example 2.42 generators ω_1 and ω_2 of $H^1_{\mathrm{dR}}(X, Z)$. With respect to these bases the period matrix is

$$\begin{pmatrix} \int_{\sigma_2} \omega_1 & \int_{\sigma_2} \omega_2 \\ \int_{\sigma_1} \omega_1 & \int_{\sigma_1} \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & \log(q) \\ 0 & 2\pi i \end{pmatrix},$$

which shows that logarithms of rational numbers are periods.

2.3.5. *Compatibility with complex conjugation.* We finish this section by stating a result which will be used in Chapter 4. Let $c\colon \mathbb{C} \to \mathbb{C}$ denote complex conjugation and assume that $k \subseteq \mathbb{R}$. Then c induces a continuous map $X(\mathbb{C}) \to X(\mathbb{C})$, hence an involution at the level of Betti cohomology $\rho\colon H^i_{\mathrm{B}}(X) \to H^i_{\mathrm{B}}(X)$. The functoriality of (2.61) implies:

PROPOSITION 2.73. *Assume that $k \subseteq \mathbb{R}$. Then the comparison isomorphism (2.61) is equivariant for the action of $\mathbb{Z}/2$ by $\mathrm{id} \otimes c$ on the left-hand side and by $\rho \otimes c$ on the right.*

We illustrate the proposition in the case of $\mathbb{G}_m$ (see Exercise 2.75 below for another instance). We know from Example 2.69 that the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ sends dt/t to $\sigma^\vee \otimes (2\pi i)$. The differential form being rational, it is invariant under complex conjugation, so $\sigma^\vee \otimes (2\pi i)$ should also be invariant. For this, observe that the image of σ by complex conjugation is the clockwise oriented unit circle, whose cohomology class is $-\sigma$. Thus,

$$(\rho \otimes c)(\sigma \otimes (2\pi i)) = -\sigma \otimes (-2\pi i) = \sigma \otimes (2\pi i).$$

⋆ ⋆ ⋆

EXERCISE 2.74. In this exercise, we show that all algebraic numbers are naive periods in the sense of Definition 2.58. For example, the integral representation

$$\sqrt{2} = \int_{\substack{x^2 \leq 2 \\ x \geq 0}} dx$$

shows that $\sqrt{2}$ is a naive period.

(a) Let $P \in \mathbb{Q}[x]$ be an irreducible polynomial and let $\alpha_1, \dots, \alpha_r$ be its real roots. Generalize the above example to show that all α_i are naive periods.
(b) Using that the real and the imaginary part of a complex algebraic number are real algebraic numbers, deduce that all algebraic numbers are naive periods.

EXERCISE 2.75. Let $C \subset \mathbb{A}^2_{\mathbb{Q}}$ be the affine conic given by $x^2 + y^2 = 1$.

(a) Show that the de Rham cohomology group $H^1_{\mathrm{dR}}(C)$ is generated by the class of the differential form $xdy - ydx$ and that the singular homology $H_1(C(\mathbb{C}), \mathbb{Q})$ is generated by the chain
$$\sigma\colon [0,1] \longrightarrow C(\mathbb{R}), \quad t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$
(b) Prove that the associated period is equal to
$$\int_\sigma xdy - ydx = 2\pi$$
and check Proposition 2.73 in this case.
(c) Find generators of the singular homology of the conics C defined by the equations $x^2 + y^2 = -1$ and $x^2 - y^2 = 1$ and check Proposition 2.73 in these cases as well.

2.4. Multiple zeta values as periods. The previous examples show that algebraic numbers, logarithms of rational numbers, and the ubiquitous $2\pi i$ are all periods. From the integral representation (1.109), it follows immediately that multiple zeta values are periods in the sense of Kontsevich and Zagier (Definition 2.58). However, it is not so easy to exhibit the corresponding algebraic varieties. The main goal of this section is to work out in detail the example of $\zeta(2)$ in order to give an idea of the difficulties involved.

2.4.1. *The example of* $\zeta(2)$. Recall from Example 1.103 that $\zeta(2)$ admits the integral representation
$$\zeta(2) = \int_{1 \geq t_1 \geq t_2 \geq 0} \frac{dt_1}{t_1} \wedge \frac{dt_2}{1 - t_2}. \tag{2.76}$$

The integrand is the differential form on the affine plane
$$\omega = \frac{dt_1}{t_1} \wedge \frac{dt_2}{1 - t_2},$$
which is singular along the union of the lines
$$\ell_0 = \{t_1 = 0\} \quad \text{and} \quad \ell_1 = \{t_2 = 1\}.$$
Thus, ω is a global differential 2-form on $Y = \mathbb{A}^2 \setminus (\ell_0 \cup \ell_1)$.

The domain of integration is the simplex
$$\sigma = \{(t_1, t_2) \mid 1 \geq t_1 \geq t_2 \geq 0\} \subset \mathbb{A}^2.$$

However, if we want to consider the integral (2.76) as a period of Y, relative to some divisor containing the boundary of σ, we immediately face the technical problem that σ is not contained in Y, as the points $p = (0,0)$ and $q = (1,1)$ belong to $\sigma \cap (\ell_0 \cup \ell_1)$ (see Figure 5).

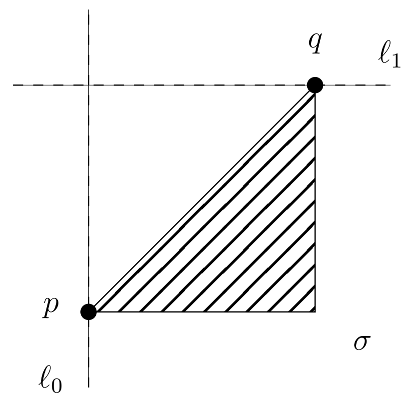


FIGURE 5. The simplex σ and the singular locus $\ell_0 \cup \ell_1$

A way to remedy this is to perform a geometric construction called *blow-up*, which replaces a point on a variety by a divisor called the *exceptional divisor*. It is a very useful technique in the study of singularities. In our case, we have to blow up the two problematic points p and q. More precisely, the blow-up of $\mathbb{A}^2$ along p and q is the closed subvariety $X \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equations

$$t_1\alpha_1 = t_2\beta_1,$$
$$(t_1 - 1)\alpha_2 = (t_2 - 1)\beta_2,$$

where $[\alpha_i : \beta_i]$ are homogeneous coordinates on the two copies of $\mathbb{P}^1$. The projection onto the first factor induces a proper surjective map

$$\pi \colon X \to \mathbb{A}^2.$$

It is easy to verify that $\pi^{-1}(p)$ is the projective line

$$E_p = (0,0) \times \mathbb{P}^1 \times [1:1] \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

while $\pi^{-1}(q)$ is the projective line

$$E_q = (1,1) \times [1:1] \times \mathbb{P}^1 \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Moreover, the restriction

$$\pi\,|_{X \setminus (E_p \cup E_q)} \colon X \setminus (E_p \cup E_q) \longrightarrow \mathbb{A}^2 \setminus \{p, q\}$$

is an isomorphism. For any closed subset $C \subset \mathbb{A}^2$, the *strict transform* $\widehat{C}$ of C is the closed subset of X given by

$$\widehat{C} = \overline{\pi^{-1}(C \setminus \{p, q\})}.$$

In words: we first remove the points p and q if they are in C, then we pull-back by π, and finally we take the Zariski closure. The strict transform is contained in the *total transform* $\pi^{-1}(C)$ but it may be smaller. For instance, the strict transform of ℓ_0 is the affine line

$$L_0 = \widehat{\ell}_0 = \{((0,t_2),[1:0],[1-t_2:1]) \mid t_2 \in \mathbb{A}^1\},$$

while the total transform is $L_0 \cup E_p$. Note that L_0 and E_p have only one common point:

$$L_0 \cap E_p = \{((0,0),[1:0],[1:1])\}. \tag{2.77}$$

Similarly, the strict transform of ℓ_1 is the affine line

$$L_1 = \widehat{\ell}_1 = \{((t_1,1),[1:t_1],[0:1]) \mid t_1 \in \mathbb{A}^1\},$$

which is disjoint from the exceptional divisor E_p, intersects L_0 at the point $((0,1),[1:0],[0:1])$, and E_q at $((1,1),[1:1],[0:1])$.

In principle, the pull-back $\pi^*(\omega)$ of ω might have singularities along the total transform of $\ell_0 \cup \ell_1$, which would only worsen the initial situation. Fortunately, it is only singular on the strict transform $L_0 \cup L_1$. This can be seen using local coordinates in X. For instance, a local patch of X around the intersection of L_0 and E_p is given by the coordinates

$$t = \frac{\beta_1}{\alpha_1} = \frac{t_1}{t_2}, \quad s = t_2,$$

in which E_p and L_0 have local equations $s=0$ and $t=0$, respectively. Then

$$\pi^*(\omega) = \frac{d(st)}{st} \wedge \frac{ds}{1-s} = \frac{ds}{s} \wedge \frac{ds}{1-s} + \frac{dt}{t} \wedge \frac{ds}{1-s} = \frac{dt}{t} \wedge \frac{ds}{1-s},$$

where we have used the Leibniz rule and the fact that $ds \wedge ds = 0$. It follows that $\pi^*(\omega)$ is smooth along E_p. An analogous computation shows that $\pi^*(\omega)$ has singularities along L_1 but not along E_q.

The closed points of the exceptional divisor E_p can be interpreted as lines passing through the point p. This allows us to find the points of E_p that are contained in $\widehat{\sigma}$:

$$\widehat{\sigma} \cap E_p = \{((0,0),[m:1],[1:1]) \mid 0 \le m \le 1\}.$$

Combined with (2.77), this implies that $\widehat{\sigma} \cap L_0 = \emptyset$. A similar argument shows that $\widehat{\sigma} \cap L_1 = \emptyset$, so, after passing to the blow-up X, the singular locus of $\pi^*(\omega)$ and the domain of integration $\widehat{\sigma}$ are disjoint (Figure 6).

Write $L = L_0 \cup L_1$. The complement $X \setminus L$ is still an affine variety; in fact, it is the closed subvariety of $\mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{A}^1$ defined by

$$\begin{aligned} t_1 t &= t_2, \\ (t_1 - 1) &= (t_2 - 1)s, \end{aligned}$$

where t, s are the coordinates of the first and the second affine lines. By the previous discussion, $\pi^*(\omega)$ is an element of $\Omega^2(X \setminus L)$.

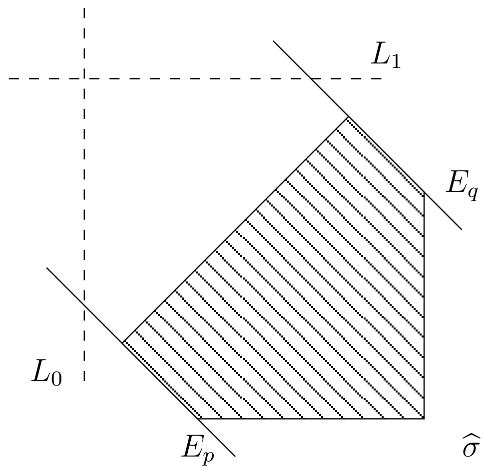


FIGURE 6. The strict transform of σ and the singular locus $L_0 \cup L_1$

The next issue one needs to deal with is that σ is not a closed chain. Its boundary is contained in the union of the affine lines

$$m_2 = \{t_1 = t_2\},\ m_3 = \{t_2 = 0\},\ m_4 = \{t_1 = 1\},$$

so we are naturally led to consider the normal crossings divisor

$$M = \pi^{-1}(m_2 \cup m_3 \cup m_4) = E_p \cup E_q \cup M_2 \cup M_3 \cup M_3 \subset X,$$

where M_i denotes the strict transform of m_i. One easily checks that the intersection $L \cap M$ is reduced to the points $L_0 \cap E_p$ and $L_1 \cap E_q$ which we have already computed.

Since $\widehat{\sigma}$ is contained in $X \backslash L$ and its boundary lies in M, using Remark 2.9 we see that $\widehat{\sigma}$ determines a relative homology class

$$\widehat{\sigma} \in H_2(X \setminus L, M \setminus (L \cap M)).$$

Besides, the restriction of $\pi^*(\omega)$ to every irreducible component of M is zero for dimension reasons, so it defines a relative cohomology class

$$\pi^*(\omega) \in H^2_{\mathrm{dR}}(X \setminus L, M \setminus (L \cap M)).$$

Pairing these classes through the comparison isomorphism (2.67) yields, as we wanted, the period

$$\int_{\widehat{\sigma}} \pi^*(\omega) = \int_{\pi_*(\widehat{\sigma})} \omega = \int_{\sigma} \omega = \zeta(2).$$

2.4.2. *Multiple zeta values as periods of the moduli spaces $\overline{M}_{0,n}$.* For each integer $n \geq 3$, let $M_{0,n}$ be the moduli space of n ordered distinct points in $\mathbb{P}^1$ up to projective equivalence. In other words, two tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are identified if there exists an element $g \in \mathrm{PGL}_2$ such that $g(x_i) = y_i$ for all i. Since there exists a unique automorphism of $\mathbb{P}^1$ sending any given three points to $0, 1$ and ∞, we can fix an identification

$$(x_1, \dots, x_n) = (0, 1, \infty, t_1, \dots, t_{n-3})$$

to get rid of the quotient. This induces an isomorphism

$$M_{0,n} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \{(t_1, \dots, t_{n-3}) \mid t_i = t_j \text{ for some } i \neq j\},$$

which shows that $M_{0,n}$ is a smooth variety of dimension $n-3$. In particular, $M_{0,3}$ is reduced to a point and $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Deligne, Mumford and Knudsen [**Knu83**] constructed a smooth compactification $\overline{M}_{0,n}$ of $M_{0,n}$ by a normal crossings divisor. The irreducible components of the boundary are in one-to-one correspondence with the partitions of the marked points into subsets of cardinality at least 2. We refer the reader to [**KV07**] for a nice introduction to these spaces and their compactifications.

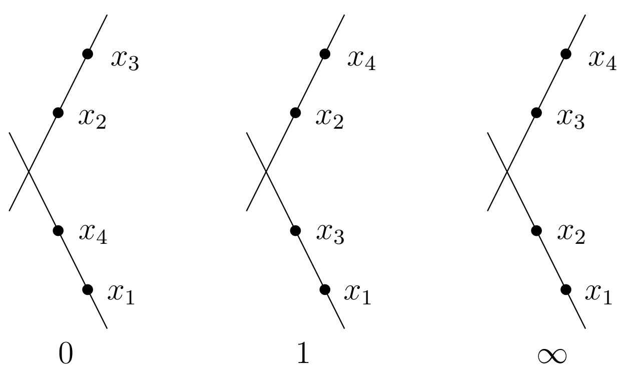


FIGURE 7. Boundary of the moduli space $M_{0,4}$

REMARK 2.78. The blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0,0)$, $(1,1)$ and (∞, ∞) is isomorphic to the Deligne-Mumford compactification $\overline{M}_{0,5}$ of the moduli space of genus zero curves with 5 marked points. The boundary $\overline{M}_{0,5} \setminus M_{0,5}$ consists of 10 smooth divisors intersecting transversally. The previous constructions shows that $\zeta(2)$ is a period of

$$H^2(\overline{M}_{0,5} \setminus A, B \setminus (A \cap B)),$$

where A is the union of 5 irreducible components of the boundary and B consists of the remaining ones.

Although the technical difficulties to transform the integral representation of any multiple zeta value into a period are the same we have encountered for $\zeta(2)$, one needs a more systematic method to deal with all of them. This was accomplished by Goncharov and Manin:

THEOREM 2.79 (Goncharov-Manin [**GM04**]). *Given an admissible multi-index $\boldsymbol{s}$ of weight n, there exists two normal crossings divisors $A_{\boldsymbol{s}}$ and B, supported on the boundary of $M_{0,n+3}$ and with no common irreducible components, such that $\zeta(\boldsymbol{s})$ is a period of*

$$H^n(\overline{M}_{0,n+3} \setminus A_{\boldsymbol{s}}, B \setminus (A_{\boldsymbol{s}} \cap B)). \tag{2.80}$$

REMARK 2.81. A converse to this theorem, due to Brown [**Bro09**], affirms that, for any choice of boundary divisors A and B, all periods of the cohomology groups $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$ are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values. This can now be seen as a consequence of Brown's theorem characterizing the periods of mixed Tate motives over $\mathbb{Z}$.

In these notes, we will rather follow Deligne and Goncharov [**DG05**] to show that multiple zeta values are periods associated to the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. A reason to prefer this approach is that it becomes easier to study the question whether relations between multiple zeta values come from geometry. A third way to see multiple zeta values as periods was proposed by Terasoma in [**Ter02**].

⋆ ⋆ ⋆

EXERCISE 2.82. Show that the boundary of the Deligne-Mumford compactification of $M_{0,n}$ has $2^{n-1} - n - 1$ irreducible components.

2.5. Mixed Hodge structures. Thanks to the comparison isomorphism (2.61), the Betti cohomology of algebraic varieties has richer properties than the singular cohomology of a random topological space. As we will explain in this section, it is endowed with a *mixed Hodge structure*, which can be thought of as a first approximation to the notion of motive. Usually, the study of a period in the sense of Definition 2.70 begins by understanding the mixed Hodge structure on the cohomology of the pair of varieties from which it comes. This theory was developed by Deligne in the 70s, taking as source of inspiration on the one hand Hodge's theorem for compact Kähler manifolds and, on the other hand, ℓ-adic cohomology of varieties over finite fields. For a more systematic treatment, we refer the reader to Deligne's original papers [**Del71, Del74**] or the monographs [**Voi02**] and [**PS08**].

2.5.1. *Pure Hodge structures.* Let M be a compact Kähler manifold of dimension d (for instance a projective complex manifold). For each pair of integers (p, q), let $H^{p,q}(M) \subseteq H^{p+q}(M, \mathbb{C})$ be the subspace of cohomology classes that can be represented by a $\mathcal{C}^\infty$-closed differential $(p+q)$-form of type (p, q), *i.e.* that can be locally written as

$$\sum_{I,J} f_{I,J}(z_1, \dots, z_d) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where the sum runs over subsets $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$ of $\{1, \dots, d\}$, and $f_{I,J}$ are $\mathcal{C}^\infty$-functions.

The starting point of Hodge theory is the fundamental theorem by Hodge.

Theorem 2.83 (Hodge). *There is a direct sum decomposition*

$$H^n(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}(M). \tag{2.84}$$

Complex conjugation acts on the right-hand side of (2.84) through the action on the coefficients of the left-hand side, that is,

$$\overline{\sigma \otimes w} = \sigma \otimes \overline{w} \quad \sigma \in H^n(M, \mathbb{Q}),\ w \in \mathbb{C}.$$

This action sends $H^{p,q}(M)$ to $H^{q,p}(M)$, a property commonly referred to as *Hodge symmetry.*

Remark 2.85. Abstractly, what appears in Hodge's theorem is a finite-dimensional $\mathbb{Q}$-vector space H, together with a bigrading

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

satisfying $\overline{H^{p,q}} = H^{q,p}$. This is called a *pure Hodge structure of weight* n, and the set of pairs (p, q) for which $H^{p,q} \neq 0$ is called the *Hodge type.* As you will prove in Exercise 2.113, these data are equivalent to a decreasing filtration $F^\bullet$ on $H_{\mathbb{C}}$ (the *Hodge filtration*) such that, for all integers p,

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{n-p+1} H_{\mathbb{C}}}. \tag{2.86}$$

This is the definition that one usually finds in textbooks about Hodge structures. However, for the purpose of studying periods it is important to remember that the filtration $F^\bullet$ in Hodge's theorem comes from de Rham cohomology. If M arises as the complex points of a variety X defined over a subfield $k \subseteq \mathbb{C}$, then

$$H^n(M, \mathbb{C}) = H^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C}$$

and the Hodge filtration is already defined on the k-vector space $H^n_{\mathrm{dR}}(X, k)$.

The following definition keeps track of all these elements:

Definition 2.87. Let k be a subfield of $\mathbb{C}$ and n an integer. A *pure Hodge structure* over k is the datum

$$H = (H_B, (H_{\mathrm{dR}}, F^\bullet), \mathrm{comp}_{\mathrm{B,dR}})$$

of a finite-dimensional $\mathbb{Q}$-vector space H_B, a finite-dimensional k-vector space H_{dR}, together with a decreasing filtration $F^\bullet$, and an isomorphism of complex vector spaces

$$\mathrm{comp}_{\mathrm{B,dR}} \colon H_{\mathrm{dR}} \otimes_k \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C},$$

such that the induced filtration on $H_{\mathbb{C}} = H_B \otimes_{\mathbb{Q}} \mathbb{C}$, still denoted by $F^\bullet$, satisfies that there exists an integer n such that, for all p,

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{n-p+1} H_{\mathbb{C}}}. \tag{2.88}$$

We call n the *weight* of H. For simplicity, we will often say that H_B carries a pure Hodge structure.

Definition 2.89. A *morphism of pure Hodge structures over* k

$$f \colon H \to H'$$

is a pair $f = (f_B, f_{\mathrm{dR}})$ consisting of a $\mathbb{Q}$-linear map $f_B \colon H_B \to H'_B$ and a k-linear map $f_{\mathrm{dR}} \colon H_{\mathrm{dR}} \to H'_{\mathrm{dR}}$ such that $f_{\mathrm{dR}}(F^\bullet H_{\mathrm{dR}}) \subseteq F^\bullet H'_{\mathrm{dR}}$ and the following diagram commutes:

$$\begin{tikzcd}
H_{\mathrm{dR}} \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}_{\mathrm{B,dR}}"] \arrow[d, "f_{\mathrm{dR}} \otimes_k \mathrm{id}_{\mathbb{C}}"'] & H_B \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, "f_B \otimes_{\mathbb{Q}} \mathrm{id}_{\mathbb{C}}"] \\
H'_{\mathrm{dR}} \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}'_{\mathrm{B,dR}}"] & H'_B \otimes_{\mathbb{Q}} \mathbb{C}.
\end{tikzcd}$$

It follows from this definition that if H and H' have different weights, then every morphism of Hodge structures between them is zero (Exercise 2.114).

Example 2.90 (Hodge-Tate structures). Let k be a subfield of $\mathbb{C}$. For each integer $n \in \mathbb{Z}$, we define

$$\mathbb{Q}(n) = (\mathbb{Q}, (k, F^\bullet), \mathrm{comp}_{\mathrm{B,dR}}),$$

where the filtration reads $k = F^{-n}k \supseteq F^{-n+1}k = \{0\}$, and the isomorphism $\mathrm{comp}_{\mathrm{B,dR}} \colon \mathbb{C} \to \mathbb{C}$ is given by multiplication by $(2\pi i)^{-n}$. Then $\mathbb{Q}(n)$ is a one-dimensional pure Hodge structure of weight $-2n$ over k. The Hodge structure $\mathbb{Q}(1)$ is known as the *Tate* Hodge structure. We will call all the $\mathbb{Q}(n)$ *Hodge-Tate structures*. Observe that we have already encountered the Hodge-Tate structure $\mathbb{Q}(-1)$. By Example 2.69, it is isomorphic to the triple

$$H^1(\mathbb{G}_m) = (H^1_{\mathrm{B}}(\mathbb{G}_m), (H^1_{\mathrm{dR}}(\mathbb{G}_m), F^\bullet), \mathrm{comp}_{\mathrm{B,dR}}),$$

where $F^\bullet$ is the trivial filtration concentrated in degree 1, and $\mathrm{comp}_{\mathrm{B,dR}}$ is Grothendieck's comparison isomorphism from Theorem 2.60.

Once we have introduced these notions, we can state the following variant of Hodge's theorem:

THEOREM 2.91. *Let k be a subfield of $\mathbb{C}$ and X a smooth projective variety over k. Then the Betti cohomology $H^n_{\mathrm{B}}(X)$ carries a pure Hodge structure of weight n over k, functorial for morphisms of algebraic varieties.*

More precisely, we consider the triple

$$H^n(X) = (H^n_{\mathrm{B}}(X), (H^n_{\mathrm{dR}}(X), F^\bullet), \mathrm{comp}_{\mathrm{B,dR}}).$$

As in the previous example, $\mathrm{comp}_{\mathrm{B,dR}}$ is the comparison isomorphism of Theorem 2.60. The Hodge filtration $F^\bullet$ is given by

$$F^p H^n_{\mathrm{dR}}(X) = \mathrm{Im}(\mathbb{H}^n(X, \Omega_X^{\geq p}) \longrightarrow \mathbb{H}^n(X, \Omega_X^\bullet)),$$

where $\Omega_X^{\geq p}$ stands for the *bête* truncation of the de Rham complex, namely

$$\Omega_X^{\geq p}\colon \quad 0 \to \cdots 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots .$$

That the Hodge structure on $H^n(X)$ is *functorial* means that, for any morphism $f\colon X \to Y$ of smooth projective varieties, the induced map on cohomology $f^* : H^n(Y) \to H^n(X)$ is a morphism of Hodge structures.

As we have already mentioned, by Exercise 2.114, there are no non-zero morphisms between pure Hodge structures of different weight. However, such maps naturally occur in geometry. For example, if $Z \hookrightarrow X$ is a smooth closed subvariety of codimension c, then there is a *Gysin morphism*

$$H^n(Z) \to H^{n+2c}(X).$$

In order to turn the Gysin morphism into a morphism of Hodge structures, we introduce *Tate twists*: given a pure Hodge structure H of weight n and another integer m, we denote by $H(m)$ the pure Hodge structure of weight $n - 2m$ with the same underlying $\mathbb{Z}$-module and k-vector space, filtration shifted by m and comparison isomorphism multiplied by $(2\pi i)^{-m}$. In fact (see Exercise 2.115) there is a tensor product of Hodge structures and $H(m) = H \otimes \mathbb{Q}(m)$. With this notation, the Gysin map becomes a morphism of Hodge structures $H^n(Z)(-c) \to H^{n+2c}(X)$.

EXAMPLE 2.92. As Hodge structure, the cohomology of the projective space $\mathbb{P}^n$ is given by

$$H^j(\mathbb{P}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q}(-j/2) & 0 \leq j \leq 2n \text{ even}, \\ 0 & \text{else}. \end{cases}$$

2.5.2. *Mixed Hodge structures.* Before discussing mixed Hodge structures, we recall some terminology concerning filtrations and morphisms.

DEFINITION 2.93. Let k be a field and let (V, F) and (V', F) be filtered k-vector spaces. A morphism $f\colon V \to V'$ is called *filtered* if $f(F^pV) \subset F^pV'$ and *strict* (with respect to F) if, in addition,

$$f(F^pV) = F^pV' \cap \mathrm{Im}(f).$$

Hodge's Theorem says that the cohomology in degree n of a smooth projective complex variety carries a pure Hodge structure of weight n. This theorem is no longer true when X fails to be smooth or projective. For instance, we saw in Example 2.3 that $H^1(\mathbb{G}_m)$ is one-dimensional, so it cannot carry a pure Hodge structure of weight one. Nevertheless, Deligne proved that the cohomology of any quasi-projective complex variety is an "iterated extension" of pure Hodge structures.

Theorem 2.94 (Deligne). *Let X be a quasi-projective variety over the field of complex numbers.*

(a) There exists an increasing filtration

$$W_{-1} = 0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{2n} = H^n(X),$$

and a decreasing filtration

$$F^0 = H^n(X, \mathbb{C}) \supseteq F^1 \supseteq \cdots \supseteq F^n \supseteq F^{n+1} = 0$$

such that $F^\bullet$ induces a pure Hodge structure of weight m on each graded piece

$$\mathrm{Gr}_m^W H^n(X) = W_m / W_{m-1}.$$

(b) Moreover, if $f\colon X \to Y$ is a morphism of quasi-projective varieties, the induced map on cohomology $f^*\colon H^n(Y) \to H^n(X)$ is a filtered morphism with respect to both filtrations, i.e.

$$f^*(W_m H^n(Y)) \subseteq W_m H^n(X),$$
$$f^*_{\mathbb{C}}(F^p H^n(Y)) \subseteq F^p H^n(X).$$

(c) If X is smooth, then $\mathrm{Gr}_m^W H^n(X) = 0$ for all $m < n$ and, if X is projective, $\mathrm{Gr}_m^W H^n(X) = 0$ for all $m > n$.

This motivates the following definition:

Definition 2.95. Let k be a subfield of $\mathbb{C}$. A *mixed Hodge structure over k* is a triple

$$H = ((H_{\mathrm{B}}, W_\bullet^{\mathrm{B}}), (H_{\mathrm{dR}}, F^\bullet, W_\bullet^{\mathrm{dR}}), \mathrm{comp}_{\mathrm{B,dR}})$$

consisting of:

- a finite-dimensional $\mathbb{Q}$-vector space H_{B}, together with an increasing filtration $W_\bullet^B$,
- a finite-dimensional k-vector space H_{dR}, together with an increasing filtration $W_\bullet^{\mathrm{dR}}$ and a decreasing filtration $F^\bullet$,
- an isomorphism of complex vector spaces

$$\mathrm{comp}_{\mathrm{B,dR}}\colon H_{\mathrm{dR}} \otimes_k \mathbb{C} \xrightarrow{\sim} H_B \otimes_{\mathbb{Q}} \mathbb{C}$$

  that is filtered with respect to the weight filtration. That is,

$$\mathrm{comp}_{\mathrm{B,dR}}(W_\bullet^{\mathrm{dR}} \otimes_k \mathbb{C}) = W_\bullet^B \otimes_{\mathbb{Q}} \mathbb{C}.$$

We require that these data verify the following: for each integer m,

$$\operatorname{Gr}_m^W H = (\operatorname{Gr}_m^W H_B, (\operatorname{Gr}_m^W H_{\mathrm{dR}}, F^\bullet), \mathrm{comp}_{\mathrm{B,dR}}) \tag{2.96}$$

is a pure Hodge structure over k of weight m.

DEFINITION 2.97. A morphism $f\colon H \to H'$ of mixed Hodge structures over k is a pair $f = (f_{\mathrm{B}}, f_{\mathrm{dR}})$ consisting of

- a morphism of $\mathbb{Q}$-vector spaces $f_{\mathrm{B}}\colon H_{\mathrm{B}} \to H'_{\mathrm{B}}$,
- a morphism of k-vector spaces $f_{\mathrm{dR}}\colon H_{\mathrm{dR}} \to H'_{\mathrm{dR}}$

such that f_{B} is filtered with respect to the weight filtration, while f_{dR} is filtered with respect to the weight and the Hodge filtrations, and both maps are compatible with the comparison isomorphisms. In other words

$$\begin{aligned} f_{\mathrm{B}}(W_\bullet^{\mathrm{B}} H_{\mathrm{B}}) &\subseteq W_\bullet^{\mathrm{B}} H'_{\mathrm{B}}, \\ f_{\mathrm{dR}}(F^\bullet H_{\mathrm{dR}}) &\subseteq F^\bullet H'_{\mathrm{dR}}, \\ f_{\mathrm{dR}}(W_\bullet^{\mathrm{dR}} H_{\mathrm{dR}}) &\subseteq W_\bullet^{\mathrm{dR}} H'_{\mathrm{dR}}, \\ f_{\mathrm{dR}} \circ \mathrm{comp}'_{\mathrm{B,dR}} &= \mathrm{comp}_{\mathrm{B,dR}} \circ (f_B \otimes \mathrm{id}_{\mathbb{C}}). \end{aligned}$$

We shall denote by $\mathbf{MHS}(k)$ the category of mixed Hodge structures over k. When $k = \mathbb{C}$, we shall simply write $\mathbf{MHS}$.

DEFINITION 2.98. A mixed Hodge structure over k is called *split* if there is an isomorphism of mixed Hodge structures

$$H \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} \operatorname{Gr}_m^W H,$$

therefore it is a direct sum of pure Hodge structures.

THEOREM 2.99 (Deligne). *The category* $\mathbf{MHS}(k)$ *is abelian.*

The proof of this theorem is sometimes called "a masterpiece of linear algebra". The main difficulty comes from the fact that the category of bifiltered vector spaces is not abelian. The key property that makes everything work is that any morphism of mixed Hodge structures is *strict* with respect to the weight and Hodge filtrations. More precisely we have the following lemma.

LEMMA 2.100. *Let* $f\colon H \to H'$ *be a morphism of mixed Hodge structures, then* f_{B} *is strict with respect to the weight filtration and* f_{dR} *is strict with respect to the weight and Hodge filtrations.*

DEFINITION 2.101. The category $\mathbf{MHS}(k)$ comes naturally with two forgetful functors

$$\begin{aligned} \omega_B\colon \mathbf{MHS}(k) &\to \mathbf{Vec}_{\mathbb{Q}}, \\ \omega_{\mathrm{dR}}\colon \mathbf{MHS}(k) &\to \mathbf{Vec}_k \end{aligned}$$

sending H to H_B and H_{dR} respectively. These functors are called the *Betti fibre functor* and the *de Rham fibre functor*.

2.5.3. *Mixed Hodge-Tate structures.*

DEFINITION 2.102. A mixed Hodge structure H over k is said to be of *Tate type* if $\mathrm{Gr}^W_{2m+1} H = 0$ and $\mathrm{Gr}^W_{2m} H$ is a sum of copies of the pure Hodge-Tate structure $\mathbb{Q}(-m)$ for all m. Mixed Hodge structures of Tate type are also called mixed Hodge-Tate structures.

We shall denote by $\mathbf{MHTS}(k)$ the full subcategory of $\mathbf{MHS}(k)$ consisting of mixed Hodge structures of Tate type over k.

REMARK 2.103. One can think of mixed Hodge structures as "iterated extensions" of the pure ones. Indeed, given two successive steps of the weight filtration, there is an exact sequence of vector spaces

$$0 \to W_{m-1}H \to W_mH \to \mathrm{Gr}^W_m H \to 0.$$

When m is the highest weight of H (*i.e.* $W_mH = H$), this exhibits H as an extension of the pure Hodge structure $\mathrm{Gr}^W_m H$ by $W_{m-1}H$, which in turn is an extension of $\mathrm{Gr}^W_{m-1} H$ by $W_{m-2}H$, and so on. Then mixed Hodge-Tate structures are those obtained as iterated extensions of the simplest ones, that is, sums of $\mathbb{Q}(n)$.

2.5.4. *Ind and pro-mixed Hodge structures.* Inductive and projective limits are important operations in category theory. In many categories such limits may not exist. This is the case of the category of mixed Hodge structures. To remedy this situation, given a category $\mathcal{C}$, one can define categories $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ of inductive and projective systems in $\mathcal{C}$, where inductive or projective limits in $\mathcal{C}$ can be constructed.

Since we do not want to enter in deep set theoretical questions, we will assume that all categories we consider ar essentially small.

DEFINITION 2.104. A *directed set* is a partially ordered set $(I, \leq)$ such that, given elements $i, j \in I$, there exists $n \in I$ such that $i \leq n$ and $j \leq n$. A subset $I' \subseteq I$ is called *cofinal* if, for every $i \in I$, there exists an element $n \in I'$ such that $i \leq n$.

DEFINITION 2.105. Let $\mathcal{C}$ be a category and I a directed set.

(1) An *inductive system* X in $\mathcal{C}$ indexed by I is a collection of objects $(X_i)_{i\in I}$ and morphisms $\varphi_{ij} \colon X_i \to X_j$ for all $i, j \in I$ with $i \leq j$ such that $\varphi_{ii} = \mathrm{Id}_{X_i}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for $i \leq j \leq k$.
(2) Let $X = ((X_i)_{i\in I}, (\varphi_{ij})_{i,j})$ and $Y = ((Y_i)_{i\in J}, (\phi_{ij})_{i,j})$ be inductive systems indexed by I and J respectively. A morphism f from X to Y is the data of an order preserving map $f_\sharp \colon I \to J$ and a collection

of morphisms $f_i\colon X_i \to Y_{\psi(i)}$ such that, for all $i \leq j$, the diagram

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\varphi_{ij}"'] & Y_{f_\sharp(i)} \arrow[d, "\phi_{f_\sharp(i)f_\sharp(j)}"] \\ X_j \arrow[r, "f_j"] & Y_{f_\sharp(j)}. \end{tikzcd}$$

is commutative. Two morphisms of inductive systems $f, g\colon X \to Y$ are equivalent, if for every $i \in I$ there exists an $n_i \in J$, with $n_i \geq f_\sharp(i)$ and $n_i \geq g_\sharp(i)$ such that

$$\phi_{f_\sharp(i),n_i} \circ f_i = \phi_{g_\sharp(i),n_i} \circ g_i.$$

(3) A *projective system* X in $\mathcal{C}$ indexed by I is a collection of objects $(X_i)_{i\in I}$ and morphisms $\varphi_{ij}\colon X_j \to X_i$ for all $i, j \in I$ with $i \leq j$ such that $\varphi_{ii} = \mathrm{Id}_{X_i}$ and $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ for $i \leq j \leq k$.

(4) If $X = ((X_i)_{i\in I}, (\varphi_{ij})_{i,j})$ is a projective system indexed by I and $Y = ((Y_i)_{i\in J}, (\phi_{ij})_{i,j})$ is a projective system indexed by J, then a morphism f between X and Y is an order preserving map $f_\sharp\colon I \to J$ and a collection of morphisms $f_i\colon X_i \to Y_{\psi(i)}$ such that, for all $i \leq j$, the diagram

$$\begin{tikzcd} X_j \arrow[r, "f_j"] \arrow[d, "\varphi_{ij}"'] & Y_{f_\sharp(j)} \arrow[d, "\phi_{f_\sharp(i)f_\sharp(j)}"] \\ X_i \arrow[r, "f_i"] & Y_{f_\sharp(i)}. \end{tikzcd}$$

is commutative. Two morphisms of projective systems $f, g\colon X \to Y$ are equivalent, if for every $i \in J$ there exists an $n_i \in I$, with $f_\sharp(n_i) \geq i$ and $g_\sharp(n_i) \geq i$ such that

$$\phi_{j,f_\sharp(n_i)} \circ f_{n_i} = \phi_{j,g_\sharp(n_i)} \circ g_{n_i}.$$

Definition 2.106. Let $\mathcal{C}$ be a category, I a directed set and $(X_i)_{i\in I}$ an inductive system. An *inductive limit* of this system in a universal solution to the problem: find an object X in $\mathcal{C}$ together with morphisms $\varphi_i\colon X_i \to X$ satisfying the commutativity relations $\varphi_i = \varphi_j \circ \varphi_{i,j}$ for all $i \leq j \in I$. If X is such universal solution it is written as

$$X = \varinjlim_I X_i.$$

Let now $(X_i)_{i\in I}$ be a projective system. A *projective limit* of this system in a universal solution to the problem: find an object X in $\mathcal{C}$ together with morphisms $\varphi_i\colon X \to X_i$ satisfying the commutativity relations $\varphi_i = \varphi_{i,j} \circ \varphi_j$ for all $i \leq j \in I$. If X is such universal solution it is written as

$$X = \varprojlim_I X_i.$$

Remark 2.107. Inductive limits are also called *direct limits* or *colimits*, while projective limits are also called *inverse limits* or just *limits*.

Remark 2.108. As you will prove in exercise 2.120, a morphism of inductive systems induce a morphism of the corresponding inductive limits (if they exist) and two equivalent morphisms of inductive systems induce the same morphism in the limit. Similar result is true for projective systems.

Definition 2.109. Let $\mathcal{C}$ be any category. The *Ind-category* of $\mathcal{C}$ is the smallest category that "contains" $\mathcal{C}$ and is closed under inductive limits. More precisely, is a category $\mathrm{Ind}(\mathbb{C})$ together with a functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ such that, for any category $\mathcal{A}$, closed under inductive limits, with a functor $\mathcal{C} \to \mathcal{A}$, there exists a unique functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{A}$ making the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{A} \\ & \mathrm{Ind}(\mathcal{C}) \arrow[u] \end{tikzcd}$$

commutative.

The *pro-category* of $\mathcal{C}$ is a category $\mathrm{Pro}(\mathbb{C})$ together with a functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ such that, for any category $\mathcal{A}$, closed under projective limits, with a functor $\mathcal{C} \to \mathcal{A}$, there exists a unique functor $\mathcal{A} \to \mathrm{Pro}(\mathcal{C})$ making the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{A} \arrow[d] \\ & \mathrm{Pro}(\mathcal{C}) \end{tikzcd}$$

commutative.

We refer the reader to [**KS06**] for a general construction of $\mathrm{Ind}(\mathcal{C})$ and of $\mathrm{Pro}(\mathcal{C})$. We give here an elementary ad-hoc construction that works in many cases like for the category of mixed Hodge structures, the category of finite-dimensional vector spaces or the category of finitely generated abelian groups. More generally, these constructions work for categories that are Noetherian and Artinian.

Definition 2.110. A category $\mathcal{C}$ is called *Noetherian* if every object satisfies the *ascending chain condition*. In other words, for every object C in $\mathcal{C}$, every ascending chain of subobjects of C

$$C_0 \subset C_1 \subset \cdots \subset C_i \subset C_{i+1} \subset \cdots \subset C$$

becomes stationary. That is, there is a i_0 such that

$$C_i = C_{i_0}, \quad \text{for all } i \geq i_0.$$

A category $\mathcal{C}$ is called *Artinian* if every object satisfies the *descending chain condition*. In other words, for every object C in $\mathcal{C}$, every descending chain of subobjects

$$C \supset C_0 \supset C_1 \supset \cdots \supset C_i \supset C_{i+1} \supset \ldots$$

becomes stationary. That is, there is a i_0 such that

$$C_i = C_{i_0}, \qquad \text{for all } i \geq i_0.$$

EXAMPLE 2.111. The category of finite dimensional vector spaces and the category of mixed Hodge structures are both Noetherian and Artinian.

We state without proof the next result.

PROPOSITION 2.112. *Let $\mathcal{C}$ be a category.*

(1) If $\mathcal{C}$ is Noetherian, then the category $\mathrm{Ind}(\mathcal{C})$ *is the category of whose objects are inductive systems in $\mathcal{C}$ and whose morphisms are equivalence classes of morphisms of inductive systems.*

(2) If $\mathcal{C}$ is Artinian, then the category $\mathrm{Pro}(\mathcal{C})$ *is the category of whose objects are projective systems in $\mathcal{C}$ and whose morphisms are equivalence classes of morphisms of projective systems.*

⋆⋆⋆

EXERCISE 2.113. Prove the claim of Remark 2.85.

[*Hint*: to get the direct sum decomposition starting from the filtration, define $H^{p,q} = F^p H_{\mathrm{dR}} \cap \overline{F^q H_{\mathrm{dR}}}$. Conversely, consider $F^p H_{\mathrm{dR}} = \bigoplus_{r \geq p} H^{r,n-r}$].

EXERCISE 2.114. Let H and H' be pure Hodge structures over k of weights n and m respectively.

(1) Show that the vector space $\mathrm{Hom}_{\mathbb{Q}}(H_{\mathrm{B}}, H'_{\mathrm{B}})$ admits a pure Hodge structure over k of weight $m-n$, denoted $\underline{\mathrm{Hom}}(H, H')$.

(2) Show that the group of morphisms of Hodge structures between H and H' agrees with the subspace $\underline{\mathrm{Hom}}(H, H')^{(0,0)}$.

(3) Conclude that, if $n \neq m$, then any morphism of Hodge structures between H and H' is zero.

EXERCISE 2.115. Let H and H' be mixed Hodge structures over k. Define a natural mixed Hodge structure on $H \otimes H'$. Show that for any pure Hodge structure H, we have

$$H(m) = H \otimes \mathbb{Q}(m).$$

EXERCISE 2.116. There are two possible ways of inducing $F^{\bullet}$ on $\mathrm{Gr}_m^W H$. Show that they are equivalent.

EXERCISE 2.117. Given a morphism $f : H \to H'$ of mixed Hodge structures, prove that the induced maps $f_m : \mathrm{Gr}_m^W H \to \mathrm{Gr}_m H'$ are morphism of pure Hodge structures.

EXERCISE 2.118. Let $H = (H_{\mathrm{dR}}, H_{\mathrm{B}}, \alpha)$ be a triple consisting of

- a finite-dimensional $\mathbb{Q}$-vector space H_{B}, equipped with an increasing filtration $W_{2\bullet}H_{\mathrm{B}}$ indexed by even integers,
- a finite-dimensional $\mathbb{Q}$-vector space H_{dR}, together with a grading indexed by even integers $H_{\mathrm{dR}} = \bigoplus_n (H_{\mathrm{dR}})_{2n}$,
- a comparison isomorphism $\alpha\colon H_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$,

subject to the condition that α maps $(H_{\mathrm{dR}})_{2n} \otimes_{\mathbb{Q}} \mathbb{C}$ to $W_{2n}H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$, and induces an isomorphism

$$\alpha_n\colon (H_{\mathrm{dR}})_{2n} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \big(W_{2n}H_{\mathrm{B}}/W_{2(n-1)}H_{\mathrm{B}}\big) \otimes_{\mathbb{Q}} \mathbb{C}$$

which sends $(H_{\mathrm{dR}})_{2n}$ to $\big(W_{2n}H_{\mathrm{B}}/W_{2(n-1)}H_{\mathrm{B}}\big) \otimes_{\mathbb{Q}} (2\pi i)^n\mathbb{Q}$.

Prove that the category $\mathbf{MHTS}(\mathbb{Q})$ is equivalent to the category whose objects are such triples and whose morphisms are the obvious ones.

Exercise 2.119. In this exercise, we see that the condition of being Noetherian is needed for the description of the morphisms of $\mathrm{Ind}(\mathcal{C})$ given by Proposition 2.112. Let $\mathcal{C}$ be a category and $(C_i)_{i\in\mathbb{N}}$ an inductive system. Assume that the limit

$$C = \varinjlim_{\mathbb{N}} C_i$$

exists in $\mathcal{C}$. Consider the constant inductive system (C). Prove that, if $\mathcal{C}$ is Noetherian, then there exists a morphism of inductive systems $(C) \to (C_i)_{i\in\mathbb{N}}$ that is the inverse in $\mathrm{Ind}(\mathcal{C})$ to the natural morphism $(C_i)_{i\in\mathbb{N}} \to (C)$.

Give an example of a non-Noetherian category where the morphism $(C) \to (C_i)_{i\in\mathbb{N}}$ does not exist.

Exercise 2.120. Let $\mathcal{C}$ be a category.

(1) Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be inductive systems and f a morphism of inductive systems. Assume that the inductive limits X and Y of the inductive systems exist. Then f induces a morphism, also detoted f between X and Y. Prove that, if g is an equivalent morphism, the the morphisms induced by f and g agree.
(2) Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be projective systems and f a morphism of projective systems. Assume that the projective limits, X and Y, of the projective systems exists. Then f induces a morphism, also detoted f between X and Y. Prove that, if g is an equivalent morphism, the the morphisms induced by f and g agree.

Exercise 2.121. Let $\mathbf{Vec}_k$ be the category of finite-dimensional vector spaces over k.

(1) Prove that, if V is an ind-vector space, then its dual $V^\vee$ is a pro-vector space.
(2) If $f\colon V \to W$ is a morphism of ind-vector spaces, show that it induces a morphism $f^\vee$ of pro-vector spaces.
(3) Show that, if f and g are equivalent morphisms of ind-vector spaces, then $f^\vee$ and $g^\vee$ are equivalent.

2.6. Examples of mixed Hodge structures. We now explain the ideas behind the construction of the weight and the Hodge filtration in two important cases: smooth but not necessarily projective varieties and normal crossings varieties. We refer the reader to [**Dur83**] for a user-friendly introduction to the subject.

2.6.1. *The smooth case.* Let X be a smooth quasi-projective variety over a subfield k of $\mathbb{C}$. By Theorem 2.60, there is a canonical isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} \colon H^n_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{2.122}$$

We want to endow $H^n_B(X)$ with a filtration $W^B_\bullet$ and $H^n_{\mathrm{dR}}(X)$ with two filtrations $W^{\mathrm{dR}}_\bullet$ and $F^\bullet$ making the triple

$$((H^n_B(X), W^B_\bullet), (H^n_{\mathrm{dR}}(X), F^\bullet, W^{\mathrm{dR}}_\bullet), \mathrm{comp}_{\mathrm{B,dR}})$$

into a mixed Hodge structure over k. However, if de Rham cohomology is computed using the complex $\Omega^\bullet_X$ as in Definition 2.37, we face two problems:

(a) One may define a Hodge filtration using the *bête* truncation $\Omega^{\bullet \geq p}_X$, but it will not give much information. For example, if X is affine, we saw in Remark 2.38 that $H^n_{\mathrm{dR}}(X)$ is the cohomology of the global de Rham complex, so in this case the definition would yield the trivial filtration $F^n H^n_{\mathrm{dR}}(X) = H^n_{\mathrm{dR}}(X)$.

(b) There is no obvious way to get the weight filtration from $\Omega^\bullet_X$.

To solve these difficulties, we shall use the complex of logarithmic differentials instead. Recall that, by resolution of singularities, there exists a smooth projective variety $\overline{X}$ over k and an open immersion $j : X \hookrightarrow \overline{X}$ such that $D = \overline{X} \setminus X$ is a simple normal crossings divisor (see Definition 2.46).

Definition 2.123 (Deligne). The *complex of sheaves of logarithmic differentials along* D is the smallest subcomplex $\Omega^\bullet_{\overline{X}}(\log D)$ of $j_*\Omega^\bullet_X$ stable under wedge product and containing $\Omega^\bullet_{\overline{X}}$ and the logarithmic derivatives df/f of all local sections f of $j_*\mathcal{O}^\times_X$ with poles along D.

It follows from the definition that $\Omega^1_{\overline{X}}(\log D)$ is a locally free $\mathcal{O}_{\overline{X}}$-module of rank $d = \dim X$. Indeed, if $(z_1, \dots, z_d)$ are local coordinates such that D is given by $z_1 \cdots z_r = 0$, then $\Omega^1_{\overline{X}}(\log D)$ is locally generated by

$$\tfrac{dz_1}{z_1}, \dots, \tfrac{dz_r}{z_r}, dz_{r+1}, \dots, dz_d.$$

Moreover, one has $\Omega^p_{\overline{X}}(\log D) = \Lambda^p \Omega^1_{\overline{X}}(\log D)$ for all $p \geq 0$.

Proposition 2.124. *The inclusion of complexes* $\Omega^\bullet_{\overline{X}}(\log D) \hookrightarrow j_*\Omega^\bullet_X$ *is a quasi-isomorphism, hence*

$$\mathbb{H}^n(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D)) \simeq \mathbb{H}^n(\overline{X}, j_*\Omega^\bullet_X) \simeq \mathbb{H}^n(X, \Omega^\bullet_X) \simeq H^n_{\mathrm{dR}}(X). \tag{2.125}$$

In view of the proposition, the strategy is to define the Hodge and the weight filtrations on the complex $\Omega^\bullet_{\overline{X}}(\log D)$ and look at their images in $H^n_{\mathrm{dR}}(X)$. Then one can transport the weight filtration to Betti cohomology using the comparison isomorphism. The resulting filtration is a priori only defined on $H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$, but one can prove that it comes from $H^n_B(X)$.

More precisely, the Hodge filtration is given by the *bête* filtration of the complex of logarithmic differentials, that is

$$F^p H^n_{\mathrm{dR}}(X) = \mathrm{Im}(\mathbb{H}^n(\overline{X}, \Omega^{\bullet \geq p}_{\overline{X}}(\log D)) \longrightarrow H^n_{\mathrm{dR}}(X)). \tag{2.126}$$

Note that $F^\bullet$ is defined over k. The advantage of working with logarithmic differentials is that now we can also filter by the order of poles:

$$W_m \Omega^p_{\overline{X}}(\log D) = \begin{cases} 0 & m < 0, \\ \Omega^{p-m}_{\overline{X}} \wedge \Omega^m_{\overline{X}}(\log D) & 0 \leq m \leq p, \\ \Omega^p_{\overline{X}}(\log D) & m \geq p. \end{cases}$$

Consider the filtration on cohomology

$$W^{\mathrm{dR}}_m H^n_{\mathrm{dR}}(X) = \mathrm{Im}(\mathbb{H}^n(X, W_{m-n}\Omega^\bullet_{\overline{X}}(\log D)) \to H^n_{\mathrm{dR}}(X)). \tag{2.127}$$

Through the comparison isomorphism (2.122), $W^{\mathrm{dR}}_\bullet$ induces a filtration on $H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$. It is a non-trivial fact, which can be proved using the Leray spectral sequence for the inclusion $X \hookrightarrow \overline{X}$, that this filtration is defined over $\mathbb{Q}$, in the sense that there exists a filtration $W^B_\bullet H^n_B(X)$ inducing $W^{\mathrm{dR}}_\bullet$ on $H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$. We refer the reader e.g. to [**PS08**, §4] for a proof that the filtrations we have introduced define a mixed Hodge structure on $H^n_B(X)$.

Definition 2.128. We say that a mixed Hodge structure H has weights in a subset $I \subseteq \mathbb{Z}$ if $\mathrm{Gr}^W_m H = 0$ whenever $m \notin I$.

It follows from (2.127) that the cohomology group $H^n_B(X)$ of a smooth variety X has weights in $[n, 2n]$. Moreover, noting that $W_0 \Omega^\bullet_{\overline{X}}(\log D) = \Omega^\bullet_{\overline{X}}$ and the shift of indices in (2.127), one finds that the first step in the weight filtration is the piece of the cohomology coming from the compactification:

$$W_n H^n_B(X) = \mathrm{Im}(H^n_B(\overline{X}) \to H^n_B(X)).$$

In contrast, when X is projective, the mixed Hodge structure $H^n(X)$ defined in [**Del74**] has weights in $[0, n]$. The combination of these two statements implies that the cohomology of a smooth projective variety carries a pure Hodge structure.

The definition of de Rham cohomology involves hypercohomology of sheaves, therefore, to compute it concretely, in general we can not use directly the algebraic de Rham complex but we need a resolution of it. As we have seen in Remark 2.38 for an affine variety X, every coherent sheaf is acyclic and we can represent de Rham cohomology with algebraic differentials directly. Nevertheless, the Hodge struture involves a hypercohomology computed on a projective compactification of X therefore, even in the case of affine varieties, in order to compute the Hodge structure we will need an acyclic resolution of the complex of logarithmic differentials, compatible with the weight and the Hodge filtrations.

For an arbitrary subfield $k \subseteq \mathbb{C}$ one has to remain in the algebraic situation, and we refer the reader to [**NA87**] for a general method to construct the needed resolutions. We will discuss here briefly the case of a smooth variety X over $\mathbb{C}$, were we can use smooth differential forms. We denote $X^{\mathrm{an}} = X(\mathbb{C})$ and $\overline{X}^{\mathrm{an}} = \overline{X}(\mathbb{C})$ the associated complex varieties. Similarly we will denote $\Omega^{\bullet}_{\overline{X}^{\mathrm{an}}}(\log D)$ for the sheaf of holomorphic logarithmic differential. It has the same definition as $\Omega^{\bullet}_{\overline{X}}(\log D)$ but with holomorphic forms instead of algebraic forms. It is a complex of sheaves with respect to the analytic topology of $\overline{X}^{\mathrm{an}}$, while $\Omega^{\bullet}_{\overline{X}}(\log D)$ is a complex of sheaves for the Zariski topology of $\overline{X}$. By the GAGA principle

$$\mathbb{H}^n(\overline{X}, \Omega^{\bullet}_{\overline{X}}(\log D)) = \mathbb{H}^n(\overline{X}^{\mathrm{an}}, \Omega^{\bullet}_{\overline{X}^{\mathrm{an}}}(\log D))$$

and the same is true for the different pieces of the weight and Hodge filtrations. Thus we can use holomorphic forms and the usual topology.

We now denote by $\mathscr{E}^{p,q}_{\overline{X}^{\mathrm{an}}}$ for the sheaf of smooth complex valued differential forms on $\overline{X}^{\mathrm{an}}$ of type (p,q), and we write

$$\mathscr{E}^{p,q}_{\overline{X}^{\mathrm{an}}}(\log D) = \mathscr{E}^{0,q}_{\overline{X}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{X}^{\mathrm{an}}}} \Omega^{p}_{\overline{X}^{\mathrm{an}}}(\log D).$$

The anti-holomorphic derivative $\bar{\partial}\colon \mathscr{E}^{0,q} \to \mathscr{E}^{0,q+1}$ induce anti-holomorphic derivatives

$$\bar{\partial}\colon \mathscr{E}^{p,q}_{\overline{X}^{\mathrm{an}}}(\log D) \longrightarrow \mathscr{E}^{p,q+1}_{\overline{X}^{\mathrm{an}}}(\log D).$$

The sequence of sheaves

$$0 \to \Omega^{p}_{\overline{X}^{\mathrm{an}}}(\log D) \to \mathscr{E}^{p,0}_{\overline{X}^{\mathrm{an}}}(\log D) \xrightarrow{\bar{\partial}} \mathscr{E}^{p,1}_{\overline{X}^{\mathrm{an}}}(\log D) \to \cdots$$

is exact. Thus, writing

$$\mathscr{E}^{*}_{\overline{X}^{\mathrm{an}}}(\log D) = \bigoplus_{p,q} \mathscr{E}^{p,q}_{\overline{X}^{\mathrm{an}}}(\log D),$$

the map

$$\Omega^{\bullet}_{\overline{X}^{\mathrm{an}}}(\log D) \longrightarrow \mathscr{E}^{*}_{\overline{X}^{\mathrm{an}}}(\log D) \tag{2.129}$$

is a quasi-isomorphism. Moreover the Hodge and weight filtrations of the complex $\Omega^{\bullet}_{\overline{X}^{\mathrm{an}}}(\log D)$ induce Hodge and weight filtrations on $\mathscr{E}^{*}_{\overline{X}^{\mathrm{an}}}(\log D)$ in such a way that the quasi-isomorphism (2.129) induce quasi-isomorphism of the graded pieces. In other words the quasi-isomorphism (2.129) is a bi-filtered quasi-isomorphism.

We will denote the space of global sections as

$$E^{p,q}_{\overline{X}^{\mathrm{an}}}(\log D) = \Gamma(\overline{X}, \mathscr{E}^{p,q}_{\overline{X}^{\mathrm{an}}}(\log D))$$

and

$$E^{*}_{\overline{X}^{\mathrm{an}}}(\log D) = \bigoplus_{p,q} E^{p,q}_{\overline{X}^{\mathrm{an}}}(\log D).$$

By the aciclicity of the sheaf of smooth differential forms, we deduce

$$\begin{aligned} H^n_{\mathrm{dR}}(X) &= H^n(E^*_{\overline{X}^{\mathrm{an}}}(\log D)), \\ F^p H^n_{\mathrm{dR}}(X) &= \mathrm{Im}(H^n(F^p E^*_{\overline{X}^{\mathrm{an}}}(\log D)) \to H^n_{\mathrm{dR}}(X)), \\ W_m H^n_{\mathrm{dR}}(X) &= \mathrm{Im}(H^n(W_{m-n} E^*_{\overline{X}^{\mathrm{an}}}(\log D)) \to H^n_{\mathrm{dR}}(X)). \end{aligned}$$

EXAMPLE 2.130. Let us compute everything for $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, viewed as a variety over $\mathbb{Q}$. As for any smooth curve, there is a canonical smooth compactification, in this case $\overline{X} = \mathbb{P}^1$. Write $D = \{0, 1, \infty\}$ for the divisor at infinity. Recall that $\mathcal{O}_{\mathbb{P}^1}(D)$ stands for the sheaf of rational functions having at most simple poles at D and nowhere else. We have:

$$\Omega^0_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}, \qquad \Omega^1_{\mathbb{P}^1}(\log D) = \mathcal{O}_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}.$$

Since $\Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, one sees that $\Omega^1_{\mathbb{P}^1}(\log D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. By the standard computation of the cohomology of line bundles on $\mathbb{P}^1$ [**Har77**, III, §5], none of the terms in the complex of logarithmic differentials has higher cohomology. Besides, setting $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{1-t}$, one has:

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{Q}, \qquad H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)) = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1$$

(note that these differentials ω_0 and ω_1 have a simple pole at ∞ as well). From the spectral sequence (2.16), it follows that

$$\begin{aligned} H^*_{\mathrm{dR}}(X) &= \mathbb{H}^*(\mathcal{O}_{\mathbb{P}^1} \xrightarrow{d} \mathcal{O}_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}) \\ &= H^*(\mathbb{Q} \longrightarrow \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1), \end{aligned}$$

where the differential in the second complex is the zero map. Thus,

$$H^1_{\mathrm{dR}}(X) = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1.$$

We now turn to the filtrations. For the Hodge filtration, (2.126) gives

$$H^1_{\mathrm{dR}}(X) = F^0 = F^1 \supseteq F^2 = \{0\}.$$

Moreover, the weight filtration on the complex of logarithmic differentials is given by $\Omega^\bullet_{\mathbb{P}^1} = W_0 \subseteq W_1 = \Omega^\bullet_{\mathbb{P}^1}(\log D)$. Since $H^1_{\mathrm{dR}}(\mathbb{P}^1)$ vanishes, we find:

$$\{0\} = W_1 \subseteq W_2 = H^1_{\mathrm{dR}}(X).$$

On the other hand, the first homology group $H_1(X(\mathbb{C}), \mathbb{Q})$ has as a basis the classes of two loops σ_0 and σ_1 winding once counterclockwise around the punctures 0 and 1. By Cauchy's residue theorem, the period matrix reads:

$$\begin{pmatrix} \int_{\sigma_0} \omega_0 & \int_{\sigma_1} \omega_0 \\ \int_{\sigma_0} \omega_1 & \int_{\sigma_1} \omega_1 \end{pmatrix} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

In other words, if $\sigma_0^\vee$ and $\sigma_1^\vee$ are the dual elements in cohomology, the isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ sends ω_0 to $\sigma_0^\vee \otimes 2\pi i$ and ω_1 to $\sigma_1^\vee \otimes 2\pi i$. Comparing with Example 2.90, one concludes that

$$H^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \simeq \mathbb{Q}(-1)^{\oplus 2}$$

as mixed Hodge structures over $\mathbb{Q}$.

Observe that all the information in the mixed Hodge structure over $\mathbb{Q}$ of the variety $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ can be read from the complex

$$A^* = A^0 \oplus A^1, \quad A^0 = \mathbb{Q}, \quad A^1 = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1,$$

together with the trivial differential and the filtrations

$$\begin{aligned} &F^0 = A^* \supset F^1 = A^1 \supset F^2 = \{0\}, \\ &W_{-1} = 0 \subset W_0 = A^0 \subset W_1 = A^*. \end{aligned} \tag{2.131}$$

Note that A^* has an algebra structure given by $\omega_i \wedge \omega_j = 0$, for $i, j \in \{0, 1\}$.

For later reference, we summarize the results of this example in a proposition. We say that a morphism $f\colon (A^\bullet, W, F) \to (A'^\bullet, W', F')$ between two complexes provided with two filtrations is a *bifiltered quasi-isomorphism* if f is compatible with the filtrations and the induced maps

$$\operatorname{Gr}^p_F \operatorname{Gr}^W_n A \longrightarrow \operatorname{Gr}^q_{F'} \operatorname{Gr}^{W'}_n A'$$

are quasi-isomorphisms for all p and n.

Proposition 2.132. *Let $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. The algebraic de Rham cohomology of X is given by*

$$H^*_{\mathrm{dR}}(X) = H^*(A^*).$$

The Hodge and the weight filtration are induced by the filtrations (2.130):

$$\begin{aligned} F^p H^*_{\mathrm{dR}}(X) &= H^*(F^p A^*), \\ W_k H^n_{\mathrm{dR}}(X) &= H^n(W_{k-n} A^*). \end{aligned}$$

Moreover, the inclusion of algebras

$$A^* \longrightarrow E^*_{\mathbb{P}^1(\mathbb{C})}(\log D)$$

induces a bifiltered quasi-isomorphism

$$(A^* \otimes_{\mathbb{Q}} \mathbb{C}, W, F) \longrightarrow (E^*_{\mathbb{P}^1(\mathbb{C})}(\log D), W, F).$$

2.6.2. *Normal crossings varieties.* Let $X = X_1 \cup X_2$ be the union of two smooth projective varieties such that $X_1 \cap X_2$ is smooth as well. We shall put a mixed Hodge structure on $H^n(X)$ by means of the Mayer-Vietoris long exact sequence

$$\begin{aligned} \cdots \to H^{n-1}(X_1 \cap X_2) &\longrightarrow H^n(X) \\ &\to H^n(X_1) \oplus H^n(X_2) \longrightarrow H^n(X_1 \cap X_2) \to \cdots \end{aligned} \tag{2.133}$$

Indeed, defining

$$\begin{aligned} A &= \operatorname{Coker}\left[H^{n-1}(X_1) \oplus H^{n-1}(X_2) \longrightarrow H^{n-1}(X_1 \cap X_2)\right], \\ B &= \operatorname{Ker}\left[H^n(X_1) \oplus H^n(X_2) \longrightarrow H^n(X_1 \cap X_2)\right], \end{aligned}$$

the long exact sequence (2.133) yields a short exact sequence

$$0 \longrightarrow A \longrightarrow H^n(X) \longrightarrow B \longrightarrow 0.$$

Note that A and B carry pure Hodge structures of weights $n-1$ and n respectively, since X_1, X_2 and $X_1 \cap X_2$ are smooth projective varieties. In this case, the weight filtration is given by

$$W_{n-2} = 0 \subseteq W_{n-1} = A \subseteq W_n = H^n(X).$$

2.6.3. *Mixed Hodge structures on relative cohomology.* Similarly, one can endow the cohomology with compact support and the relative cohomology with mixed Hodge structures, in such a way that the maps in the usual long exact sequences are compatible with the weight and the Hodge filtrations. The following result is very useful:

PROPOSITION 2.134 (Gysin long exact sequence). *Let X be a smooth variety over k and $Z \subseteq X$ a smooth closed subvariety of codimension c. Set $U = X \setminus Z$. There is a long exact sequence of mixed Hodge structures*

$$\cdots \to H^{j-1}(X) \xrightarrow{\alpha} H^{j-1}(U) \xrightarrow{\beta} H^{j-2c}(Z)(-c) \xrightarrow{\gamma} H^j(X) \to \cdots \quad (2.135)$$

where α is the usual restriction map and γ is the Gysin map.

2.6.4. *More examples.*

EXAMPLE 2.136 (Smooth open curves). Let $\overline{C}$ be a smooth projective complex curve and $S \subset \overline{C}$ a (non-empty) finite subset of s points. We describe the mixed Hodge structure on the first cohomology of the open curve $C = \overline{C} \setminus S$. The Gysin exact sequence (2.135) reads

$$0 \to H^1(\overline{C}) \to H^1(C) \to H^0(S)(-1) \xrightarrow{\gamma} H^2(\overline{C}) \to 0,$$

where γ is the "sum" map $\mathbb{Q}(-1)^{\oplus s} \to \mathbb{Q}(-1)$. From this we get

$$0 \to H^1(\overline{C}) \to H^1(C) \to \mathbb{Q}(-1)^{\oplus(s-1)} \to 0.$$

The weight filtration is given by

$$0 = W_0 H^1(C) \subset W_1 H^1(C) = H^1(\overline{C}) \subset W_2 H^1(C) = H^1(C),$$

so the graded pieces are

$$\mathrm{Gr}_1^W H_1(C) \simeq H^1(\overline{C}), \quad \mathrm{Gr}_2^W H^1(C) \simeq \mathbb{Q}(-1)^{\oplus(s-1)},$$

which are indeed pure Hodge structures of weights 1 and 2 respectively. In particular, the mixed Hodge structure $H^1(C)$ is of Tate type if and only if $H^1(\overline{C}) = 0$, which is equivalent to $C = \mathbb{P}^1 \setminus S$.

EXAMPLE 2.137 (Moduli spaces $M_{0,n}$). Let us compute the Hodge structure on the cohomology of the moduli spaces $M_{0,n}$ from paragraph 2.4.2.

PROPOSITION 2.138. *The cohomology group $H^i(M_{0,n})$ carries a pure Hodge-Tate structure of weight $2i$. More precisely,*

$$H^i(M_{0,n}) = \mathbb{Q}(-i)^{\oplus b_{i,n}}$$

where the Betti numbers $b_{i,n}$ are given by

$$\sum_{i\geq 0} b_{i,n}t^i = (1+2t)(1+3t)\cdots(1+(n-2)t).$$

PROOF. We proceed by induction on n. When $n=3$, the moduli space is reduced to a point, hence the only non-zero cohomology group is $H^0(M_{0,3}) = \mathbb{Q}(0)$. The case $n=4$ was settled in the previous example, where we saw that the non-trivial cohomology groups are $H^0(M_{0,4}) = \mathbb{Q}(0)$ and $H^1(M_{0,4}) = \mathbb{Q}(-1)^{\oplus 2}$. Let $(0,1,\infty,t_1,\dots,t_{n-3})$ denote the coordinates on $M_{0,n}$. For each $n\geq 5$, the map

$$\begin{aligned} M_{0,n} &\longrightarrow (M_{0,4}\times M_{0,n-1}) \\ (0,1,\infty,t_1,\dots,t_{n-3}) &\longmapsto ((0,1,\infty,t_1),(0,1,\infty,t_2,\dots,t_{n-3})) \end{aligned}$$

induces an isomorphism between $M_{0,n}$ and the complement of the smooth closed subvariety $Z\subset M_{0,4}\times M_{0,n-1}$ given by

$$Z = \bigsqcup_{i=2}^{n-3}\{t_i = t_1\} \simeq \bigsqcup_{i=2}^{n-3} M_{0,n-1}.$$

We shall compute the cohomology of $M_{0,n}\simeq (M_{0,4}\times M_{0,n-1})\setminus Z$ by combining the Gysin exact sequence, the Künneth formula and the induction hypothesis. First, the Gysin sequence (2.135) gives

$$\begin{aligned} \cdots \longrightarrow H^{i-2}(Z)(-1) &\xrightarrow{a} H^i(M_{0,4}\times M_{0,n-1}) \longrightarrow H^i(M_{0,n}) \\ \longrightarrow H^{i-1}(Z)(-1) &\xrightarrow{b} H^{i+1}(M_{0,4}\times M_{0,n-1}) \longrightarrow \cdots \end{aligned} \tag{2.139}$$

By the Künneth formula and the induction hypothesis, we have

$$\begin{aligned} H^i(M_{0,4}\times M_{0,n-1}) &\cong \bigoplus_{a+b=i} H^a(M_{0,4})\otimes H^b(M_{0,n-1}) \\ &\cong H^i(M_{0,n-1})\oplus H^{i-1}(M_{0,n-1})(-1)^{\oplus 2} \\ &\cong \mathbb{Q}(-i)^{\oplus(b_{i,n-1}+2b_{i-1,n-1})}. \end{aligned}$$

It follows that the maps a and b in (2.139) are morphisms between pure Hodge structures of different weights, hence the zero maps. From this we derive the short exact sequence

$$0\to H^i(M_{0,4}\times M_{0,n-1})\to H^i(M_{0,n})\to H^{i-1}(M_{0,n-1})(-1)^{\oplus(n-4)}\to 0.$$

Therefore, $H^i(M_{0,n}) = \mathbb{Q}(-i)^{b_{i,n}}$ with $b_{i,n} = b_{i,n-1} + (n-2)b_{i-1,n-1}$. One immediately checks that this recurrence relation amounts to the expression for the Betti numbers given in the statement. □

2.6.5. *Graph hypersurfaces.* Let $G = (V, E)$ be a finite graph with vertex and edge sets V and E, respectively. Assume that G is connected. A subgraph $T \subseteq G$ is called a *spanning tree* if T is a tree (*i.e.* connected with no loops) and contains all vertices of G. Consider a collection of variables $(x_e)_{e\in E}$ indexed by the edges of G. The first *Symanzik polynomial* of the graph is defined as

$$\Psi_G = \sum_{T\subseteq G} \prod_{e\notin T} x_e \in \mathbb{Z}[(x_e)_{e\in E}], \tag{2.140}$$

where the sum runs over all spanning trees in G. Let n_G be the number of edges of G and h_G the number of loops. It is easy to see that Ψ_G is a homogenous polynomial of degree h_G (Exercise 2.150).

DEFINITION 2.141. The *graph hypersurface* $X_G \subseteq \mathbb{P}^{n_G-1}$ is the vanishing locus of the polynomial Ψ_G.

Graph hypersurfaces appear in perturbative quantum field theory, a major goal of which is to compute Feynman amplitudes. These are the probabilities that a particle interaction is described by a given graph. The easiest case is when $n_G = 2h_G$ and $n_\gamma > 2h_\gamma$ for all non-empty strict subgraphs $\gamma \subsetneq G$. Then the corresponding Feynmann integral is given, up to a normalization factor, by the convergent integral [**BEK06**, Prop. 5.2],

$$I_G = \int_\sigma \frac{\Omega}{\psi_G^2}, \tag{2.142}$$

where we have chosen a numbering of the vertices, so that ψ_G becomes a polynomial in the variables $x_0, \ldots, x_{n_G-1}$, the differential form Ω is given by

$$\Omega = \sum_{j=0}^{n_G-1} (-1)^j x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n_G-1},$$

and one integrates over the real coordinate simplex

$$\sigma = \{[x_0 \colon \cdots : x_{n_G-1}] \in \mathbb{P}^{n_G-1}(\mathbb{R}) \mid x_i \geq 0\}.$$

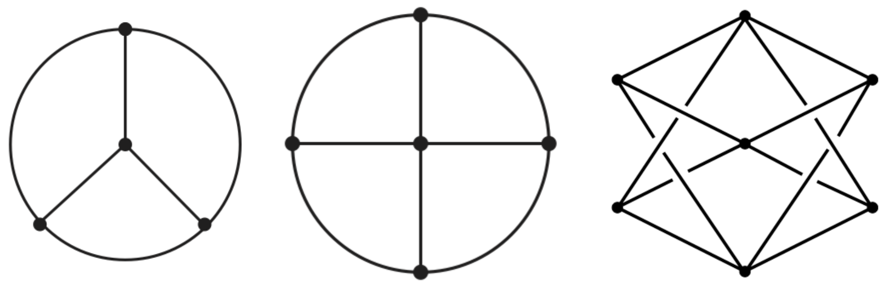
Note that the condition $n_G = 2h_G$ implies that the integrand of (2.142), that is written in homogeneous coordinates, is well defined. So I_G can be written as the affine integral

$$I_G = \int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty \frac{dt_{n_G-1}}{\psi_G^2(1, t_1, t_2, \ldots, t_{n_G-1})},$$

where $t_i = x_i/x_0$.

Graphs satisfying the above conditions are called *primitive log divergent*. Figure 8 gives examples of primitive log divergent graphs and the associated Feynman amplitudes.

It was conjectured for some time that the amplitudes I_G of primitive log divergent graphs were always linear combinations of multiple zeta values. This happens to be the case for graphs with $h_G \leq 6$. [**BS12**]



$6\zeta(3)$ $\qquad$ $20\zeta(5)$ $\qquad$ $\frac{45}{4}\zeta(3,5)+\frac{333}{20}\zeta(5,3)-\frac{9}{5}\zeta(8)$

FIGURE 8. Three examples of primitive log divergent graphs and the corresponding Feynman amplitudes

The integrand of (2.142) is a global top-degree differential form ω_G on $\mathbb{P}^{n_G-1}\setminus X_G$, and the boundary of the simplex σ is contained in the union D of the coordinate hyperplanes $\{x_i=0\}$. In general, σ intersects the graph hypersurface X_G, so, as in Section 2.4, one is faced with the problem that the integration cycle does *not* define an element in the naive relative cohomology group

$$H^{n_G-1}(\mathbb{P}^{n_G-1}\setminus X_G, D\setminus D\cap X_G).$$

However, the fact that the coefficients of ψ_G are positive makes this intersection easy to describe. In fact,

$$X_G(\mathbb{C})\cap\sigma=\bigcup_{h_\gamma>0}L_\gamma(\mathbb{R}_{\geq 0}),$$

where, if γ is a subgraph of G, then L_γ is the linear subvariety of $\mathbb{P}^{n_G-1}$ of equations $x_e=0$, for e vertex of γ and

$$L_\gamma(\mathbb{R}_{\geq 0})=\{[x_e]_{e\in E}\in L_\gamma \mid x_e\in\mathbb{R}_{\geq 0}\}.$$

This allowed Bloch, Esnault and Kreimer to prove the following in [**BEK06**, Prop. 7.3]

THEOREM 2.143 (Bloch-Esnault-Kreimer). *There exists a tower*

$$\pi\colon P=P_r\longrightarrow\cdots\longrightarrow P_0=\mathbb{P}^{n_G-1}$$

of blow-ups such that each P_i is obtained by blowing up P_{i-1} along the strict transform of a coordinate linear space L_i and the following conditions hold:

(1) *The differential $\pi^*\omega_G$ has no poles along the exceptional divisors associated to the blow-ups.*

(2) *The total transform B of D is a normal crossings divisor such that none of the non-empty intersections of its irreducible components is contained in the strict transform Y of X_G.*

(3) *The strict transform of σ does not meet Y.*

Corollary 2.144. *Keeping the notation from the previous theorem, the Feynman amplitude I_G is a period of the mixed Hodge structure*

$$H^{n_G-1}(P \setminus Y, B \setminus (B \cap Y)).$$

⋆⋆⋆

Exercise 2.145. Let X be a smooth complex variety, $Z \subset X$ a smooth subvariety of codimension c and write $U = X \setminus Z$. Use the Gysin long exact sequence (2.135) to prove that the restriction map $H^i(X) \to H^i(U)$ is an isomorphism for $i < 2c-1$, and is injective for $i = 2c-1$.

Exercise 2.146 (Varieties which admit a compactification by a smooth divisor). Let U be a smooth complex variety. In this exercise, we show that the existence of a smooth compactification by a smooth divisor imposes strong restrictions on the mixed Hodge structure of U.

(1) Use the Gysin exact sequence (2.135) to show that if $U = X \setminus D$, with X smooth and projective and D smooth, then $H^n(U)$ has only weights in $[n, n+1]$.

(2) Give an example of a smooth surface which does not admit a smooth projective compactification by a smooth divisor.

Exercise 2.147. Let $\overline{X}$ be a smooth projective complex variety and $Y_0, Y_1 \subseteq \overline{X}$ two smooth divisors such that $Y_0 \cup Y_1$ has normal crossings. Set $X = \overline{X} \setminus Y_0$ and $Y = Y_1 \setminus (Y_0 \cap Y_1)$. Show that the weight filtration on the relative cohomology group $M = H^n(X, Y)$ is given by

$$\begin{aligned}
W_{n-2}M &= 0,\\
W_{n-1}M &= \mathrm{Im}(H^{n-1}(Y_1) \to M),\\
W_nM &= \mathrm{Ker}(M \to H^{n-1}(Y_0)(-1)),\\
W_{n+1}M &= M.
\end{aligned}$$

[*Hint*: Consider a diagram of mixed Hodge structures whose rows are Gysin long exact sequences and whose columns are long exact sequences of relative cohomology. Use the fact that W_m is an exact functor and Lemma 2.100.]

Exercise 2.148. The graded pieces of the mixed Hodge structure of a smooth variety

Let X be a smooth projective variety, D a simple normal crossings divisor an $U = X \setminus D$. Following Construction 2.47, we form

$$D^0 = X, \quad D^p = \coprod_{i_1, i_2, \dots, i_p} D_{i_1} \cap \dots \cap D_{i_p}.$$

Prove that the weight filtration of $H^n(U)$ is given by

$$\operatorname{Gr}_m^W H^n(U) = H^{n-m}(\dots \to H^{m-2}(D^1)(-1) \to H^m(X) \to 0), \tag{2.149}$$

where the term $H^{n-2p}(D^p)(-p)$ sits in degree $-p$.

Exercise 2.150. Prove that the first Symanzik polynomial of a graph, as defined in (2.140), is homogeneous of degree the number of loops in G.

Exercise 2.151 (Deletion-contraction relations). Let G be a connected graph and e an edge of G. We denote by $G \setminus e$ the graph obtained by deleting the edge e and by G/e the graph obtained by contracting the edge e. Assume that $G \setminus e$ is still connected and that the two end points of e are different. Show that the following relation holds:

$$\Psi_G = x_e \Psi_{G \setminus e} + \Psi_{G/e}.$$

Exercise 2.152 (The trivial Feynman amplitude). Consider the graph G with two vertices and two edges connecting them, as in Figure 9. Compute the Feynman amplitude I_G and write down a Hodge structure for which it is a period (no blow-up is needed in this case).

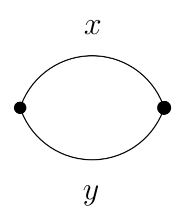


Figure 9. A simple graph

2.7. Extensions. We now turn to the question of describing the extension groups in the category of mixed Hodge structures over $\mathbb{C}$ of structures of Tate type. Recall that, by a mixed Hodge structure without explicitly mentioning the field of definition, we mean a mixed Hodge structure over $\mathbb{C}$.

Definition 2.153. Let A and B be two mixed Hodge structures.

(1) An *extension of A by B* is a short exact sequence

$$0 \to B \xrightarrow{\beta} H \xrightarrow{\alpha} A \to 0,$$

where α and β are morphisms of mixed Hodge structures. Such an extension is said to be *split* if there exists a morphism of mixed Hodge structures $s \colon A \to H$ such that $\alpha \circ s = \mathrm{id}_A$.

(2) Two extensions are *equivalent* if there exists a morphism of mixed Hodge structures $f\colon H \to H'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & H' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commutes. This defines indeed an equivalence relation whose set of equivalence classes will be denoted by

$$\mathrm{Ext}^1_{\mathbf{MHS}}(A, B).$$

THEOREM 2.154. *Let m and n be two integers. Then*

$$\mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Q}(m), \mathbb{Q}(n)) = \begin{cases} \mathbb{C}/(2\pi i)^{n-m}\mathbb{Q} & m < n, \\ 0 & \textit{otherwise}. \end{cases}$$

PROOF. Tensoring the extension by $\mathbb{Q}(-n)$, we can assume without loss of generality that $n = 0$. So let us consider an extension

$$0 \to \mathbb{Q}(0) \xrightarrow{\beta} H \xrightarrow{\alpha} \mathbb{Q}(m) \to 0.$$

Let us first assume that $m > 0$. Then $W_{-2m}H \subseteq H$ is a rank one sub-Hodge structure and the composition

$$W_{-2m}H \hookrightarrow H \xrightarrow{\alpha} \mathbb{Q}(m)$$

is an isomorphism. Thus the extension is necessarily split.

For $m = 0$, the weight and the Hodge filtration of H are trivial (the corresponding subobjects are either zero or everything), hence any section s_{B} of the map $\alpha_{\mathrm{B}}\colon H_{\mathrm{B}} \to \mathbb{Q}(0)_{\mathrm{B}}$ induces a morphism of Hodge structures $s\colon \mathbb{Q}(0) \to H$, so the extension is again split.

Now assume that $m < 0$. The complex vector space H_{dR} has a canonical splitting

$$H_{\mathrm{dR}} = W_0 H_{\mathrm{dR}} \oplus F^{-m} H_{\mathrm{dR}}.$$

Choose a basis e_0, e_1 of H_{B} satisfying $e_0 = \beta(1)$, where 1 is the generator of $\mathbb{Q}(0)_{\mathrm{B}}$ and $\alpha(e_1) = e$, where e is the generator of $\mathbb{Q}(m)_{\mathrm{B}}$. This basis determines uniquely a basis f_0, f_1 of H_{dR} by the conditions

$$\begin{aligned} f_0 &\in W_0 H_{\mathrm{dR}}, & \mathrm{comp}_{\mathrm{B,dR}}(f_0) &= e_0, \\ f_1 &\in F^{-m} H_{\mathrm{dR}}, & \mathrm{comp}_{\mathrm{B,dR}}(f_1) &\in (2\pi i)^{-m} e_1 + W_{2m}. \end{aligned}$$

In these bases the morphism $\mathrm{comp}_{\mathrm{B,dR}}$ can be written as

$$\begin{pmatrix} 1 & a \\ 0 & (2\pi i)^{-m} \end{pmatrix}$$

for a complex number a that determines the class of the extension.

We have the right to change the basis (e_0, e_1) by an upper triangular basis with ones in the diagonal and a rational coefficient in the upper right corner.

The basis (f_0, f_1) remains unchanged. In this new basis, the comparison isomorphism will be given by

$$\begin{aligned}\begin{pmatrix}1 & a' \\ 0 & (2\pi i)^{-m}\end{pmatrix} &= \begin{pmatrix}1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a \\ 0 & (2\pi i)^{-m}\end{pmatrix} \\ &= \begin{pmatrix}1 & a + (2\pi i)^{-m} b \\ 0 & (2\pi i)^{-m}\end{pmatrix}.\end{aligned}$$

Thus, two complex numbers $a, a' \in \mathbb{C}$ determine the same extension if and only if $a - a' \in (2\pi i)^{-m}\mathbb{Q}$, from which the result follows. □

REMARK 2.155. A similar argument yields Carlson's formula for the extensions between any two mixed Hodge structures [**Car80**]:

$$\mathrm{Ext}^1_{\mathbf{MHS}}(A, B) = \frac{W_0 \operatorname{Hom}(A, B)_{\mathbb{C}}}{W_0 \cap F^0 \operatorname{Hom}(A, B)_{\mathbb{C}} + W_0 \operatorname{Hom}(A, B)}.$$

Moreover, Beilinson showed that the category of mixed Hodge structures has cohomological dimension one, meaning that all higher extension groups vanish. This follows from:

THEOREM 2.156. *For any mixed Hodge structures A, B, we have*

$$\mathrm{Ext}^2_{\mathbf{MHS}}(A, B) = 0.$$

PROOF. See [**Beĭ86**, Corollary 1.10]. □

2.7.1. *Examples.* By Theorem 2.154, the extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ are parametrized by elements in $\mathbb{C}/(2\pi i)^n\mathbb{Q}$. It follows that, for each $n \geq 2$, there is a mixed Hodge structure $\zeta^{MHS}(n)$ sitting in an exact sequence

$$0 \to \mathbb{Q}(n) \to \zeta^{MHS}(n) \to \mathbb{Q}(0) \to 0, \tag{2.157}$$

whose extension class corresponds to the zeta value $\zeta(n)$. Hence, this extension is split if and only if $\zeta(n) \in (2\pi i)^n\mathbb{Q}$. By Theorem 1.3 and the fact that elements of $(2\pi i)^n\mathbb{Q}$ are purely imaginary for odd n, the extension (2.157) is split if and only if n is even. It is a hard problem to construct geometrically these extensions, *e.g.* as a relative cohomology group.

We now show that, when $n = 1$, all the extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ have geometric origin.

EXAMPLE 2.158 (Kummer mixed Hodge structure). Given a complex number $t \in \mathbb{C}^\times \setminus \{1\}$, consider the relative cohomology

$$H = H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\}).$$

The long exact sequence (2.45) gives

$$0 \to H^0(\mathbb{P}^1 \setminus \{0, \infty\}) \to H^0(\{1, t\}) \to H \to H^1(\mathbb{P}^1 \setminus \{0, \infty\}) \to 0.$$

By Example 2.136, one has $H^1(\mathbb{P}^1 \setminus \{0, \infty\}) = \mathbb{Q}(-1)$, hence

$$0 \to \mathbb{Q}(0) \to H \to \mathbb{Q}(-1) \to 0.$$

The *Kummer mixed Hodge structure* K_t^{H} is defined to be the dual of H, so

$$K_t^{\mathrm{H}} \in \mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1)).$$

For $t = 1$, the Kummer extension is defined as the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. This yields a map $\mathbb{C}^\times \to \mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{C})}(\mathbb{Q}(0), \mathbb{Q}(1))$.

Example 2.159. As another example of how arithmetic information can be encoded through extensions of mixed Hodge structures, let us consider extensions of the first cohomology of a smooth projective curve C by $\mathbb{Q}(-1)$. Then Carlson's theorem implies that

$$\mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Q}(-1), H^1(C)) = \mathrm{Jac}(C)(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

By Example 2.136, the cohomology of $C \setminus \{p, q\}$ for any pair of points gives such an extension. Through the above isomorphism, the class of the extension is given by the class of the divisor $[p] - [q]$ in $\mathrm{Jac}(C)(\mathbb{C})$. In particular, the extension splits if and only if this divisor is torsion.

2.8. Back to $\zeta(2)$ and irrationality proofs. We end the chapter[6] by showing that the relative cohomology group attached to $\zeta(2)$ in 2.4.1 is an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$. We then discuss the problem of constructing other extensions and a potential application to irrationality proofs.

2.8.1. *The extension associated to $\zeta(2)$.* We prove that the relative cohomology group constructed in paragraph 2.4.1 from the integral representation of $\zeta(2)$ is indeed an extension of $\mathbb{Q}(-2)$ by $\mathbb{Q}(0)$. Recall that we considered the blow-up X of $\mathbb{A}^2$ at the points $p = (0, 0)$ and $q = (1, 1)$, together with the normal crossings divisors

$$L = L_0 \cup L_1, \quad M = M_0 \cup M_1 \cup M_2 \cup M_3 \cup M_4,$$

where L_0 and L_1 are the strict transforms of $\{t_1 = 0\}$ and $\{t_2 = 1\}$ (affine lines), $M_0 = E_p$ and $M_1 = E_q$ are the exceptional divisors (projective lines), and M_2, M_3 and M_4 are the strict transforms of $\{t_1 = t_2\}$, $\{t_2 = 0\}$ and $\{t_1 = 1\}$ (again affine lines).

Proposition 2.160. *There exists a short exact sequence of mixed Hodge structures*

$$0 \to \mathbb{Q}(0) \to H^2(X \setminus L, M \setminus (L \cap M)) \to \mathbb{Q}(-2) \to 0. \tag{2.161}$$

Proof. Let X be any smooth complex variety, and L and M two normal crossings divisors on X with no common irreducible components and such

[6] Many thanks to Clément Dupont and Peter Jossen for their help with this section.

that $L \cup M$ has normal crossings as well. By [Dup17, App. A.1], there is a spectral sequence of mixed Hodge structures

$$E_1^{p,q} = \bigoplus_{\substack{j-i=p \\ |I|=i \\ |J|=j}} H^{q-2i}(L_I \cap M_J)(-i) \Longrightarrow \mathrm{gr}_\bullet^W H^{p+q}(X \setminus L, M \setminus (M \cap L)), \tag{2.162}$$

where the differential $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ is the sum of

libelindent=0pt the restriction maps

$$H^{q-2i}(L_I \cap M_J)(-i) \to H^{q-2i}(L_I \cap M_{J\cup\{s\}})(-i)$$

induced from the inclusions $L_I \cap M_{J\cup\{s\}} \hookrightarrow L_I \cap M_J$, multiplied by the signs $\varepsilon(J, J \cup \{s\})$;

liibeliindent=0pt the Gysin morphisms

$$H^{q-2i}(L_I \cap M_J)(-i) \to H^{q-2i+2}(L_{I\setminus\{r\}} \cap M_J)(-i+1)$$

associated to the inclusions $L_I \cap M_J \hookrightarrow L_{I\setminus\{r\}} \cap M_J$, multiplied by the signs $\varepsilon(I \setminus \{r\}, I)$.

(Recall from 2.2.6 that $\varepsilon(I, J) = (-1)^\ell$ whenever $J = \{j_0, \dots, j_r\}$ with $j_0 < \cdots < j_r$ and I is obtained from I by removing the index j_ℓ).

Assume that all $E_1^{p,q}$ carry a pure Hodge structure of weight q. The second page of the spectral sequence is given by

$$E_2^{p,q} = \frac{\mathrm{Ker}(d_1 \colon E_1^{p,q} \to E_1^{p+1,q})}{\mathrm{Im}(d_1 \colon E_1^{p-1,q} \to E_1^{p,q})},$$

together with a differential $d_2 \colon E_2^{p,q} \to E_2^{p+1,q-1}$. Thus, $E_2^{p,q}$ has a pure Hodge structure of weight q as well, which implies $d_2 = 0$ since there are no non-trivial morphisms between Hodge structures of different weight. It follows that the spectral sequence degenerates at E_2 and

$$E_2^{p,q} = \mathrm{gr}_q^W H^{p+q}(X \setminus L, M \setminus (M \cap L)). \tag{2.163}$$

Let us now turn to our particular situation. Setting

$$r = L_0 \cap L_1, \quad s = L_0 \cap E_p, \quad t = L_1 \cap E_q, \quad M_{ij} = M_i \cap M_j,$$

the spectral sequence takes the form of Figure 10. By way of illustration, the piece $E_1^{1,2}$ is the sum of all possible $H^{2-2i}(L_I \cap M_J)(-i)$ with $j = i+1$. Then necessarily $i = 0$ or $i = 1$, and the second case does not appear since there are no non-empty intersections of one component of L and two components of M. For $i = 0$, we get $\bigoplus H^2(M_j) = H^2(E_p) \oplus H^2(E_q)$, taking into account that the remaining components are affine lines. Observe that odd values of q do not need to be considered, since all intersections $L_I \cap M_J$ have only cohomology in even degrees. For the same reason, the assumption that $E_1^{p,q}$ has pure weight q is satisfied in our case.

$$\begin{array}{ccccc}
\boxed{H^0(r)(-2)} & 0 & 0 & 0 & 0 \\
\\
0 & 0 & 0 & 0 & 0 \\
\\
0 & \begin{array}{c} H^0(L_0)(-1) \\ \oplus \\ H^0(L_1)(-1) \end{array} & \boxed{\begin{array}{c} H^2(X) \\ \oplus \\ H^0(s)(-1) \\ \oplus \\ H^0(t)(-1) \end{array}} & \begin{array}{c} H^2(E_q) \\ \oplus \\ H^2(E_p) \end{array} & 0 \\
\\
0 & 0 & 0 & 0 & 0 \\
\\
0 & 0 & H^0(X) & \bigoplus H^0(M_i) & \boxed{\bigoplus H^0(M_{ij})}
\end{array}$$

FIGURE 10. The first page of the spectral sequence computing $\mathrm{gr}_\bullet^W H^2(X \setminus L, M \setminus (L \cap M))$

We need to prove that

$$\mathrm{gr}_\bullet^W H^2(X \setminus L, M \setminus (L \cap M)) = \mathbb{Q}(0) \oplus \mathbb{Q}(-2). \tag{2.164}$$

The piece $\mathbb{Q}(-2)$ comes from the top-left corner, while $\mathbb{Q}(0)$ arises as the cokernel of the map $\bigoplus H^0(M_i) \to \bigoplus H^0(M_{ij})$, which has rank 4. Indeed, it is given by

$$(a, b, c, d, e) \longmapsto (c - a, d - a, c - b, e - b, e - d).$$

Since the map $H^0(X) \to \bigoplus H^0(M_i)$ sends a to (a, a, a, a, a), the cohomology of the bottom line is concentrated in $E_2^{2,0} = \mathbb{Q}(0)$.

We are thus reduced to show that the complex $E_1^{\bullet,2}$ is exact at the middle term. For this, we first observe that the Gysin maps induce an isomorphism of Hodge structures

$$H^0(E_p)(-1) \oplus H^0(E_q)(-1) \xrightarrow{\sim} H^2(X). \tag{2.165}$$

This is an instance of the general computation of the Hodge structure of a blow-up, see e.g. [**Voi02**, 7.3.3]. In the case at hand, it can be seen as follows: the Gysin long exact sequence (2.135) for $U = X \setminus (E_p \cup E_q)$ reads

$$\cdots \to H^1(U) \to H^0(E_p)(-1) \oplus H^0(E_q)(-1) \to H^2(X) \to H^2(U) \to \cdots$$

But U is isomorphic, via the blow-up map, to $\mathbb{A}^2 \setminus \{p,q\}$, and hence (using Exercise 2.145) $H^1(U) = H^2(U) = 0$. It follows that the differential $d_1 \colon E_1^{0,2} \to E_1^{1,2}$ in the spectral sequence is given, in suitable bases compatible with the isomorphism (2.165), by

$$\begin{aligned} H^2(X) \oplus H^0(s)(-1) \oplus H^0(t)(-1) &\longrightarrow H^2(E_p) \oplus H^2(E_q) \\ (a,b,c,d) &\longmapsto (a+c, b+d). \end{aligned} \tag{2.166}$$

To compute the remaining map, one needs to know the cohomology classes $[L_i] \in H^2(X)$. We claim that $[L_0] = -[E_p]$. Indeed, since the total transform of ℓ_0 is the union $L_0 \cup E_p$, we get

$$[L_0] + [E_p] = [\pi^{-1}(\ell_0)] = \pi^*[\ell_0] = 0,$$

where the last equality follows from the fact that $[\ell_0]$ lives in $H^2(\mathbb{A}^2) = 0$. Similarly, $[L_1] = -[E_q]$, so that the differential $d_1 \colon E_1^{-1,2} \to E_1^{0.2}$ is given by

$$\begin{aligned} H^0(L_0)(-1) \oplus H^0(L_1)(-1) &\longrightarrow H^2(X) \oplus H^0(s)(-1) \oplus H^0(t)(-1) \\ (a,b) &\longmapsto (-a,-b,a,b). \end{aligned} \tag{2.167}$$

It is now obvious that the middle row of the spectral sequence is exact. Indeed, its whole second page reads

$$\begin{matrix} \mathbb{Q}(-2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{Q}(0). \end{matrix}$$

This concludes the proof of the equality (2.164) and shows, moreover, that the group $H^i(X \setminus L, M \setminus (L \cap M))$ vanishes in all degrees $i \neq 2$. $\square$

Remark 2.168. A byproduct of the proof is that we have canonical identifications (see Exercise 2.172).

$$\begin{aligned} \mathrm{gr}_4^W H^2(X \setminus L, M \setminus (L \cap M)) &= H^2(X \setminus L) = \mathbb{Q}(-2), \\ \mathrm{gr}_0^W H^2(X \setminus L, M \setminus (L \cap M)) &= H^2(X, M) = \mathbb{Q}(0). \end{aligned} \tag{2.169}$$

Recall from paragraph 2.4.1 that the differential form $\pi^*(\omega)$ is an element of $H^2_{\mathrm{dR}}(X \setminus L)$, while the simplex $\widehat{\sigma}$ belongs to $H^2(X, M)$. By Theorem 2.154, the class of the extension

$$[H^2(X \setminus L, M \setminus (L \cap M))] \in \mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Q}(-2), \mathbb{Q}(0)) = \mathbb{C}/(2\pi i)^2\mathbb{Q}$$

is thus given by $\int_{\widehat{\sigma}} \pi^*(\omega) = \zeta(2)$. One would like to use this information as follows: imagine that we knew by "pure thought" that all such extensions given by relative cohomology of varieties defined over $\mathbb{Q}$ are split. Then $\zeta(2)$ would have to vanish in the quotient $\mathbb{C}/(2\pi i)^2\mathbb{Q}$, which would yield a more conceptual explanation of why $\zeta(2)$ is a rational multiple of π^2. To carry out this program, one needs however to leave the category of mixed Hodge structures and work with the more abstract notion of mixed Tate motives which will be introduced in Chapter 4.

2.8.2. *Odd extensions.* In general, it is a difficult question to geometrically construct extensions of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$...[**Dup18**] [**Bro16**] complete

More precisely, Dupont starts with affine space $\mathbb{A}^n$ and the hypersurfaces

$$\ell_n = \{x_1 \cdots x_n = 1\}, \quad m_n = \bigcup_{1\leq i\leq n} \{x_i = 0\} \cup \bigcup_{1\leq i\leq n} \{x_i = 1\}.$$

The divisor ℓ_n is smooth and m_n has normal crossings. However, their union $\ell_n \cup m_n$ fails to have normal crossings because of the point $p_n = (1, \ldots, 1)$, where $n+1$ components intersect. Let $\pi_n \colon X_n \to \mathbb{A}^n$ be the blow-up of $\mathbb{A}^n$ at p_n and let E_n denote the exceptional divisor. We write L_n for the strict transform of ℓ_n and M_n for the union of the strict transform of m_n and E_n. We form the relative cohomology:

$$\mathcal{Z}_n = H^n(X_n \setminus L_n, M_n \setminus (L_n \cap M_n)).$$

Dupont proves that $\mathcal{Z}_n$ fits into an exact sequence of mixed Hodge structures

$$0 \to \mathbb{Q}(0) \longrightarrow \mathcal{Z}_n \to \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n)$$

and that there is a natural isomorphism

$$\mathcal{Z}_n/\mathbb{Q}(0) \xrightarrow{\sim} H^{n-1}(\ell_n, \bigcup_{1\leq i\leq n} \{x_i = 1\})(-1).$$

To separate the even and the odd weights, one uses the involution

$$\tau(x_1, \ldots, x_n) = (x_1^{-1}, \ldots, x_n^{-1}).$$

Indeed, if $p \colon H_n \to H_n/\mathbb{Q}(0)$ denotes the quotient map, one defines H_n^{odd} as $p^{-1}((H_n/\mathbb{Q}(0))^{\tau=1})$. It then fits into an exact sequence

$$0 \to \mathbb{Q}(0) \to H_n^{\mathrm{odd}} \bigoplus_{3\leq 2k+1\leq n} \mathbb{Q}(-(2k+1)) \to 0$$

2.8.3. *Irrationality proofs.* Here is how a typical irrationality proof works. To show that a real number α is irrational, we proceed in three steps:

(1) we construct linear forms
$$I_n = a_n + b_n\alpha, \quad a_n, b_n \in \mathbb{Q}, \tag{2.170}$$
such that $0 < |I_n| < \varepsilon^n$ for some $0 < \varepsilon < 1$ and n sufficiently big;

(2) if d_n is the common denominator of a_n and b_n, then we require that $d_n < D^n$ for some real number D, again when n is big enough;

(3) ε and D should be related by the inequality $\varepsilon D < 1$.

If one succeds in carrying out these three steps, then α is irrational. Indeed, assume that $\alpha = \frac{p}{q}$. Multiplying by $d_n q$, we get

$$0 < |d_n a_n q + b_n d_n p| < q d_n \varepsilon^n < q(\varepsilon D)^n,$$

so the sequence inside the absolute value converges to zero by the assumption that $\varepsilon D < 1$. But then, for n sufficiently big, we would find integers strictly bigger than 0 and smaller than 1, which is of course a contradiction!

Algebraic geometry could be useful in producing the linear forms (2.170). Indeed, assume that we can construct a mixed Hodge structure over $\mathbb{Q}$ which is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ with period matrix

$$\begin{pmatrix} 1 & \alpha \\ 0 & (2\pi i)^n \end{pmatrix}$$

with respect to some bases $\{\omega_0, \omega_1\}$ of H_{dR} and $\{\sigma_0, \sigma_1\}$ of H_B. Then, given any $\omega \in H_{\mathrm{dR}}$, there exist rational numbers a and b such that $a\omega_0 + b\omega_1$, and the integral $\int_{\sigma_0} \omega$ is equal to $a + b\alpha$. Typically, H is given by a relative cohomology group and one considers a sequence $\omega_n = f^n\omega$ where ω is a fixed differential and f is a function vanishing on the boundary.

Example 2.171. Consider the differential form

$$\omega_{a,b,c} = \frac{(x-1)^a(t-x)^b}{x^{c+1}}dx,$$

where $a, b, c \geq 1$ and $t \geq 2$ are integers. Since $\omega_{a,b,c}$ is only singular along $x = 0$ and has top degree, it defines a class in $H^1_{\mathrm{dR}}(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, t\})$. By Example 2.42, a basis of this relative cohomology group is given by the differentials $\omega_1 = (0, 1, 0)$ and $\omega_2 = (\frac{dx}{x}, 0, 0)$, so there exists rational numbers A and B such that

$$\omega_{a,b,c} = A\omega_1 + B\omega_2.$$

Indeed, elementary manipulations of the complex (2.43) yield the values

$$A = \sum_{\substack{0 \leq i \leq a \\ 0 \leq j \leq b \\ i+j \neq c}} \frac{\binom{a}{i}\binom{b}{j}(-1)^{a-i-j}(t^{b-c+i} - t^{b-j})}{i+j-c},$$

$$B = \sum_{\substack{0 \leq i \leq a \\ 0 \leq j \leq b \\ i+j = c}} \binom{a}{i}\binom{b}{j}(-1)^{a-i-j}t^{b-j},$$

Note that B is an integer. In view of Example 2.72, it follows that

$$\int_1^t \omega_{a,b,c} = A + B\log(t),$$

and choosing the parameters a, b, c as functions of n gives a sequence of linear forms in 1 and $\log(t)$ as in Step (1).

Let us specialize to the case $a = b = c = n$ and $t = 2$. Then

$$I_n = \int_1^2 \omega_{n,n,n} = a_n + b_n \log(2),$$

where b_n is an integer and a_n is given by the formula

$$a_n = \sum_{\substack{0 \le i \le n \\ 0 \le j \le n \\ i+j \neq n}} \frac{\binom{n}{i}\binom{n}{j}(-1)^{a-i-j}(t^i - t^{n-j})}{i+j-n}.$$

Since the denominators of the summands in a_n run through $[-n, n]$, one can take $d_n = \mathrm{lcm}(1, 2, \ldots, n)$. We have:

$$d_n = \prod_{\substack{p \le n \\ \text{prime}}} p^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} < \prod_{\substack{p \le n \\ \text{prime}}} p^{\frac{\log n}{\log p}} = n^{\pi(n)},$$

where $\pi(n)$ is the number of primes smaller than n. Here is where some deep arithmetic input enters: the *prime number theorem* asserts that

$$\pi(n) \sim \frac{n}{\log n} \quad \text{as} \quad n \to +\infty,$$

see e.g [**IK04**, Chap. 2]. It follows that $n^{\pi(n)} \sim e^{(1+\varepsilon)n}$ for all $\varepsilon > 0$ and, being generous, $D = 3$ works in Step (2).

Next observe that, by the choice of the parameters, I_n can be written as

$$I_n = \int_1^2 f^n \frac{dx}{x}, \qquad f(x) = \frac{(x-1)(2-x)}{x}.$$

The function f is strictly positive on the open interval $(1, 2)$ and bounded above by its maximal value $3 - 2\sqrt{2}$. Therefore,

$$0 < I_n < (3 - 2\sqrt{2})^n \log(2) < (3 - 2\sqrt{2})^n,$$

so $\varepsilon = 3 - 2\sqrt{2}$ satisfies the assumptions. Luckily, $\varepsilon D = 0,5147186\ldots < 1$ and, all in all, we have proved that $\log(2)$ is irrational!

⋆ ⋆ ⋆

Exercise 2.172. Specialize the spectral sequence (2.162) to the cases $I = \emptyset$ and $J = \emptyset$. Deduce the identifications (2.169).

Exercise 2.173. Let $L = L_0 \cup L_1 \cup L_2$ and $M = M_0 \cup M_1 \cup M_2$ be two triangles in $\mathbb{P}^2$ such that no three lines intersect at a common point. Use the spectral sequence (2.162) to show that

$$\mathrm{gr}_\bullet H^2(\mathbb{P}^2 \setminus L, M \setminus (L \cap M)) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)^{\oplus 4} \oplus \mathbb{Q}(-2).$$

The question of what happens when the lines are not in general position is studied in great detail in [**BGSV90**].

Exercise 2.174 (Irrationality of $\zeta(3)$). The goal of this exercise is to prove that $\zeta(3)$ is irrational following the approach by Beukers [**Beu79**].

$$I_n = \int_{[0,1]^3} \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} dxdydz.$$

3. Multiple zeta values and the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In this chapter, we start moving towards the goal of upgrading multiple zeta values to their motivic counterparts, which are functions on an algebro-geometric construction associated with the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. To this end, we first look for homotopy functionals on the space of paths of a differentiable manifold M. By Stokes' theorem, examples are given by line integrals of closed 1-forms. However, the corresponding functions on the fundamental group always factor through its abelianization and thus cannot detect loops whose homology classes are trivial. Trying to go further, K-T. Chen had the fundamental insight that iterated integrals yield finer invariants, which are in fact sufficient to recover all finite-dimensional unipotent representations of $\pi_1(M)$ and not only the abelian ones. More precisely, his celebrated π_1-de Rham theorem asserts that the ring of regular functions on the pro-unipotent completion of the fundamental group is isomorphic, as a Hopf algebra, to the zeroth cohomology of the bar complex of any connected model of the algebra of differential forms. This has a number of important consequences, notably the fact—due to Hain—that the pro-unipotent completion carries a mixed Hodge structure.

In Section 3.1, we review the definition and algebraic properties of iterated integrals. A basic question is when an iterated integral only depends on the homotopy class of a path relative to its endpoints. By relating the parallel transport of connections on the trivial bundle to iterated integrals, we answer the question in length two.

3.1. Iterated integrals and parallel transport. Our presentation follows closely Hain's survey [**Hai87a**]. Other nice references are Cartier's Bourbaki seminar [**Car88**] and Brown's notes [**Bro13b**].

3.1.1. *The fundamental groupoid.* Let M be a connected differentiable manifold. We say that a continuous function $\gamma\colon [0,1] \to M$ is *piecewise smooth* if there is a partition $0 = a_0 < a_1 < \ldots < a_{n+1} = 1$ of the unit interval such that the restriction of γ to each $[a_i, a_{i+1}]$ is smooth, meaning that it can be extended to a smooth function on an open neighborhood of $[a_i, a_{i+1}]$. Similarly, a continuous map $F\colon [0,1]^2 \to M$ is said to be piecewise smooth if there exists a finite polyhedral decomposition $[0,1]^2 = \bigcup_i C_i$ such that all the restrictions $F|_{C_i}$ are smooth, in the sense that they extend to a smooth function on an open neighbourhood of C_i.

We call a continuous piecewise smooth map from $[0,1]$ to M simply a *path* (see Remark 3.8 below), and denote the space of paths by

$$\mathcal{P}(M) = \{\gamma\colon [0,1] \to M \mid \gamma \text{ continuous and piecewise smooth}\}.$$

Given two points x and y in M, the subspace of $\mathcal{P}(M)$ consisting of paths from x to y will be denoted by

$${}_y\mathcal{P}(M)_x = \{\gamma \in \mathcal{P}(M) \mid \gamma(0) = x,\ \gamma(1) = y\}.$$

Definition 3.1. Two paths $\gamma_1, \gamma_2 \in {}_y\mathcal{P}(M)_x$ are said to be *homotopic* if there exists a continuous piecewise smooth function

$$\begin{array}{rccc} F\colon & [0,1]^2 & \longrightarrow & M \\ & (t,s) & \longmapsto & F(t,s) \end{array}$$

such that:

$$\begin{array}{lll} F(t,0) = \gamma_1(t), & F(t,1) = \gamma_2(t), & t \in [0,1], \\ F(0,s) = x, & F(1,s) = y, & s \in [0,1]. \end{array} \tag{3.2}$$

In other words, F is a continuous family of paths

$$\begin{array}{rccc} f_s\colon & [0,1] & \longrightarrow & M, \\ & t & \longmapsto & f_s(t) = F(t,s), \end{array}$$

parameterized by $s \in [0,1]$, that interpolates between γ_1 and γ_2 while keeping the end points fixed (see Figure 11).

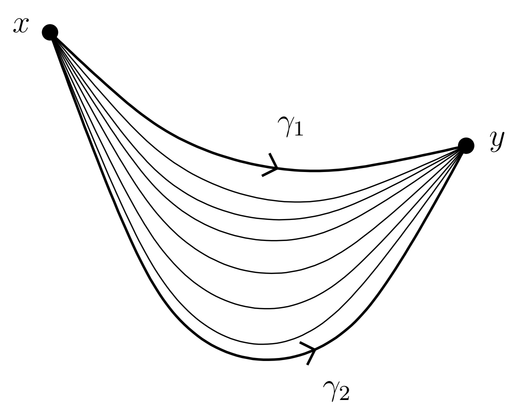


Figure 11. A homotopy between two paths

It is straightforward to check that "being homotopic" defines an equivalence relation $\sim$ on ${}_y\mathcal{P}(M)_x$. We write

$$\pi_1(M; y, x) = \{\gamma \in {}_y\mathcal{P}(M)_x\}/\sim$$

for the set of equivalence classes. When the two endpoints agree, we will abbreviate this notation to $\pi_1(M, x)$.

Note that there are is a *reversal of paths* operation

$$\begin{array}{ccc} {}_y\mathcal{P}(M)_x & \longrightarrow & {}_x\mathcal{P}(M)_y \\ \gamma & \longmapsto & \gamma^{-1} \end{array}$$

defined as $\gamma^{-1}(t) = \gamma(1-t)$. Moreover, given a third point z in M, we have a *composition of paths*

$$\begin{array}{ccc} {}_z\mathcal{P}(M)_y \times {}_y\mathcal{P}(M)_x & \longrightarrow & {}_z\mathcal{P}(M)_x, \\ (\gamma_1, \gamma_2) & \longmapsto & \gamma_1\gamma_2 \end{array}$$

given by first going along γ_2, then along γ_1:

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_2(2t) & 0 \leq t \leq \frac{1}{2}, \\ \gamma_1(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{3.3}$$

Both the reversal and the composition of paths are compatible with the homotopy equivalence relation, hence induce operations

$$\pi_1(M; y, x) \longrightarrow \pi_1(M; x, y), \tag{3.4}$$

$$\pi_1(M; z, y) \times \pi_1(M; y, x) \longrightarrow \pi_1(M; z, x), \tag{3.5}$$

which are called "inverse" and "composition" respectively. It is a simple matter of verification to see that (3.5) is associative and that the class of the constant path $\gamma(t) = x$ in $\pi_1(M, x)$ is a neutral element. As such, it will be usually denoted by 1.

The above operations endow $\pi_1(M, x)$ with the structure of a group: the *fundamental group* of M. In general, when we allow the endpoints to be distinct, we only obtain a groupoid. We recall below the definition, which is in fact tailored to study this example.

Definition 3.6. A *groupoid* G is the data of a set G_0 of "objects" and a set G_1 of "arrows", together with the following five operations:

- a *source* map $s\colon G_1 \to G_0$;
- a *target* map $t\colon G_1 \to G_0$;
- a *unit* map $u\colon G_0 \to G_1$ such that $s(u(x)) = t(u(x)) = x$ for all objects $x \in G_0$;
- a *composition* map $m\colon G_1 \,{}_s\!\times_t G_1 \to G_1$ defined on
$$G_1 \,{}_s\!\times_t G_1 = \{(f, g) \in G_1 \times G_1 \mid s(f) = t(g)\}$$
such that $s(m(f, g)) = s(g)$ and $t(m(f, g)) = t(f)$ for all arrows $f, g \in G_1$, and that u is a two-sided unit for m. Moreover, the composition is required to be associative.
- an *inverse* map $i\colon G_1 \to G_1$ such that, for all arrows $f \in G_1$, $s(i(f)) = t(f)$ and $t(i(f)) = s(f)$ and which is a two-sided inverse for the composition.

Equivalently, a groupoid can be viewed as a small category where all morphisms are isomorphisms (see Exercise 3.36).

Example 3.7 (The fundamental groupoid). The *fundamental groupoid* of M is the groupoid where G_0 is the set of points of M and G_1 is the set of homotopy classes of paths in M, that is:

$$G_1 = \coprod_{x, y \in M} \pi_1(M; y, x).$$

The source, the target and the unit are defined in the obvious way, and the inverse and the composition maps are given by (3.4) and (3.5) respectively.

REMARK 3.8. When doing homotopy theory on a differentiable manifold, one can choose to work with continuous or piecewise smooth path. The resulting group or groupoid is the same in both cases. The most convenient thing for our purposes will be to work with piecewise smooth paths because it allows us to make the link with differential forms.

3.1.2. *Homotopy functionals.* We would like to construct functions on the fundamental groupoid of a manifold.

DEFINITION 3.9. A function on $\mathcal{P}(M)$ is called a *homotopy functional* if the image of every element in $\mathcal{P}(M)$ depends only on its homotopy class, hence induces a function on $\pi_1(M;y,x)$ for all $x,y\in M$.

The simplest method to construct homotopy functionals is by means of differential forms, as we now recall. Let k be either the real or the complex numbers. We consider the k-algebra

$$E^*(M,k)=\bigoplus_{p=0}^{\dim M}E^p(M,k)$$

of smooth k-valued differential forms in M. Given $\omega\in E^1(M,k)$ and a path $\gamma\in\mathcal{P}(M)$, the pullback of ω to the interval $[0,1]$ takes the form $\gamma^*\omega=f(t)dt$ for some function f. The *line integral* of ω along γ is defined as

$$\int_\gamma\omega=\int_0^1\gamma^*\omega=\int_0^1 f(t)dt. \tag{3.10}$$

This yields a function

$$\begin{array}{cccc}\int\omega: & \mathcal{P}(M) & \longrightarrow & k\\ & \gamma & \longmapsto & \int_\gamma\omega.\end{array}$$

LEMMA 3.11. *The function $\int\omega$ is a homotopy functional if and only if the 1-form ω is closed.*

PROOF. The result follows easily from Stokes' theorem. First assume that ω is closed, and that we are given two paths γ_1 and γ_2 and a homotopy F between them. Using the conditions (3.2) in the definition of F, we find

$$\int_{\gamma_1}\omega-\int_{\gamma_2}\omega=\int_{[0,1]}\gamma_1^*\omega-\int_{[0,1]}\gamma_2^*\omega=\int_{\partial[0,1]^2}F^*\omega,$$

where $\partial[0,1]^2$ stands for the boundary of the square $[0,1]^2$. Since F is piecewise smooth, there exists a polyhedral decomposition $[0,1]^2=\bigcup_i C_i$ such that $F|_{C_i}$ is smooth. By Stokes' theorem and the commutativity of F^* with the differential,

$$\int_{\partial[0,1]^2}F^*\omega=\sum_i\int_{\partial C_i}F^*\omega=\sum_i\int_{C_i}F^*(d\omega)=0,$$

thus proving that the line integral is a homotopy functional.

Conversely, assume that the 1-form ω is *not* closed. Then we can find a smooth map $f\colon D \to M$ from the unit disc $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ to M such that

$$\int_D f^* d\omega \neq 0.$$

Consider the paths from $x = f(1, 0)$ to $y = f(-1, 0)$ given by

$$\gamma_1(t) = f(\cos(\pi t), \sin(\pi t)), \quad \gamma_2(t) = f(\cos(\pi t), -\sin(\pi t)).$$

They are homotopic through the homotopy

$$F(x, y) = f(\cos(\pi x), (1 - 2y) \sin(\pi x)).$$

On the contrary, another application of Stokes's theorem gives

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial D} f^* \omega = \int_D f^* d\omega \neq 0,$$

which proves that ω being closed is a necessary condition as well. □

Line integrals of closed 1-forms produce, however, only a very special kind of homotopy functionals. Indeed, from (3.10) we get the relations

$$\int_{\gamma_1\gamma_2} \omega = \int_{\gamma_2} \omega + \int_{\gamma_1} \omega, \qquad \int_{\gamma^{-1}} \omega = -\int_{\gamma} \omega,$$

which together imply that, for any pair of loops $\gamma_1, \gamma_2 \in \pi_1(M, x)$, one has:

$$\int_{\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2} \omega = 0. \tag{3.12}$$

Recall that, given a group G, the commutators $[g, h] = g^{-1}h^{-1}gh$ generate a normal subgroup $[G, G]$.

Definition 3.13. The *abelianization* of G is the quotient

$$G^{\mathrm{ab}} = G/[G, G].$$

It is an abelian group satisfying the universal property that any homomorphism from G to an abelian group factors through G^{ab}. In particular, for every closed 1-form ω the homomorphism

$$\int \omega\colon \pi_1(M, x) \longrightarrow k$$

factors through $\pi_1(M, x)^{\mathrm{ab}}$. Now, viewing a loop $\gamma\colon [0, 1] \to M$ as a closed singular 1-chain yields a canonical group homomorphism

$$h\colon \pi_1(M, x) \longrightarrow H_1(M, \mathbb{Z}),$$

which is often called the *Hurewicz map*. The following is a basic result from algebraic topology, see e.g. [**Hat02**, Thm. 2A.1]:

THEOREM 3.14. *The kernel of h consists exactly of the commutator subgroup $[\pi_1(M,x), \pi_1(M,x)]$. Moreover, if M is connected, then h is surjective and thus induces an isomorphism*

$$\pi_1(M,x)^{\mathrm{ab}} \simeq H_1(M,\mathbb{Z}).$$

Summarizing, line integrals of closed 1-forms always factors through homology. Since the fundamental group is a finer invariant than the first homology group, we would like to construct other homotopy functionals which are able to detect the extra information carried by $\pi_1(M,x)$.

3.1.3. *Iterated integrals.* The theory of iterated integrals started with the fundamental observation, due to K.T. Chen [**Che77**], that homotopy functionals given by successive integration of 1-forms can detect elements of $\pi_1(M,x)$ which are trivial in $H_1(M,\mathbb{Z})$.

DEFINITION 3.15. Let $\omega_1, \dots, \omega_r$ be smooth k-valued 1-forms on M. The *iterated integral* of $\omega_1, \dots, \omega_r$ is the function

$$\begin{array}{rccc} \int \omega_1 \cdots \omega_r \colon & \mathcal{P}(M) & \longrightarrow & k \\ & \gamma & \longmapsto & \int_\gamma \omega_1 \cdots \omega_r \end{array} \tag{3.16}$$

defined as follows:

$$\int_\gamma \omega_1 \cdots \omega_r = \int\limits_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r,$$

where $\gamma^* \omega_i = f_i(t)dt$ is the pullback of ω_i to $[0,1]$.

More generally, we will call *iterated integral* any function on $\mathcal{P}(M)$ obtained as a k-linear combination of (3.16) and the constant function 1, which we viewed as an iterated integral of *length* 0. We say that an iterated integral has *length* $\leq s$ if each summand is of the form $\int \omega_1 \cdots \omega_r$ with $r \leq s$.

REMARK 3.17. Here is an explanation of the term "iterated integral" after [**Del13**, p.3]. Let S be the operator that transforms a 1-form η on the interval $[0,1]$ into the function $S[\eta](t) = \int_0^t \eta$. To obtain the iterated integral we apply S to $\gamma^*\omega_r$, then multiply the resulting function by $\gamma^*\omega_{r-1}$, apply S again, multiply by $\gamma^*\omega_{r-2}$, etc., and finally evaluate at $t = 1$:

$$\int_\gamma \omega_1 \cdots \omega_r = S[\gamma^*\omega_1 \cdot S[\gamma^*\omega_2 \cdots S[\gamma^*\omega_r] \cdots](1).$$

Observe that we have already encountered iterated integrals in Chapter 1: the integral representations of multiple zeta values (Theorem 1.108) and polylogarithms (Theorem 1.117) are both examples of iterated integrals.

3.1.4. *Basic properties of iterated integrals.* The first important property is that iterated integrals are functorial and independent of the parametrization of the path. The proof is left to the reader (see Exercise 3.41).

PROPOSITION 3.18 (Functoriality). *Let $f\colon N \to M$ be a smooth map of differentiable manifolds. If $\gamma \in \mathcal{P}(N)$ and $\omega_1, \dots, \omega_r \in E^1(M,k)$, then*

$$\int_\gamma f^*\omega_1 \cdots f^*\omega_r = \int_{f\circ\gamma} \omega_1 \cdots \omega_r.$$

In particular, the iterated integral $\int_\gamma \omega_1 \cdots \omega_r$ does not depend on the choice of parametrization of the path γ.

We now prove the basic algebraic properties of iterated integrals, which are formulas for the reversal and composition of paths, as well as for the product of two iterated integrals.

THEOREM 3.19. *Let $\omega_1, \dots, \omega_{r+s}$ be smooth k-valued 1-forms on M and let $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths in M such that $\gamma_2(1) = \gamma_1(0)$. Then the following three equalities hold:*

$$\int_\gamma \omega_1 \cdots \omega_r = (-1)^r \int_{\gamma^{-1}} \omega_r \cdots \omega_1, \tag{3.20}$$

$$\int_{\gamma_1\gamma_2} \omega_1 \cdots \omega_r = \sum_{i=0}^{n} \int_{\gamma_1} \omega_1 \cdots \omega_i \int_{\gamma_2} \omega_{i+1} \cdots \omega_n, \tag{3.21}$$

$$\int_\gamma \omega_1 \cdots \omega_r \int_\gamma \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma \in ⧢(r,s)} \int_\gamma \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}. \tag{3.22}$$

In the last identity, the sum runs over the subset $⧢(r,s)$ of the symmetric group $\mathfrak{S}_{r+s}$ consisting of shuffles of type (r,s), as in Definition 1.120.

PROOF. The first identity (3.20) is a simple computation using that $\gamma^*\omega_i = f_i(t)dt$ implies $(\gamma^{-1})^*\omega_i = -f_i(1-t)dt$, hence

$$\begin{aligned}\int_{\gamma^{-1}} \omega_r \cdots \omega_1 &= (-1)^r \int\limits_{1\geq t_1 \geq \cdots \geq t_r \geq 0} f_r(1-t_1)\cdots f_1(1-t_r)dt_1\cdots dt_r \\ &= (-1)^r \int\limits_{1\geq u_1 \geq \cdots \geq u_r \geq 0} f_r(u_r)\cdots f_1(u_1)du_1\cdots du_r \\ &= (-1)^r \int_\gamma \omega_1 \cdots \omega_r.\end{aligned}$$

To pass from the first line to the second we made the change of variables $u_i = 1 - t_{r-i+1}$, whose Jacobian has absolute value 1.

We next prove (3.21). If one writes

$$(\gamma_1\gamma_2)^*\omega_i = f_i(t)dt, \quad \gamma_1^*\omega_i = g_i(t)dt, \quad \gamma_2^*\omega_i = h_i(t)dt,$$

then the three above functions are related by

$$f_i(t) = \begin{cases} 2h_i(2t) & 0 \le t \le \frac{1}{2}. \\ 2g_i(2t-1) & \frac{1}{2} \le t \le 1. \end{cases} \tag{3.23}$$

We decompose the domain of integration as a union $\Delta^r = \bigcup_{i=0}^r C_i$, where

$$C_i = \{(t_1, \dots, t_r) \in \mathbb{R}^r \mid 1 \ge t_1 \ge \cdots \ge t_i \ge \tfrac{1}{2} \ge t_{i+1} \cdots \ge t_r \ge 0\}.$$

Observe that $C_i \simeq \Delta^i \times \Delta^{r-i}$, as Figure 12 shows in the case $r = 2$.

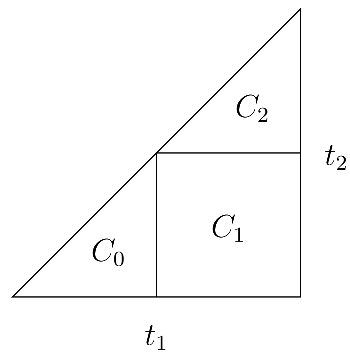


FIGURE 12. The decomposition $\Delta^2 = C_0 \cup C_1 \cup C_2$

Now equation (3.21) follows from the computation

$$\int_{C_i} f_1(t_1) \cdots f_r(t_t) dt_1 \cdots dt_r = \int_{\substack{1 \ge t_1 \ge \dots \ge t_i \ge 1/2 \\ 1/2 \ge t_{i+1} \ge \cdots \ge t_r \ge 0}} f_1(t_1) \cdots f_r(t_t) dt_1 \cdots dt_r$$

$$= \frac{2^r}{2^r} \int_{\substack{1 \ge u_1 \ge \cdots \ge u_i \ge 0 \\ 1 \ge u_{i+1} \ge \cdots \ge u_r \ge 0}} g_1(u_1) \cdots g_i(u_i) h_{i+1}(u_{i+1}) h_r(u_r) du_1 \cdots du_r$$

$$= \int_{\gamma_1} \omega_1 \cdots \omega_i \int_{\gamma_2} \omega_{i+1} \cdots \omega_r,$$

together with the fact that the overlaps of the C_i do not contribute to the integral because they all have codimension at least 2. The second equality is obtained by the change of variables

$$u_j = \begin{cases} 2t_j - 1 & j \le i, \\ 2t_j & j > i. \end{cases}$$

The 2^r in the numerator comes from equation (3.23), whereas the 2^r in the denominator is the Jacobian of the change of variables.

Finally, the formula (3.22) is a consequence of the decomposition

$$\varDelta^r \times \varDelta^s = \bigcup_{\sigma \in \sqcup\!\sqcup(r,s)} \{(t_1, \dots, t_{r+s}) \mid 1 \ge t_{\sigma^{-1}(1)} \ge \cdots \ge t_{\sigma^{-1}(r+s)} \ge 0\},$$

that was already used in Proposition 1.123. □

3.1.5. *When are iterated integrals homotopy functionals?* We have seen that iterated integrals do not depend on the parametrization of the path (Proposition 3.18). However, even when all the ω_i are closed, they do not always give rise to homotopy functionals, as the example below shows:

EXAMPLE 3.24. Take $M = \mathbb{R}^2$ with the standard coordinates x and y. Let $a, b > 0$ be real numbers and consider the path $\gamma_{a,b}\colon [0,1] \to \mathbb{R}^2$ from $(0,0)$ to $(1,1)$ given by $\gamma_{a,b}(t) = (t^a, t^b)$. Let $\omega_1 = dx$ and $\omega_2 = dy$. Then

$$\gamma_{a,b}^* \omega_1 = at^{a-1}dt, \qquad \gamma_{a,b}^* \omega_2 = bt^{b-1}dt,$$

so one has the iterated integral

$$\int_{\gamma_{r,s}} \omega_1\omega_2 = \int_0^1 \left(at_1^{a-1} \int_0^{t_1} bt_2^{b-1}dt_2 \right) dt_1 = \frac{a}{a+b},$$

which obviously depends on the choice of a and b. However, all the paths $\gamma_{a,b}$ are homotopic to each other!

A natural question is thus when an iterated integral is invariant under homotopy. Theorem 3.135 will give a complete solution to this problem in terms of a construction called the *bar complex*. For the moment, we content ourselves with a partial answer by linking iterated integrals to connections on trivial bundles through the notion of parallel transport.

3.1.6. *Iterated integrals and connections on trivial bundles.* We continue writing k for either the real or the complex numbers. Let

$$V = k^n \times M$$

be the trivial rank n vector bundle over M. Recall that sections of V are functions $x\colon M \to k^n$. We denote by $C^\infty(V)$ the space of all smooth sections.

DEFINITION 3.25. A *connection* on V is a k-linear map

$$\nabla\colon C^\infty(V) \to C^\infty(V) \otimes_{C^\infty(M)} E^1(M,k)$$

which satisfies the Leibniz rule

$$\nabla(fx) = x \otimes df + f\nabla x$$

for each smooth function $f \in C^\infty(M)$ and each smooth section $x \in C^\infty(V)$.

A connection ∇ canonically extends to a k-linear map on p-forms, still denoted by ∇, as follows:

$$\begin{aligned} C^\infty(V) \otimes_{C^\infty(M)} E^p(M,k) &\longrightarrow C^\infty(V) \otimes_{C^\infty(M)} E^{p+1}(M,k) \\ x \otimes \eta &\longmapsto x \otimes d\eta + \nabla(x) \wedge \eta \end{aligned}$$

The operator $\nabla^2 = \nabla \circ \nabla$ is called the *curvature* and one says that the connection ∇ is *flat* (or integrable) if ∇^2 vanishes.

We call *global canonical frame* of V the tuple $e = (e_1, \dots, e_n)$ consisting of the constant functions $e_i\colon M \to k^n$ with value the i-th standard basis vector $(0, \dots, 1, \dots, 0)$. By virtue of the Leibniz rule, the connection ∇ is determined by the image of the global canonical frame. Write

$$\nabla e_j = \sum_{i=1}^{n} e_i \otimes \eta_{ij}, \qquad j = 1, \dots, n$$

with $\eta_{ij} \in E^1(M, k)$. The matrix

$$\omega = (\eta_{ij}) \in E^1(M, k) \otimes_{C^\infty(M)} \mathrm{End}(V) = E^1(M, k) \otimes_k \mathrm{End}(k^n),$$

whose entries are smooth k-valued 1-forms on M, is called the matrix of the connection in the global canonical frame e.

Seeing a section $x\colon M \to k^n$ as a column vector of smooth functions and invoking the Leibniz rule again, the connection is given by

$$\nabla x = dx + \omega x.$$

From this one easily computes the curvature:

$$\begin{aligned} \nabla^2 x &= \nabla(dx + \omega x) \\ &= (d^2 x + d(\omega x) + \omega dx + \omega \wedge \omega x) \\ &= (d\omega + \omega \wedge \omega)x, \end{aligned}$$

where $\omega \wedge \omega$ stands for the product of matrices of 1-forms induced by the usual wedge product. In explicit terms, if $\omega = \sum M_i \eta_i$ with $\eta_i \in E^1(M, k)$ and $M_i \in \mathrm{GL}_n(k)$, then

$$\omega \wedge \omega = \tfrac{1}{2} \sum_{i,j} [M_i, M_j] \eta_i \wedge \eta_j,$$

where we have used that wedge products anti-commute. The matrix $R = d\omega + \omega \wedge \omega$ is called the curvature matrix of ∇.

Any rank n vector bundle has an associated principal bundle $\mathrm{GL}(V)$ with group $\mathrm{GL}_n(k)$. Since V is trivial the same is true for $\mathrm{GL}(V)$, hence $\mathrm{GL}(V) \cong \mathrm{GL}_n(k) \times M$. A connection ∇ on V lifts to a connection on $\mathrm{GL}(V)$ that, in this trivialization, is given by the formula

$$\nabla X = dX + \omega X.$$

3.1.7. *Parallel transport.* Given a path $\gamma\colon [0, 1] \to M$ and a section

$$\begin{aligned} X\colon [0,1] &\to \mathrm{GL}_n(k) \\ t &\mapsto X(t) \end{aligned}$$

of $\mathrm{GL}(V)$ along γ, we say that X is *horizontal* if

$$\nabla X(t) = 0. \tag{3.26}$$

Equation (3.26) is equivalent to the condition $dX(t) = -\gamma^*(\omega)X(t)$. If we write $\gamma^*(\omega) = A(t)dt$, then (3.26) becomes the linear differential equation

$$X'(t) + A(t)X(t) = 0.$$

The *parallel transport function*

$$T\colon \mathcal{P}(M) \to \mathrm{GL}_n(k)$$

associated with the connection ∇ is defined as follows: if $\gamma\colon [0,1] \to M$ is a smooth path, then $T(\gamma) = X(1)$, where $X\colon [0,1] \to \mathrm{GL}_n(k)$ is the unique section along the path $\gamma\colon [0,1] \to M$ that is horizontal with respect to ∇ and has initial value $X(0) = \mathrm{Id}_n$, the identity $n \times n$ matrix.

PROPOSITION 3.27. *Let γ, γ' be smooth paths in M with $\gamma'(1) = \gamma(0)$. Then the following holds:*

(1) $T(\gamma)$ is independent of the parametrization of γ.
(2) $T(\gamma\gamma') = T(\gamma)T(\gamma')$.

Using Proposition 3.27 we can extend the definition of parallel transport to piecewise smooth paths by reparametrizing them as a finite composition of smooth paths.

We now state the main result which relates connections and homotopy functionals. Recall that the connection ∇ is flat if the associated curvature matrix $R = d\omega + \omega \wedge \omega = 0$ is zero.

THEOREM 3.28. *The connection ∇ is flat if and only if the parallel transport function is a homotopy functional, in the sense that each component is a homotopy functional.*

PROOF. See for instance [**DK90**, Theorem 2.2.1]. □

3.1.8. *Parallel transport and iterated integrals.* Using iterated integrals, one can give the following explicit formula for the parallel transport function:

PROPOSITION 3.29. *Let $\nabla = d + \omega$ be a connection on the trivial bundle $k^n \times M \to M$. Then the parallel transport function is given by*

$$T(\gamma) = \mathrm{Id}_n - \int_\gamma \omega + \int_\gamma \omega\omega - \int_\gamma \omega\omega\omega + \dots,$$

where the products in the integrands are formal products of matrices of 1-forms and the iterated integrals are computed componentwise.

PROOF. Let $\gamma^*\omega = A(t)dt$. Then the iterated integrals of formal products of matrices of 1-forms are given by

$$\int_\gamma \underbrace{\omega\omega\cdots\omega}_{r} = \int\limits_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} A(t_1)A(t_2)\cdots A(t_r)dt_1\cdots dt_r.$$

Moreover, the parallel transport function is $T(\gamma) = X(1)$, where $X(t)$ is the unique solution of the differential equation

$$X'(t) + A(t)X(t) = 0 \tag{3.30}$$

with initial condition $X(0) = \mathrm{Id}_n$. Observe that the function $X(t)$ satisfies (3.30) and $X(0) = \mathrm{Id}_n$ if and only if the following integral equation holds

$$X(t) = \mathrm{Id}_n - \int_0^t A(s)X(s)ds. \tag{3.31}$$

We will solve (3.31) by applying the method of Picard-Lindelöf. For this, we define recursively a sequence of approximations to the solution:

$$\begin{aligned} X_0(t) &= \mathrm{Id}_n, \\ X_r(t) &= \mathrm{Id}_n - \int_0^t A(s)X_{r-1}(s)ds, \quad r \geq 1. \end{aligned}$$

We need to show that the sequence $\{X_r(t)\}$ converges. First we prove by induction that, for all $r \geq 1$, one has:

$$X_r(t) - X_{r-1}(t) = (-1)^r \int\limits_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r. \tag{3.32}$$

Indeed, by definition

$$X_1(t) - X_0(t) = -\int_0^t A(s)ds,$$

which settles the case $r = 1$. Assume that (3.32) holds for all indices smaller than r. By the induction hypothesis

$$\begin{aligned} X_r(t) - X_{r-1}(t) &= -\int_0^t A(s)(X_{r-1}(s) - X_{r-2}(s))ds \\ &= -\int_0^t A(s)(-1)^{r-1} \int\limits_{s \geq s_2 \geq \cdots \geq s_r \geq 0} A(s_2) \cdots A(s_r) ds_2 \cdots ds_r ds \\ &= (-1)^r \int\limits_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r. \end{aligned}$$

Using that the volume of the simplex Δ^r is $1/r!$, we deduce that there exists a constant $K > 0$ such that

$$\int\limits_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r = O\left(\frac{K^r}{r!}\right).$$

This estimate proves that $\{X_r(t)\}$ is a Cauchy sequence and that its limit is given by the convergent series

$$X_\infty(t) = \sum_{r \geq 0} (-1)^r \int\limits_{t \geq s_1 \geq \cdots \geq s_r \geq 0} A(s_1) \cdots A(s_r) ds_1 \cdots ds_r.$$

Clearly, $X_\infty(0) = \mathrm{Id}_n$, and a telescopic argument shows that $X_\infty(t)$ satisfies the differential equation (3.30). Therefore,

$$T(\gamma) = X_\infty(1) = \mathrm{Id}_n - \int_\gamma \omega + \int_\gamma \omega\omega - \dots \qquad \square$$

The entries of the parallel transport matrix involve a priori infinite series and, therefore, are not iterated integrals according to Definition 3.15. On the contrary, when ω is a nilpotent matrix, all the entries are finite sums. One can then combine Theorem 3.28 and Proposition 3.29 to give examples of iterated integrals which are homotopy functionals.

EXAMPLE 3.33. A strictly upper triangular matrix $A(t)$ is nilpotent, so there exists $r_0 \geq 1$ such that $A(s_1)\dots A(s_{r_0}) = 0$. In this case, the parallel transport function reduces to an iterated integral:

$$T = 1 - \int \omega + \dots + (-1)^{r_0 - 1} \int \underbrace{\omega\omega\cdots\omega}_{r_0 - 1}.$$

For instance, when

$$\omega = \begin{pmatrix} 0 & \omega_1 & \omega_{12} \\ 0 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix},$$

the parallel transport function is given by

$$T = \begin{pmatrix} 1 & -\int \omega_1 & \int \omega_1\omega_2 - \int \omega_{12} \\ 0 & 1 & -\int \omega_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and the curvature of the connection is equal to

$$d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & d\omega_1 & \omega_1 \wedge \omega_2 + d\omega_{12} \\ 0 & 0 & d\omega_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\nabla = d + \omega$ is flat if and only if the following two equalities hold

$$d\omega_1 = d\omega_2 = 0, \qquad d\omega_{12} + \omega_1 \wedge \omega_2 = 0. \tag{3.34}$$

It follows that the iterated integral $\int \omega_1\omega_2 - \int \omega_{12}$ is a homotopy functional if and only if the conditions (3.34) are satisfied.

More generally, one has the following result [**Hai87a**, Prop. 3.1]:

PROPOSITION 3.35. *Let $\omega, \omega_1, \dots, \omega_r$ be smooth k-valued 1-forms on M. Assume that all the ω_i are closed. An iterated integral of length two*

$$\sum_{1 \leq i,j \leq r} a_{ij} \int \omega_i\omega_j - \int \omega$$

is a homotopy functional if and only if $d\omega + \sum_{1 \leq i,j \leq r} a_{ij}\omega_1 \wedge \omega_j = 0$.

We can generalize the previous example to nilpotent flat connections to obtain plenty of iterated integrals that are homotopy functionals. Note that nilpotent flat connections define unipotent local systems, which are nothing else but finite-dimensional unipotent representations of the fundamental group. The next two sections are devoted to detailing this relation.

⋆⋆⋆

Exercise 3.36 (Groupoids as categories). Let $\mathcal{C}$ be a small category in which all morphisms are isomorphisms. Show that $\mathcal{C}$ yields a groupoid in the sense of Definition 3.6. Conversely, given a groupoid, construct such a category. Note that groups correspond to the case where the set of objects consists of a single element.

Exercise 3.37 (Integration by parts). Let $\omega_1, \dots, \omega_r$ be smooth k-valued 1-forms on a differentiable manifold M and f a smooth function. Prove that the following three equalities hold for any path $\gamma \in \mathcal{P}(M)$:

$$\int_\gamma df\omega_1 \cdots \omega_r = (f \circ \gamma)(1) \int_\gamma \omega_1 \cdots \omega_r - \int_\gamma (f\omega_1)\omega_2 \cdots \omega_r, \tag{3.38}$$

$$\int_\gamma \omega_1 \cdots \omega_{i-1} df \omega_i \cdots \omega_r = \int_\gamma \omega_1 \cdots (f\omega_{i-1})\omega_i \cdots \omega_r - \int_\gamma \omega_1 \cdots \omega_{i-1}(f\omega_i)\omega_{i+1} \cdots \omega_r, \tag{3.39}$$

$$\int_\gamma \omega_1 \cdots \omega_r df = \int_\gamma \omega_1 \cdots \omega_{r-1}(f\omega_r) - (f \circ \gamma)(0) \int_\gamma \omega_1 \cdots \omega_r. \tag{3.40}$$

Exercise 3.41. Prove Proposition 3.18.

Exercise 3.42. As we have seen in Example 3.24, the iterated integral of the 1-forms $\omega_1 = dx$ and $\omega_2 = dy$ on $\mathbb{R}^2$ is not a homotopy functional. According to Proposition 3.35, this is explained by the fact that $\omega_1 \wedge \omega_2$ does not vanish. Find a 1-form ω_{12} such that $d\omega_{12} + \omega_1 \wedge \omega_2 = 0$ and check that the value of the iterated integral

$$\int \omega_1\omega_2 - \int \omega_{12} \colon \mathcal{P}(\mathbb{R}^2) \to \mathbb{R}$$

on the paths $\gamma_{a,b}$ from Example 3.24 is now independent of a and b.

Exercise 3.43 (Another proof of (3.20) and (3.21)). Let $\omega_1, \dots, \omega_r$ be 1-forms on a differentiable manifold M. Consider the connection on the rank $(r+1)$ trivial bundle $\nabla = d + d\omega$ given by the matrix

$$\omega = \begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 \\ 0 & 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \omega_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Show that the parallel transport associated with ∇ is the $r+1$ by $r+1$ matrix $T = (T_{ij})$ with entries

$$T_{ij} = \begin{cases} \int \omega_i \cdots \omega_{j-1} & i < j, \\ 1 & i = j, \\ 0 & i > j. \end{cases}$$

Using that $T(\gamma_1\gamma_2) = T(\gamma_1)T(\gamma_2)$ by Proposition 3.27, deduce from this computation another proof of formulas (3.20) and (3.21).

3.2. Affine group schemes, Lie algebras and Hopf algebras. In this section, we recall the definition of affine group schemes and of two intimately related algebraic structures: Lie and Hopf algebras. The book [**Wat79**] is an excellent reference for readers unfamiliar with these notions. We also recommend [**Car07**] as an introduction to Hopf algebras.

Throughout, we fix a field k of characteristic zero (the field of coefficients), that in the applications will always be equal to $\mathbb{Q}$. All undecorated cartesian and tensor products are assumed to be over k.

3.2.1. *Affine group schemes.*

Definition 3.44. Let A be a commutative k-algebra. The corresponding affine k-scheme $G = \operatorname{Spec}(A)$ is said to be a *group scheme* if it is endowed with algebraic operations

$$\begin{aligned} \mu &\colon G \times G \to G \text{ (product)}, \\ e &\colon \operatorname{Spec}(k) \to G \text{ (unit)}, \\ \iota &\colon G \to G \text{ (inverse)}, \end{aligned}$$

satisfying the usual axioms of a group, which are expressed by the commutativity of the following three diagrams:

(1) *Associativity:*

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G. \end{tikzcd}$$

(2) *Unit:*

$$\begin{tikzcd}
G \times \operatorname{Spec}(k) \arrow[r, "\operatorname{Id} \times e"] \arrow[dr, "\mathrm{pr}_1"'] & G \times G \arrow[d, "\mu"] & \operatorname{Spec}(k) \times G \arrow[l, "e \times \operatorname{Id}"'] \arrow[dl, "\mathrm{pr}_2"] \\
& G &
\end{tikzcd}$$

(3) *Inverse:*

$$\begin{tikzcd}
& G \times G \arrow[dr, "\mu"] & \\
G \arrow[ur, "\operatorname{Id} \times \iota"] \arrow[r, "\pi"] \arrow[dr, "\iota \times \operatorname{Id}"'] & \operatorname{Spec} k \arrow[r, "e"] & G, \\
& G \times G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

where π denotes the structural map of G as a k-scheme.

If the algebra A is finitely generated, we say that G is *algebraic*. We will see below (Lemma 3.48) that every affine group scheme is in fact a projective limit of algebraic affine group schemes.

A group scheme over k defines a functor between the categories of commutative k-algebras and abstract groups. Namely, given $G = \operatorname{Spec}(A)$, one considers the functor:

$$R \mapsto G(R) = \operatorname{Hom}_{k\text{-alg}}(A, R).$$

Conversely, we will say that a functor F from commutative k-algebras to groups is *representable* if there exist an affine group scheme G and a natural isomorphism of functors between F and G.

3.2.2. *Hopf algebras.* Recall that the category of affine schemes over k is equivalent to the category of commutative k-algebras through the contravariant functors

$$A \mapsto \operatorname{Spec}(A), \qquad G \mapsto \mathcal{O}(G),$$

where $\mathcal{O}(G)$ is the ring of regular functions on G. Thus the defining properties of a group scheme can be transferred to the corresponding algebra, yielding the concept of a Hopf algebra.

Definition 3.45. Let H be an associative (not necessarily commutative) k-algebra. Let $\nabla \colon H \otimes H \to H$ be the product of H and $\eta \colon k \to H$ the unit.

(1) We say that H is a *bialgebra* if it is provided with two morphisms of algebras

$$\Delta \colon H \to H \otimes H \text{ (coproduct)},$$
$$\epsilon \colon H \to k \text{ (counit)},$$

such that the following diagrams commute:

(a) *Coassociativity.*

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\mathrm{Id} \otimes \Delta"] \\ H \otimes H \arrow[r, "\Delta \otimes \mathrm{Id}"] & H \otimes H \otimes H. \end{tikzcd}$$

(b) *Counit.*

$$\begin{tikzcd} H \otimes k & H \otimes H \arrow[l, "\mathrm{Id} \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{Id}"] & k \otimes H \\ & H \arrow[ul] \arrow[u, "\Delta"] \arrow[ur] & \end{tikzcd},$$

where the maps from H to $H \otimes k$ and $k \otimes H$ are the natural inclusions (which are isomorphisms).

(2) A bialgebra H is said to be a *Hopf algebra* if it is further equipped with a morphism of algebras

$$S \colon H \to H \text{ (antipode)}$$

such that the following diagram commutes:

(c) *Antipode.*

$$\begin{tikzcd} & H \otimes H \arrow[rr, "S \otimes \mathrm{Id}"] & & H \otimes H \arrow[dr, "\nabla"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & H. \\ & H \otimes H \arrow[rr, "\mathrm{Id} \otimes S"'] & & H \otimes H \arrow[ur, "\nabla"'] & \end{tikzcd}$$

(3) A bialgebra H is called *commutative* if the product is commutative, and *cocommutative* if the coproduct satisfies $\Delta = \tau \circ \Delta$, where $\tau \colon H \otimes H \to H \otimes H$ is the flip of the factors.

Remarks 3.46.

(1) This definition is not symmetric in the algebra and coalgebra structures of H. Note also that many compatibilities between the product, the coproduct, the unit, the counit and the antipode are hidden in the statement that Δ, ϵ and S are morphisms of algebras.

(2) A bialgebra does not always admit an antipode, see Exercise 3.76 for an example.

Clearly, the notions of coproduct, counit and antipode at the level of algebras give rise to the dual notions of product, unit and inverse at the level of spectra. We immediately obtain the following result:

PROPOSITION 3.47. *The assignment* $A \mapsto \operatorname{Spec}(A)$ *is a contravariant equivalence between the category of commutative Hopf* k*-algebras and the category of affine group schemes over* k*. The quasi-inverse equivalence is given by* $G \mapsto \mathcal{O}(G)$*. Moreover, the group scheme* G *is commutative if and only if the Hopf algebra* $\mathcal{O}(G)$ *is cocommutative.*

By way of illustration, we show how to use this correspondence to prove the promised result that affine group schemes are pro-algebraic:

LEMMA 3.48. *Every Hopf algebra is a directed union of Hopf subalgebras which are finitely generated* k*-algebras. Therefore, every affine group scheme is a projective limit of algebraic affine group schemes.*

PROOF. Let H be a Hopf algebra and $x \in H$. It suffices to show that x is contained in a finitely generated Hopf subalgebra of H. Choose a basis $\{h_i\}$ of H and write $\Delta(x) = \sum_i x_i \otimes h_i$, where only finitely many x_i are non-zero. Let $V \subseteq H$ be the vector subspace spanned by x and the x_i. We claim that $\Delta(V) \subseteq V \otimes H$, which amounts of course to saying that $\Delta(x_i) \in V \otimes H$ for all i. Indeed, if one writes $\Delta(h_i) = \sum_{j,\ell} a_{ij\ell} h_j \otimes h_\ell$ with $a_{ij\ell} \in k$, then

$$\begin{aligned}\sum_i \Delta(x_i) \otimes h_i &= (\Delta \otimes \mathrm{Id})\Delta(x) \\ &= (\mathrm{Id} \otimes \Delta)\Delta(x) \\ &= \sum_{i,j,\ell} x_i \otimes a_{ij\ell} h_j \otimes h_\ell\end{aligned}$$

by the associativity of the coproduct. Comparing the coefficients of h_ℓ yields $\Delta(x_\ell) = \sum_{i,j} x_i \otimes a_{ij\ell} h_j \in V \otimes H$, as we wanted. Now let $\{v_i\}$ be a basis of V and write $\Delta(v_j) = \sum_i v_i \otimes h_{ij}$ with $h_{ij} \in H$. By Exercise 3.75, it follows that $\Delta(h_{ij}) = \sum_\ell h_{i\ell} \otimes h_{\ell j}$, hence the vector space U generated by $\{v_i\}$ and $\{h_{ij}\}$ satisfies $\Delta(U) \subseteq U \otimes U$. If W is the vector space spanned by U and $S(U)$, then $\Delta(W) \subseteq W \otimes W$ and $S(W) \subseteq W$ using Exercise 3.75 again. Finally, let A be the subalgebra of H generated by W. Since Δ and S are morphisms of algebras, we also have $\Delta(A) \subseteq A \otimes A$ and $S(A) \subseteq A$. It is thus a finitely generated Hopf subalgebra of H containing x. □

3.2.3. *Comodules and Hopf modules.*

DEFINITION 3.49. Let H be a coalgebra over k. A *right comodule* over H is a k-vector space V, together with a coproduct

$$\Delta \colon V \longrightarrow V \otimes H$$

satisfying the following conditions:

(1) (associativity) $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$;
(2) (compatibility with the counit) $(\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id}$ once we identify V with $V \otimes k$.

Left comodules are defined in a similar way.

Definition 3.50. Let H be a commutative Hopf algebra. A *left Hopf module* is a vector space V that is a module over the algebra structure of H and a left comodule over its coalgebra structure. Moreover, both structures are compatible in the sense that the equality

$$\Delta(hv) = \Delta(h)\Delta(v)$$

holds for all $h \in H$ and for all $v \in V$.

3.2.4. *Graded Hopf algebras.*

Definition 3.51.

(1) A bialgebra H is said to be *graded* if the underlying k-algebra has a direct sum decomposition

$$H = \bigoplus_{n \geq 0} H_n$$

compatible with the operations in the sense that, for all $p, q, n \geq 0$,

$$\nabla(H_p \otimes H_q) \subseteq H_{p+q}, \quad \Delta H_n \subseteq \bigoplus_{i+j=n} H_i \otimes H_j$$

If, moreover, $H_0 = k$ we say that H is *connected*.

(2) A *graded Hopf algebra* is a Hopf algebra such that the underlying bialgebra is graded and the antipode satisfies $SH_n \subseteq H_n$.

One advantage of working with graded connected bialgebras is that they automatically admit a unique antipode turning them into (graded) Hopf algebras (see Exercise 3.77).

3.2.5. *Examples.* In this paragraph, we give a few examples of affine group schemes and their corresponding Hopf algebras. Of particular interest for the sequel is the Hoffman algebra from Example 3.56.

Examples 3.52.

(1) The *trivial* group scheme is $\mathrm{Spec}(k)$ with all operations equal to the identity. The corresponding commutative Hopf algebra is k with all operations equal to the identity once we identify $k \otimes k$ with k.

(2) The *multiplicative group* $\mathbb{G}_m$. The functor from commutative k-algebras to groups given by $R \mapsto R^\times$ is represented by an affine group scheme $\mathbb{G}_m$. The corresponding Hopf algebra is $k[x, x^{-1}]$, together with the coproduct given by

$$\Delta(x) = x \otimes x, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1},$$

the counit $\epsilon(x) = \epsilon(x^{-1}) = 1$, and the antipode determined by $S(x) = x^{-1}$ and $S(x^{-1}) = x$.

(3) The *additive group* $\mathbb{G}_a$. The functor from commutative k-algebras to groups given by $R \mapsto (R, +)$ is represented by an affine group scheme $\mathbb{G}_a$. The corresponding Hopf algebra is $k[x]$ with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x.$$

(4) The *linear group* GL_n. The functor that, to each commutative k-algebra R, assigns the group $\mathrm{GL}_n(R)$ of invertible n by n matrices with entries in R is representable by an affine group scheme GL_n. The corresponding Hopf algebra is

$$k[t, (x_{ij})_{i,j=1,\dots,n}]/(t\det(x_{ij}) - 1).$$

Recall that this means that the determinant $\det(x_{ij})$, which is a homogeneous polynomial of degree n in the entries x_{ij}, is invertible. Its inverse is the variable t. The coproduct is given by

$$\Delta t = t \otimes t, \qquad \Delta x_{ij} = \sum_{l=1}^{n} x_{il} \otimes x_{lj}. \tag{3.53}$$

The counit is the map

$$\epsilon(x_{ij}) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Finally, the antipode can be expressed using Cramer's rule for the inverse of a matrix in terms of cofactors, that is,

$$S(t) = t^{-1}, \qquad S(x_{ij}) = tC_{ji},$$

where C_{ij} is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the i-th row and the j-th column of $(x_{\ell m})_{\ell,m}$. Observe that C_{ij} is a homogeneous polynomial of degree $n-1$.

(5) Similarly, for every finite-dimensional k-vector space V, the functor

$$R \mapsto \mathrm{GL}(R \otimes V)$$

is representable by an algebraic affine k-group scheme $\mathrm{GL}(V)$. If V has dimension n, the choice of a basis of V induces an isomorphism between $\mathrm{GL}(V)$ and GL_n.

(6) One needs to be cautious when working with infinite-dimensional vector spaces. In fact, given a k-vector space V, the functor

$$R \mapsto R \otimes V$$

is representable if and only if V is finite-dimensional [**GD71**, 9.4.10]. Therefore, the group-valued functor $\mathrm{Aut}(V)$ defined by

$$R \mapsto \mathrm{Aut}_R(R \otimes V)$$

does not define an affine group scheme when V has infinite dimension. In fact, the rule (3.53) from Example (4) above does not define a coproduct in the infinite-dimensional case.

EXAMPLE 3.54. Any Zariski closed subset of GL_n which is stable under the group operations is also an affine group scheme. This includes the *classical algebraic groups* such as

$$\mathrm{SL}_n = \mathrm{Spec}(k[(x_{ij})_{i,j=1,\dots,n}]/(\det(x_{ij})-1)).$$

EXAMPLE 3.55. Let G be a finite group. The group algebra

$$k[G] = \{\sum_{g\in G} a_g g \mid a_g \in k\}$$

carries a structure of Hopf algebra. The product is determined by the group structure of G, that is:

$$\sum_{g\in G} a_g g \cdot \sum_{h\in G} b_h h = \sum_{g,h\in G} a_g b_h gh = \sum_{f\in G}\left(\sum_{g\in G} a_g b_{g^{-1}f}\right) f.$$

The coproduct is given by $\Delta g = g\otimes g$, and the antipode by $S(g) = g^{-1}$. This Hopf algebra is cocommutative but not commutative, unless the group G is abelian.

EXAMPLE 3.56. For the purpose of these notes, the main example will be the Hoffman algebra $\mathfrak{H}$ of Section 1.6. Recall that the underlying vector space of $\mathfrak{H}$ is the vector space $\mathbb{Q}\langle X\rangle$ generated by (non-commutative) words in two letters x_0, x_1. The Hopf algebra structure is given by

(1) *Shuffle product.*

$$x_{\varepsilon_1}\cdots x_{\varepsilon_r} ⧢ x_{\varepsilon_{r+1}}\cdots x_{\varepsilon_{r+s}} = \sum_{\sigma\in ⧢(r,s)} x_{\varepsilon_{\sigma^{-1}(1)}}\cdots x_{\varepsilon_{\sigma^{-1}(p+q)}}.$$

(2) *Unit.* The map $\eta\colon \mathbb{Q}\to\mathfrak{H}$ that sends 1 to the empty word.

(3) *Deconcatenation coproduct.*

$$\Delta x_{\varepsilon_1}\cdots x_{\varepsilon_n} = \sum_{j=0}^{n} x_{\varepsilon_1}\cdots x_{\varepsilon_j}\otimes x_{\varepsilon_{j+1}}\cdots x_{\varepsilon_n}.$$

(4) *Counit.* The map $\epsilon\colon\mathfrak{H}\to\mathbb{Q}$ that sends every non-empty word to 0 and the empty word to 1.

(5) *Antipode.*

$$S(x_{\varepsilon_1}\cdots x_{\varepsilon_n}) = (-1)^n x_{\varepsilon_n}\cdots x_{\varepsilon_1}$$

For convenience, if w is a word on the letters x_0 and x_1, we will also use the notation

$$w^* = S(w). \tag{3.57}$$

Consider the grading of $\mathfrak{H}$ that gives *weight* n to $x_{\varepsilon_1}\cdots x_{\varepsilon_n}$. Since all the above operations respect the weight, $\mathfrak{H}$ is a graded Hopf algebra.

3.2.6. *The dual of a Hopf algebra.* Let H be a Hopf algebra over k. If H is a finite-dimensional k-vector space, then its dual

$$H^\vee = \mathrm{Hom}(H, k)$$

is again equipped with a Hopf algebra structure, whose product is the dual of the coproduct of H, whose coproduct is the dual of the product, and the antipodes of H and $H^\vee$ are dual of each other. In other words, the axioms in Definition 3.45 are *self-dual*. This uses the canonical isomorphism

$$H^\vee \otimes H^\vee \simeq (H \otimes H)^\vee.$$

If H has infinite dimension, these two vector spaces fail to be isomorphic, hence the dual of the product does not give rise to a coproduct but only to what is called a *completed coproduct*. Let us explain why. Let V be an infinite-dimensional k-vector space and write

$$V = \varinjlim_I V_I,$$

where I runs over the directed set of finite-dimensional subspaces of V. Since $\mathrm{Hom}(\cdot, k)$ exchanges inductive and projective limits, the dual of V is

$$V^\vee = \mathrm{Hom}(V, k) = \mathrm{Hom}(\varinjlim_I V_I, k) = \varprojlim_I \mathrm{Hom}(V_I, k) = \varprojlim_I V_I^\vee.$$

Thus, $V^\vee$ has a natural structure of pro-finite-dimensional k-vector space.

Definition 3.58. Given a pro-finite-dimensional k-vector space

$$W = \varprojlim_I W_I,$$

the *completed tensor product* with itself is defined as

$$W \hat{\otimes} W = \varprojlim_I (W_I \otimes W_I).$$

Note that the definition requires a structure of pro-finite-dimensional space. When dealing with the dual of an infinite-dimensional vector space we will tacitly assume the previously described structure.

For infinite-dimensional vector spaces, the notion of tensor product is not self-dual, but the dual notion to the tensor product is the completed tensor product:

$$(V \otimes V)^\vee = V^\vee \hat{\otimes} V^\vee.$$

There is a natural morphism

$$V^\vee \otimes V^\vee \longrightarrow (V \otimes V)^\vee = V^\vee \hat{\otimes} V^\vee,$$

that, in general, is not an isomorphism.

Thus, when we dualize the product $A \otimes A \to A$ of an algebra, we only obtain a morphism

$$A^\vee \longrightarrow (A \otimes A)^\vee = A^\vee \hat{\otimes} A^\vee \tag{3.59}$$

and not necessarily a coproduct $A^\vee \to A^\vee \otimes A^\vee$. A map as in (3.59) is called a *completed coproduct*.

DEFINITION 3.60. A *completed Hopf algebra* A is a pro-finite-dimensional vector space satisfying the analogous properties of a Hopf algebra (Definition 3.45) where all tensor products are replaced by completed tensor products and all the maps are compatible with the pro-finite-dimensional structure. In particular it has a completed coproduct

$$\Delta\colon A \longrightarrow A\hat{\otimes}A,$$

the algebra product $\nabla\colon A\otimes A \longrightarrow A$ factorizes through a *completed product*

$$A\otimes A \longrightarrow A\hat{\otimes}A \xrightarrow{\hat{\nabla}} A.$$

and the antipode S is compatible with the pro-finite-dimensional structure.

The dual of an infinite-dimensional Hopf algebra is a completed Hopf algebra. Typically, we will apply this construction to a connected graded Hopf algebra with finite-dimensional graded pieces. In this case, the notion of completed Hopf algebra can be conveniently written in terms of the topology induced by the augmentation ideal.

EXAMPLE 3.61. Let $A = k[x]$ be the Hopf algebra of polynomials of Example 3.52 (3). Let $y_m \in A^\vee$ be the element determined by $\langle y_m, x^n\rangle = \delta_{n,m}$. Then

$$\begin{aligned}\langle \nabla(y_m\otimes y_n), x^j\rangle &= \langle y_m\otimes y_n, \Delta x^j\rangle\\ &= \langle y_m\otimes y_n, (1\otimes x + x\otimes 1)^j\rangle = \begin{cases}\frac{(m+n)!}{n!m!}, & \text{if } j=n+m,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

From this equation we deduce that $y_m = y_1^m/m!$ and $A^\vee$ is the algebra of formal series on divided powers. Since we are working over a field of characteristic zero, it is isomorphic to the algebra of formal power series. Thus, writing $y = y_1$, as algebra we have

$$A^\vee = k[[y]].$$

One can easily check that the completed coproduct is determined by $\Delta y = 1\otimes y + y\otimes 1$ and the antipode by $S(y) = -y$. In particular

$$\Delta y_m = \sum_{j=0}^{m} y_j\otimes y_{m-j}, \qquad S(y_m) = -y_m.$$

The completed coproduct can not be factored through a true coproduct. Consider the element $\eta = \sum_{n\geq 0} ny_n$. Then

$$\Delta\eta = \sum_{n\geq 0}\sum_{j=0}^{n} ny_j\otimes y_{n-j}.$$

This element does not belong to $A^\vee\otimes A^\vee$. This can be seen as follows. Any element

$$\sum_{i,j\geq 0} a_{i,j}y_i\otimes y_j \in A^\vee\otimes A^\vee$$

satisfies that the rank of the matrix $(a_{i,j})$ is finite. By contrast, the rank of the matrix $(b_{i,j})$ with $b_{i,j} = i + j$ is not bounded.

EXAMPLE 3.62. The dual of the Hopf algebra $\mathfrak{H}$ of Example 3.56 is the space $\mathfrak{H}^\vee = \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ of series on the non-commutative words in two letters e_0, e_1. Given a binary sequence α and an element $\gamma \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, the duality is given by the pairing

$$\langle x_\alpha, \gamma\rangle = \text{coefficient of } e_\alpha \text{ in } \gamma.$$

This duality and the Hopf algebra structure of $\mathfrak{H}$ endows $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ with the structures

(1) *Concatenation product.* The product $\Delta^\vee \colon \mathfrak{H}^\vee \otimes \mathfrak{H}^\vee \to \mathfrak{H}^\vee$ is given by

$$e_{\varepsilon_1} \cdots e_{\varepsilon_r} \cdot e_{\varepsilon_{r+1}} \cdots e_{\varepsilon_{r+s}} = e_{\varepsilon_1} \cdots e_{\varepsilon_{r+s}}.$$

(2) *Unit.* It is the morphism

$$\eta^\vee \colon \mathbb{Q} \longrightarrow \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$$

that sends 1 to the empty word.

(3) *Completed coproduct.* It is the unique morphism of algebras

$$\nabla^\vee \colon \mathfrak{H}^\vee \longrightarrow \mathfrak{H}^\vee \hat{\otimes} \mathfrak{H}^\vee$$

such that

$$\nabla^\vee e_\varepsilon = 1 \otimes e_\varepsilon + e_\varepsilon \otimes 1, \quad \varepsilon = 0, 1.$$

This implies that, for any word w on the alphabet $\{e_0, e_1\}$,

$$\nabla^\vee w = \sum_{w_1, w_2} ⧢(w_1, w_2; w) w_1 \otimes w_2,$$

where the shuffle index $⧢(w_1, w_2; w)$ was introduced in 1.153.

(4) *Counit.* The map

$$\epsilon^\vee \colon \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \longrightarrow \mathbb{Q}$$

sending all non-empty words to 0 and the empty word to 1.

(5) *Dual antipode.* It is given by

$$S^\vee(e_{\varepsilon_1} \cdots e_{\varepsilon_n}) = (-1)^n e_{\varepsilon_n} \cdots e_{\varepsilon_1}.$$

By analogy with (3.57), for a word w in the letters e_0 and e_1, we use the notation

$$w^* = S^\vee(w). \tag{3.63}$$

3.2.7. *Lie algebras.* Another important construction attached to a group scheme is its Lie algebra. The Lie algebra of a differentiable algebraic group is the tangent space at the origin together with an antisymmetric product (the Lie bracket) that reflects the non-commutativity of the group operation.

DEFINITION 3.64. A *Lie algebra* over a field k is a k-vector space L together with a bilinear product

$$[\cdot,\cdot]\colon L\otimes L\to L$$

satisfying the following two conditions:

(i) *Anti-symmetry:* $[a,b]+[b,a]=0$ for all $a,b\in L$.
(ii) *Jacobi identity:* $[[a,b],c]+[[b,c],a]+[[c,a],b]=0$ for $a,b,c\in L$.

If, moreover, the underlying vector space has a grading

$$L=\bigoplus_{n\in\mathbb{Z}}L_n$$

such that $[L_n,L_m]\subseteq L_{n+m}$, we say that L is a *graded Lie algebra.*

The dual notion of a Lie algebra is a Lie coalgebra. We let the reader explore its properties in Exercise 3.80.

In the case of affine group schemes, the Lie algebra can be directly constructed from its Hopf algebra, as we now explain. Let G be an affine group scheme over k, and $A=\mathcal{O}(G)$ the corresponding commutative Hopf algebra. We keep the notation $(\nabla,\eta,\Delta,\epsilon,S)$ from Definition 3.45.

DEFINITION 3.65. The *augmentation ideal* of A is the kernel of the counit map. It will be denoted by $I=\operatorname{Ker}(\epsilon)$.

The augmentation ideal is the maximal ideal of regular functions vanishing at the unit $e=\eta(1)$. Since $\epsilon\circ\eta=\mathrm{Id}_k$ there is a canonical direct sum decomposition $A=k\oplus I$ and a canonical projection $A\to I$.

DEFINITION 3.66. The tangent space of the affine group scheme G at the origin is $L=(I/I^2)^\vee$.

To make L into a Lie algebra, we need a bracket

$$[\cdot,\cdot]\colon L\wedge L\to L.$$

We will first define the dual map. For this we observe that the compatibilities of the coproduct with the unit and the counit imply that, if $f\in I$, then

$$\Delta f-f\otimes 1-1\otimes f\in I\otimes I. \tag{3.67}$$

We now consider the map

$$I\xrightarrow{\Delta}A\otimes A\longrightarrow (I/I^2)\otimes(I/I^2)\longrightarrow (I/I^2)\wedge(I/I^2),$$

where the second map is induced by the projection $A\to I\to I/I^2$ and the third map is the projection from the tensor product to the exterior product.

Using property (3.67), we can see that this map vanishes on I^2. Therefore, we obtain a map

$$d\colon I/I^2 \to \bigwedge^2 (I/I^2). \tag{3.68}$$

By duality, we obtain a map

$$L \wedge L = (I/I^2)^\vee \wedge (I/I^2)^\vee \longrightarrow (I/I^2 \wedge I/I^2)^\vee \xrightarrow{d^\vee} (I/I^2)^\vee = L,$$

that we denote $[\cdot,\cdot]$. Following Exercise 3.81, we have that $(I/I^2, d)$ is a Lie coalgebra.

Definition 3.69. The *Lie coalgebra* associated with the commutative Hopf algebra A is the pair $(I/I^2, d)$. The dual $(L, [\cdot,\cdot])$ is called the *Lie algebra* of G and denoted $\mathrm{Lie}(G)$.

In practice, to compute the Lie algebra of an affine group scheme G, one looks for the elements of $G(k[\varepsilon])$ mapping to the identity in $G(k)$, which is an algebraic characterization of the tangent space at the unit. Here $k[\varepsilon]$ denotes the ring of dual numbers, in which $\varepsilon^2 = 0$.

Examples 3.70.

(1) The group $G = \mathrm{GL}_n$ is an open subscheme of the affine space $\mathbb{A}^{n^2}$, the complement of the determinant hypersurface $\{\det = 0\}$. Thus, the tangent space at the origin can be identified with the space $\mathrm{Mat}_n(k)$ of all $n \times n$ matrices over k and the Lie bracket is just the usual commutator of matrices.

(2) Let $G = \mathrm{SL}_n$. It is the closed subscheme of GL_n defined by the equation $\det = 1$. The Lie algebra of G is a subalgebra of $\mathrm{Lie}(\mathrm{GL}_n)$, hence of $\mathrm{Mat}_n(k)$. To determine it, it suffices to check which matrices of the form $1 + \varepsilon M$, with $\varepsilon^2 = 0$ have determinant 1. Since

$$\det(1 + \varepsilon M) = 1 + \varepsilon \operatorname{Tr}(M),$$

we deduce that $\mathrm{Lie}(\mathrm{SL}_n)$ can be identified with the space of traceless n by n matrices.

3.2.8. *The universal enveloping algebra.* It is sometimes convenient to associate to a Lie algebra an associative algebra containing the same information. This is the universal enveloping algebra.

Definition 3.71. Let $(L, [\cdot,\cdot])$ be a Lie algebra. Its *universal enveloping algebra* is an associative algebra $U(L)$, together with a universal morphism $\iota_L\colon L \to U(L)$ such that

$$\iota_L([a,b]) = \iota_L(a)\iota_L(b) - \iota_L(b)\iota_L(a).$$

By "universal" we mean that, if A is another associative algebra with a map $\iota\colon L \to A$ satisfying $\iota([a,b]) = \iota(a)\iota(b) - \iota(b)\iota(a)$, then there exists a unique map $\varphi\colon U(L) \to A$ such that $\iota = \varphi \circ \iota_L$.

We next recall the construction of the universal enveloping algebra. Let $T(L)$ be the tensor algebra over L. That is

$$T(L) = \bigoplus_{n\geq 0} L^{\otimes n}$$

with the associative product determined by

$$a_1 \otimes \cdots \otimes a_r \cdot a_{r+1} \otimes \cdots \otimes a_{r+s} = a_1 \otimes \cdots \otimes a_{r+s}.$$

If $\dim_k L > 1$ this algebra is non-commutative.

Let $J \subseteq T(L)$ be the two-sided ideal generated by the elements

$$a \otimes b - b \otimes a - [a, b], \quad a, b \in L$$

Then

$$U(L) = T(L)/J,$$

and the map ι_L is the composition $L \to T(L) \to U(L)$.

By Exercise 3.79 below, the algebra $U(L)$ has a coproduct Δ determined by the condition

$$\Delta\iota_L(a) = \iota_L(a) \otimes 1 + 1 \otimes \iota_L(a).$$

In fact, $U(L)$ is a Hopf algebra, whose counit $\eta\colon U(L) \to \mathbb{Q}$ is induced by the zero map $L \to \mathbb{Q}$, and whose antipode is induced characterized by $S(x) = -x$ for all $x \in L$.

The main structure theorem for universal enveloping algebras is

THEOREM 3.72 (Poincaré-Birkhoff-Witt). *The map ι_L is injective. In particular, one can recover L from the universal envelopping algebra $U(L)$, together with the coproduct Δ, as the subspace of primitive elements*

$$a \in L \iff \Delta a = a \otimes 1 + 1 \otimes a.$$

If G is an affine algebraic group and $L = \mathrm{Lie}(G)$, there is a canonical map $U(L) \to \mathcal{O}(G)^\vee$, that, in general, is not an isomorphism.

For the sequel, we also need to introduce the completion of the universal enveloping algebra.

DEFINITION 3.73. The *completed universal enveloping algebra* is the completion $\widehat{U}(L)$ of $U(L)$ with respect to the ideal $\mathrm{Ker}(\eta)$, where η is the counit of $U(L)$.

EXAMPLES 3.74.

(1) Let $G = \mathbb{G}_a$ be the additive group over $\mathbb{Q}$. Then its algebra of functions is the polynomial ring $\mathcal{O}(G) = \mathbb{Q}[x]$ and its Lie algebra is the abelian one-dimensional algebra $L = \mathbb{Q}$. Its universal enveloping algebra is the algebra of polynomials $\mathbb{Q}[y]$, while its completed universal enveloping algebra is the algebra of formal power series $\mathbb{Q}[\![y]\!]$. In this particular case, this last algebra agrees with the dual of $\mathcal{O}(G)$, where the dual of x^n is the divided power $y^n/n!$.

(2) Let $G = \mathbb{G}_m$ be the multiplicative group over $\mathbb{Q}$. Then its algebra of functions is the ring of Laurent polynomials $\mathcal{O}(G) = \mathbb{Q}[x, x^{-1}]$ and its Lie algebra is the abelian one-dimensional algebra $L = \mathbb{Q}$. Its universal enveloping algebra is the algebra of polynomials $\mathbb{Q}[y]$, while its completed universal enveloping algebra is the algebra of formal power series $\mathbb{Q}[\![y]\!]$. In this particular case, this last algebra agrees with the dual of $\mathcal{O}(G)$, where the dual of x^n is the divided power $y^n/n!$.

⋆ ⋆ ⋆

Exercise 3.75. Let H be a Hopf algebra.

(a) Consider a finite-dimensional subvector space V of H satisfying $\Delta(V) \subseteq V \otimes H$. Pick a basis $\{v_i\}$ of V and write $\Delta(v_j) = \sum_i v_i \otimes h_{ij}$. Prove that $\Delta(h_{ij}) = \sum_\ell h_{i\ell} \otimes h_{\ell j}$.

(b) Show that $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$, where τ is the flip of the factors of $H \otimes H$. Concretely, if $\Delta(h) = \sum_i a_i \otimes b_i$, then

$$\Delta(S(h)) = \sum_i S(b_i) \otimes S(a_i).$$

Exercise 3.76 (A bialgebra without antipode). Let $H = k[x]$ be the polynomial algebra in one variable. The coproduct $\Delta(x) = x \otimes x$ and the counit $\epsilon(x) = 1$ endow H with a cocommutative bialgebra structure. Show that H does not have an antipode.

Exercise 3.77 (A connected graded bialgebra has an antipode). Let H be a connected graded bialgebra.

(a) Use the commutativity of diagram (2) in Definition 3.45 to prove that the counit $\epsilon\colon H \to k$ vanishes on H_n for all $n \geq 1$, hence induces an isomorphism $H_0 \simeq k$.

(b) Show that the antipode $S\colon H \to H$ is the unique algebra morphism such that $S_{|H_0} = \mathrm{Id}$ and, if $x \in H_n$ for $n \geq 1$,

$$S(x) = -x - \sum \nabla(S(x') \otimes x''),$$

where the sum runs over all elements x'' appearing in the coproduct $\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$.

Exercise 3.78. Let $\mathfrak{H}$ be the Hoffman algebra.

(a) Verify that the operations described in Example 3.56 endow $\mathfrak{H}$ with a Hopf algebra structure.

(b) Recall that $\mathfrak{H}$ is graded by assigning weight n to $x_{\varepsilon_1} \cdots x_{\varepsilon_n}$. Prove by induction on n that the recipe to compute the antipode presented in Exercise 3.77 yields $S(x_{\varepsilon_1} \cdots x_{\varepsilon_n}) = (-1)^n x_{\varepsilon_n} \cdots x_{\varepsilon_1}$.

EXERCISE 3.79. Consider the coproduct $\Delta\colon T(L) \to T(L)\otimes T(L)$, which is the unique algebra homomorphism determined by the condition $\Delta a = 1\otimes a + a\otimes 1$ for $a \in L$. Show that

$$\Delta J \subset J \otimes T(L) + T(L) \otimes J.$$

EXERCISE 3.80 (Lie coalgebras). We introduce Lie coalgebras.

a) Let L be a finite-dimensional Lie algebra over k. The dual of the Lie bracket $[\cdot,\cdot]\colon L\otimes L \to L$ is a map $d\colon L^\vee \to L^\vee \otimes L^\vee$. Write down the properties dual to the anti-symmetry and the Jacobi identity from Definition 3.64.
b) Define a Lie coalgebra over k as a k-vector space C with a map $d\colon C \to C\wedge C$ such that $d\circ d = 0$ (when d is appropriately extended to $C\wedge C$). Prove that the dual of a Lie coalgebra, not necessarily of finite dimension, is a Lie algebra.

EXERCISE 3.81. In this exercise, we show that I/I^2 is a Lie coalgebra, hence L is a Lie algebra.

(a) Check the property (3.67).
(b) Extend d to an operator

$$d\colon \bigwedge^{n}(I/I^2) \to \bigwedge^{n+1}(I/I^2)$$

by using the Leibniz rule with appropriate signs. Then show that $d^2 = 0$. This implies that I/I^2 is a Lie coalgebra. Deduce from Exercise 3.80 that L is a Lie algebra.

EXERCISE 3.82. Show that, in the Hopf algebra $k[X]$, one has

$$\Delta(X^n) = \sum_{r=0}^{n} \binom{n}{r} X^r \otimes X^{n-r}.$$

EXERCISE 3.83 (The Hopf algebra of rooted trees). In this exercise, we describe the *Hopf algebra of rooted trees* introduced by Connes and Kreimer, in connection with the renormalization of quantum field theories [**CK98**]. Another nice reference is [**Foi**].

We begin with a couple of definitions. A *rooted tree* is an oriented finite graph which is connected and simply connected (in other words, a tree), and has a distinguished vertex with no incoming edges called the root. Continuing the metaphor, the vertices with no outcoming edges are called the leaves. A *rooted forest* is a disjoint union of rooted trees.

Let $\mathcal{H}_R$ be the $\mathbb{Q}$-algebra of polynomials in rooted trees, *i.e.* $\mathcal{H}_R$ is the free commutative $\mathbb{Q}$-algebra with unit generated by (isomorphism classes of) rooted trees. The product of two rooted trees is their disjoint union and the unit is the empty tree 1. As a vector space,

$$\mathcal{H}_R = \mathbb{Q}[\text{rooted forests}].$$

Let t be a rooted tree. An *admissible cut* c of t is the choice of a subset of the edges such that any path from the root to the leaves meets at most one cut. Deleting the edges in c, one gets a rooted forest $W^c(t)$. Among the connected components of $W^c(t)$, there is a unique tree $R^c(t)$ containing the root. The rooted forest consisting of the remaining components will be denoted by $P^c(t)$. Two extremes cases of admissible cuts are the empty cut, for which $R^c(t) = t$ and $P^c(t) = 1$, and the total cut, for which $R^c(t) = 1$ and $P^c(t) = t$. We shall write $\mathrm{Adm}_*(t)$ for the set of non-total, non-empty admissible cuts of t. We define:

$$\Delta t = 1 \otimes t + t \otimes 1 + \sum_{c \in \mathrm{Adm}_*(t)} P^c(t) \otimes R^c(t). \tag{3.84}$$

Since $\mathcal{H}_R$ is the free algebra in rooted trees, (3.84) extends uniquely to a coproduct $\Delta\colon \mathcal{H}_R \to \mathcal{H}_R \otimes \mathcal{H}_R$. Figure 13 contains an example of an admissible cut and the contribution to the coproduct.

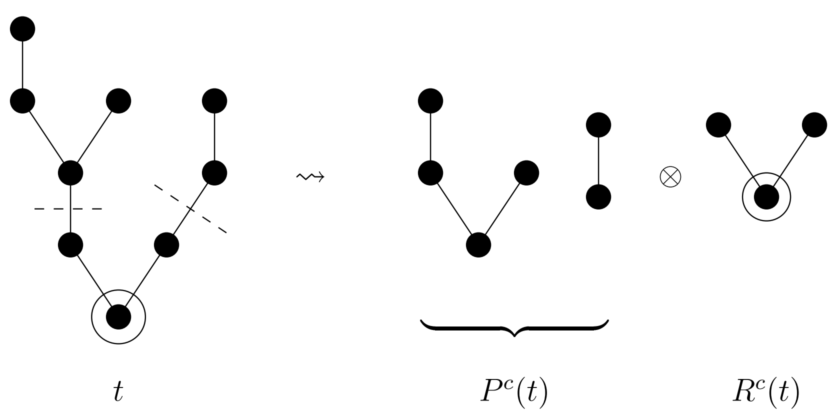


FIGURE 13. Coproduct of rooted trees

The counit is the map $\epsilon\colon \mathcal{H}_R \to \mathbb{Q}$ which sends the empty tree to 1 and everything else to zero.

(1) Prove that Δ and ϵ satisfy the associativity and counit axioms from Definition 3.45. In other words, $\mathcal{H}_R$ is a bialgebra.

(2) For each integer $n \geq 0$, let $\mathcal{H}_R(n) \subseteq \mathcal{H}_R$ be the vector subspace generated by rooted forests with n vertices, so that

$$\mathcal{H}_R = \bigoplus_{n \geq 0} \mathcal{H}_R(n).$$

Observe that $\Delta \mathcal{H}_R(n) \subseteq \bigoplus_{i+j=n} \mathcal{H}_R(i) \otimes \mathcal{H}_R(j)$. Since $\mathcal{H}_R$ is obviously a graded connected algebra, by Exercise 3.77 there is a unique antipode S turning $\mathcal{H}_R$ into a Hopf algebra.

(3) Given a rooted tree t and a cut c, write n_c for the numbers of cut edges in c. Prove that the antipode is given by

$$S(t) = -t - \sum_{c \in \mathrm{Adm}_*(t)} (-1)^{n_c} W_c(t).$$

3.3. The pro-unipotent completion of a group. In this section, we develop some abstract machinery which will be used in the sequel to rephrase the constructions of Section 3.1 in a more conceptual way. We have seen that iterated integrals carry information about the fundamental group of a differentiable manifold. The question we would like to answer is how much of it can be recovered using differential forms. Stated in a vaguer form: what information about the fundamental group is "cohomological", or even "motivic", if we are dealing with algebraic varieties?

Throughout, k will still denote a field of characteristic zero.

3.3.1. *Representations.* We first introduce the notion of representation of an abstract group and an affine group scheme. In the latter case, one needs to be careful because, as explained in Example 3.52 (6), the group-valued functor $\mathrm{Aut}(V)$ is not representable by a scheme when V is an infinite-dimensional vector space.

Definition 3.85. Let Γ be a group. A *k-linear representation* of Γ is a k-vector space V together with a group homomorphism

$$\Gamma \to \mathrm{Aut}_k(V).$$

Let G be an affine group scheme over k. A *linear representation* of G is a k-vector space V together with a natural transformation of group valued functors $G \to \mathrm{Aut}(V)$. This means that for every k-algebra R we have a group homomorphism $G(R) \to \mathrm{Aut}_R(R \otimes V)$ and for each morphism of K-algebras $R \to R'$ a commutative diagram

$$\begin{array}{ccc} G(R) & \longrightarrow & \mathrm{Aut}_R(R \otimes V) \\ \downarrow & & \downarrow \\ G(R') & \longrightarrow & \mathrm{Aut}_{R'}(R' \otimes V). \end{array}$$

Every linear representation of an affine group scheme determines a representation of the group $G(k)$, but the converse is not true. Since we will only work with linear representation, for shorthand we will omit the adjective linear.

In some cases it is more convenient to use the point of view of comodules. For a proof of the next result see, for instance [**Mil12**, Proposition VIII.6.1].

Lemma 3.86. *Let G be an affine group scheme over k, and let V be a k-vector space. There is a natural one-to-one correspondence between the* *linear representations of G on V and the right $\mathcal{O}(G)$-comodule structures on V.*

The first part of the proof of Lemma 3.48 shows the following:

LEMMA 3.87. *Every linear representation of an affine group scheme is a directed union of finite-dimensional subrepresentations.*

REMARK 3.88. Recall from Example 3.52 that, if V is a finite dimensional vector space, then the automorphisms of a vector space V form an affine group scheme $\mathrm{GL}(V)$. It turns out that, to give a finite dimensional representation of G, is equivalent to give a pair consisting of a k-vector space V and a morphism of group schemes $\rho\colon G \to \mathrm{GL}(V)$. Since we will be mainly interested in finite dimensional representations, this is the point of view that we will use the most.

3.3.2. *The abelianization of the fundamental group.* The obvious piece of information that can be recovered via differential forms is the abelianization of the fundamental group. Indeed, recall from Theorem 3.14 that

$$\pi_1(M,x)^{\mathrm{ab}} \simeq H_1(M,\mathbb{Z}),$$

so that, passing to the dual, de Rham's Theorem 2.24 yields an isomorphism

$$H^1_{\mathrm{dR}}(M,\mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(M,x)^{\mathrm{ab}},\mathbb{R}).$$

Moreover, in the case where k is a subfield of $\mathbb{C}$ and $M = X(\mathbb{C})$ is the set of complex points of a smooth variety X over k, we get

$$H^1_{\mathrm{dR}}(X)\otimes\mathbb{C} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(M,x)^{\mathrm{ab}},\mathbb{C})$$

where the left-hand side stands for algebraic de Rham cohomology (see Definition 2.37) and has thus a purely algebraic definition.

However, the abelianization of the fundamental group is a very crude invariant. In fact, the abelianization of a group Γ knows only about the abelian representations of Γ. We should be able to see much more than just the abelianization of the fundamental group using differential forms. A glimpse of this appeared in Section 3.1 where we saw that iterated integrals are related to nilpotent flat connections, that in turn are related to unipotent representations of the fundamental group. In the next paragraphs we elaborate on this idea.

3.3.3. *Unipotent and pro-unipotent groups.* Recall from Lemma 3.48 that every affine group scheme is pro-algebraic.

DEFINITION 3.89. An affine algebraic group (respectively affine group scheme) G over k is called *unipotent* (respectively *pro-unipotent*) if every non-zero representation V of G has a non-zero fixed vector.

REMARK 3.90. In view of Lemma 3.87, it is enough to check that every non-zero finite-dimensional representation has a non-zero fixed vector.

EXAMPLES 3.91.

(1) Let Up_n be the functor that associates to each k-algebra R the group of n by n upper triangular matrices with 1 in the diagonal. This functor is represented by an affine group scheme, still denoted by Up_n. The group Up_n is unipotent. Indeed, let $\rho\colon \mathrm{Up}_n \to \mathrm{GL}(V)$ be a finite-dimensional representation. It follows from the definition of Up_n that $(\rho-1)^n = 0$. Let m be an integer such that $(\rho-1)^m = 0$, but there is an element $g \in \mathrm{Up}_n(k)$ and a vector $v_1 \in V$ with

$$v \coloneqq (\rho(g) - 1)^{m-1} v_1 \neq 0.$$

By construction, $\mathrm{Up}_n \cdot v = v$, showing that Up_n is unipotent. If V is a k-vector space of dimension n, then a choice of a basis of V induces a closed immersion $\mathrm{Up}_n \to \mathrm{GL}(V)$.

(2) Passing to the limit yields the pro-unipotent group

$$\mathrm{Up}_\infty = \varprojlim_n \mathrm{Up}_n .$$

DEFINITION 3.92. Let G be either an abstract group or an affine group scheme. A finite-dimensional representation $\rho\colon G \to \mathrm{GL}(V)$ is called *unipotent* if there exists a basis of V such that $\rho(G) \subseteq \mathrm{Up}_n$. Equivalently, ρ is unipotent if there exists an integer $N > 0$ such that $(\rho(\gamma) - \mathrm{Id})^N = 0$ for all $\gamma \in G$.

It follows easily from definitions 3.89 and 3.92 that an affine algebraic group (resp. affine group scheme) G is unipotent (resp. *pro-unipotent*) if every non-zero finite-dimensional representation V of G is unipotent.

3.3.4. *The conilpotency filtration.* We give an alternative characterization of pro-unipotent groups. Let $G = \mathrm{Spec}(A)$ be an affine group scheme over k. Since A is a coalgebra, the dual $A^\vee$ is an algebra, but this time not necessarily commutative. The unit of A defines an augmentation $\varepsilon\colon A^\vee \to k$. Let $J = \mathrm{Ker}(\varepsilon)$ be the augmentation ideal. Then the *conilpotency filtration* is the filtration of A given by

$$0 \subset C_0 \coloneqq \mathrm{Ann}_A J \subset C_1 \coloneqq \mathrm{Ann}_A J^2 \subset \cdots \subset C_i \coloneqq\subset \mathrm{Ann}_A J^{i+1} \subset \cdots$$

It is easy to see that $C_0 = k \cdot 1$, where 1 is the unit of A, and that

$$\Delta C_i \subset \sum_{a+b=i} C_a \otimes C_b. \tag{3.93}$$

PROPOSITION 3.94. *An affine group scheme is pro-unipotent if and only if the conilpotency filtration is exhaustive, that is:*

$$A = \bigcup_{i=0}^{\infty} C_i.$$

PROOF. Assume that the conilpotency filtration is exhaustive and let V be a non-zero representation of $G = \operatorname{Spec}(A)$, and denote by $\Delta\colon V \to V \otimes A$ the corresponding comodule structure. Consider the exhaustive filtration $\{V_i\}_{i\geq 0}$ given by

$$V_i = \{v \in V \mid \Delta v \in V \otimes C_i\}.$$

In particular, by the axioms of a comodule, if $v \in V_0$, then $\Delta v = v \otimes 1$. Thus we have to show that V_0 is non-zero. To this end we show that $V_i = 0$ implies that $V_{i+1} = 0$. So, assume that $V_i = 0$ and let $v \in V_{i+1}$. By (3.93)

$$(1 \otimes \Delta)\Delta v \in \sum_{a+b=i+1} V \otimes C_a \otimes C_b.$$

Since a and b can not be both bigger that i, the vector v is sent to zero by the map

$$V \xrightarrow{\Delta} V \otimes A \xrightarrow{1\otimes\Delta} V \otimes A \otimes A \longrightarrow V \otimes A/C_i \otimes A/C_i.$$

But, by the associative property of comodules, this map agrees with the map

$$V \xrightarrow{\Delta} V \otimes A \xrightarrow{\Delta\otimes 1} V \otimes A \otimes A \longrightarrow V \otimes A/C_i \otimes A/C_i.$$

that is an injection, since $V_i = 0$. Thus $v = 0$, hence $V_{i+1} = 0$.

Conversely, assume that every non-zero representation of G has a non-zero fixed point. Then every representation V has a filtration $\{V_i\}_{i\geq 0}$ determined by the fact that V_{i+1}/V_i is the trivial subrepresentation of V/V_i. This filtration is exhaustive by Lemma 3.87. The conilpotency filtration agrees with this filtration in the representation given by A itself, thus it is exhaustive. □

3.3.5. *The pro-unipotent completion.* The central concept of the whole section is the following:

DEFINITION 3.95. Let Γ be an abstract group. The *pro-unipotent completion* Γ^{un} of Γ over k is the universal pro-unipotent affine group scheme G over k endowed with a morphism of abstract groups $\Gamma \to G(k)$. More precisely,

- Γ^{un} is a pro-unipotent affine group scheme over k with a morphism $\Gamma \to \Gamma^{\mathrm{un}}(k)$,
- for each pro-unipotent affine group scheme G over k with a morphism $\Gamma \to G(k)$, there is a unique morphism of affine group schemes $\Gamma^{\mathrm{un}} \to G$ such that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr] & \Gamma^{\mathrm{un}}(k) \arrow[d] \\ & G(k) \end{tikzcd} .$$

The pro-unipotent completion of Γ over $\mathbb{Q}$ will be called the pro-unipotent completion of Γ.

The pro-unipotent completion is also called *the Malcev completion* in the literature. As it is always the case with universal objects, when they exist they are unique up to unique isomorphism.

REMARK 3.96. If the pro-unipotent completion exists, then the groups Γ and Γ^{un} have the same finite-dimensional unipotent representations. Therefore, one cannot recover Γ by just looking at this kind of representations.

We now present the construction, due to Quillen, of the pro-unipotent completion of a group satisfying a finiteness condition. For the moment, let Γ be any abstract group and consider the non-commutative k-algebra

$$k[\Gamma] = \{\sum_{g\in\Gamma} a_g g \mid a_g \in k,\ a_g = 0 \text{ except for a finite subset}\},$$

with the product structure induced by the group operation of Γ.

DEFINITION 3.97. The *augmentation* of $k[\Gamma]$ is the algebra morphism

$$\begin{array}{cccc} \varepsilon\colon & k[\Gamma] & \longrightarrow & k \\ & \sum_{g\in\Gamma} a_g g & \longmapsto & \sum_{g\in\Gamma} a_g. \end{array}$$

Its kernel $J = \operatorname{Ker}(\varepsilon)$ is called the *augmentation ideal*:

$$J = \{\sum_{g\in\Gamma} a_g g \mid \sum_{g\in\Gamma} a_g = 0\}.$$

The completion of $k[\Gamma]$ with respect to J is the inverse limit

$$k[\Gamma]^\wedge = \varprojlim_N k[\Gamma]/J^{N+1}.$$

This is a topological algebra. It has a completed coproduct

$$\nabla^\vee\colon k[\Gamma]^\wedge \longrightarrow k[\Gamma]^\wedge \hat{\otimes} k[\Gamma]^\wedge$$

induced by the rule $\nabla^\vee g = g \otimes g$ for all elements $g \in \Gamma$. Moreover, there is an antipode

$$S^\vee\colon k[\Gamma]^\wedge \longrightarrow k[\Gamma]^\wedge$$

determined by $g \mapsto g^{-1}$. With these operations $k[\Gamma]^\wedge$ is a completed Hopf algebra in the sense of Definition 3.60. Its dual

$$A = (k[\Gamma]^\wedge)^\vee = \varinjlim_N (k[\Gamma]/J^{N+1})^\vee$$

with the induced structures is a Hopf algebra. The augmentation ε extends to an augmentation $\varepsilon\colon A^\vee = k[\Gamma]^\wedge \to k$, that agrees with the augmentation introduced in Paragraph 3.3.4. We will also denote by

$$J = \operatorname{Ker}(\varepsilon) = J k[\Gamma]^\wedge$$

the augmentation ideal of $k[\Gamma]^\wedge$.

Let us now assume that Γ satisfies the finiteness condition that $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is a finite-dimensional k-vector space. By Theorem 3.14, this is for instance satisfied when Γ is the fundamental group of a topological space with the homotopy type of a finite CW-complex.

Lemma 3.98. *If the vector space $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is finite-dimensional, then all the quotients $k[\Gamma]/J^{N+1}$ are finite-dimensional as well.*

Proof. Since $k[\Gamma] = k \oplus J$, it suffices to prove that J/J^{N+1} is finite-dimensional for all $N \geq 0$. Looking at the filtration

$$J^{N+1} \subseteq J^N \subseteq \cdots \subseteq J^2 \subseteq J,$$

this amounts to proving that the successive quotients J^i/J^{i+1} are finite-dimensional for all $i \geq 1$. To treat the case $i = 1$, we note that the map

$$\begin{array}{ccc} \Gamma & \longrightarrow & J/J^2 \\ g & \longmapsto & (g-1) + J^2 \end{array}$$

factors through the abelianization of Γ, as can be seen by writing $gh - 1$ as $(g-1) + (h-1) + (g-1)(h-1)$. In fact, it induces an isomorphism

$$\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k \xrightarrow{\sim} J/J^2$$

(the inverse is the map that sends the class of a generator $g-1$ to the class of g in Γ^{ab}). This proves that J/J^2 is finite-dimensional. Taking into account that the multiplication map

$$(J/J^2)^{\otimes i} \longrightarrow J^i/J^{i+1}$$

is surjective for all $i \geq 1$, the general result follows. $\square$

The following result can be deduced from [**Qui69**, Appendix A], although the language there is different. A translation into the language of algebraic groups is given in [**Hai93**, Theorem 3.3]. We sketch the proof.

Theorem 3.99 (Quillen [**Qui69**]). *Let Γ be an abstract group such that the vector space $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ has finite dimension. Then the pro-unipotent completion of Γ over k is the pro-algebraic group* $\mathrm{Spec}((k[\Gamma]^{\wedge})^{\vee})$.

Proof. The conilpotency filtration of A is given by $\mathrm{Ann}_A J^{N+1}$. Since

$$\bigcap_{n \geq 0} J^{N+1} = 0,$$

the conilpotency filtration of A is exhaustive. Therefore, by Proposition 3.94 we deduce that G is pro-unipotent.

Let now $H = \mathrm{Spec}(S)$ be a pro-unipotent group with a group morphism $\Gamma \to H(k)$. Let $S^{\vee}$ be the non-commutative algebra dual to the co-algebra S. There is an inclusion $H(k) \to S^{\vee}$ given by evaluating functions at points. The map $f \colon \Gamma \to H(k)$ extends to a map $k[\Gamma] \to S^{\vee}$ also denoted f. The augmentations of $k[\Gamma]$ and of $S^\vee$ are compatible with f. Thus we obtain maps

$$k[\Gamma]/J^{N+1} \to S^\vee/J^{N+1},$$

where J denotes the augmentation ideal in both algebras. Dualizing we obtain maps

$$\operatorname{Ann}_S J^{N+1} \longrightarrow \operatorname{Ann}_A J^{N+1} \hookrightarrow A.$$

Since H is pro-unipotent, by Proposition 3.94 the conilpotency filtration of S is exhaustive and we obtain a map $S \to A$, therefore a map of pro-unipotent groups $G \to H$. By construction, this is the only map of pro-unipotent groups that preserves the image of Γ. Thus G satisfies the universal property defining Γ^{un}. $\square$

3.3.6. *Group-like and Lie-like elements.* Let Γ be an abstract group such that $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ is finite dimensional. In this paragraph, we describe the set of rational points $\Gamma^{\mathrm{un}}(k)$, the map $\Gamma \to \Gamma^{\mathrm{un}}(k)$ and the Lie algebra $\operatorname{Lie}(\Gamma^{\mathrm{un}})$. In the applications we will always be interested in the case $k = \mathbb{Q}$.

Definition 3.100. An element $g \in k[\Gamma]^\wedge$ is said to be *group-like* if it satisfies the conditions $\varepsilon(g) = 1$ and $\nabla^\vee g = g \otimes g$.

The set of group-like elements of $k[\Gamma]^\wedge$, denoted by $\mathcal{G}(k[\Gamma]^\wedge)$, is a group. Clearly, the image of an element $g \in \Gamma$ in $k[\Gamma]^\wedge$ is group-like.

Definition 3.101. An element of $x \in k[\Gamma]^\wedge$ is called *Lie-like* if it satisfies the condition $\nabla^\vee x = 1 \otimes x + x \otimes 1$.

The set of Lie-like elements of $k[\Gamma]^\wedge$, denoted by $\mathcal{L}(k[\Gamma]^\wedge)$, is a Lie algebra. The power series exp and log are bijections, inverses to each other

$$\mathcal{G}(k[\Gamma]^\wedge) \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathcal{L}(k[\Gamma]^\wedge)\ .$$

Proposition 3.102. *Let Γ be an abstract group such that $\Gamma^{\mathrm{ab}} \otimes_{\mathbb{Z}} k$ has finite dimension. Then $\mathcal{G}(k[\Gamma]^\wedge) = \Gamma^{\mathrm{un}}(k)$, and the natural map $\Gamma \to \mathbb{Q}[\Gamma]^\wedge$ agrees with the structural map $\Gamma \to \Gamma^{\mathrm{un}}(\mathbb{Q})$. Moreover, the Lie algebra of Γ^{un} agrees with $\mathcal{L}(\mathbb{Q}[\Gamma]^\wedge)$.*

Proof. We just sketch the proof that $\mathcal{G}(k[\Gamma]^\wedge) = \Gamma^{\mathrm{un}}(k)$. We continue using the notation $A = (k[\Gamma]^\wedge)^\vee$ so that $\Gamma^{\mathrm{un}} = \operatorname{Spec}(A)$. By definition the set $\Gamma^{\mathrm{un}}(k)$ is in bijection with the set of morphisms $\operatorname{Spec}(k) \to \operatorname{Spec}(A)$, which, in turn is in bijection with the set of algebra homomorphisms $A \to k$. That is, the set of elements $g \in A^\vee = k[\Gamma]^\wedge$ that preserve the unit and the product. For an element $g \in A^\vee$, to preserve the unit is equivalent to $\varepsilon(g) = 1$ and to preserve the product is equivalent to $\nabla^\vee(g) = g \otimes g$. Thus we get bijection $\mathcal{G}(k[\Gamma]^\wedge) = \Gamma^{\mathrm{un}}(k)$. $\square$

EXAMPLE 3.103. Let us illustrate the above proposition for $\Gamma = \mathbb{Z}$. As we will see in Exercise 3.111, the pro-unipotent completion of Γ is the additive group $\mathbb{G}_a$ over $\mathbb{Q}$, so we need to show that group-like elements in $\mathbb{Q}[[x]]$ are in one-to-one correspondence with $\mathbb{Q}$. Let $\sum_{n\geq 0} a_n x^n$ be a group-like element. Then $a_0 = 1$ and

$$\nabla^\vee\left(\sum_{n\geq 0} a_n x^n\right) = \left(\sum_{n\geq 0} a_n x^n\right) \otimes \left(\sum_{n\geq 0} a_n x^n\right). \tag{3.104}$$

Since $\nabla^\vee x = 1 \otimes x + x \otimes 1$, we have

$$\nabla^\vee x^n = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

Equation (3.104) is thus equivalent to the relation

$$a_k a_m = \binom{k+m}{k} a_{k+m}$$

for all $k, m \geq 0$. In particular, all coefficients are determined by a_1 and indeed $a_n = \frac{a_1^n}{n!}$. Hence our element is of the form $\exp(a_1 x)$ and this gives the correspondence.

From the compatibility between the antipode, the product and the completed coproduct we easily deduce the following (Exercice 3.113):

LEMMA 3.105. *If x is a Lie-like element, then $S(x) = -x$, while if g is a group-like element, then it is invertible in the algebra $k[\Gamma]^\wedge$ and satisfies $S(g) = g^{-1}$.*

EXAMPLE 3.106. Let Γ be the free group on two generators γ_0 and γ_1. In this example we compute the pro-unipotent completion of Γ (over $\mathbb{Q}$). Since $\gamma_0 - 1$ and $\gamma_1 - 1$ belong to the augmentation ideal, we can define

$$\log(\gamma_0) = \log(1 + (\gamma_0 - 1)) = \gamma_0 - 1 - \frac{(\gamma_0 - 1)^2}{2} + \ldots$$
$$\log(\gamma_1) = \log(1 + (\gamma_1 - 1)) = \gamma_1 - 1 - \frac{(\gamma_1 - 1)^2}{2} + \ldots$$

as elements in $\mathbb{Q}[\Gamma]^\wedge$. Recall the algebra $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ of Example 3.62. We define a morphism of algebras $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \to \mathbb{Q}[\Gamma]^\wedge$ by sending e_0 to $\log(\gamma_0)$ and e_1 to $\log(\gamma_1)$. It is easy to verify that this map is an isomorphism compatible with all extra structures (unit, counit, completed coproduct and antipode) of both algebras. From this it follows that $\Gamma^{\mathrm{un}} = \mathrm{Spec}(\mathfrak{H})$, where $\mathfrak{H}$ is the Hoffman algebra of Example 3.56.

In particular, we can identify the group of rational points $\Gamma^{\mathrm{un}}(\mathbb{Q})$ with the set of group-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, the Lie algebra $\mathrm{Lie}(\Gamma^{\mathrm{un}})$ with the set of Lie-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ and the completed universal enveloping algebra of $\mathrm{Lie}(\Gamma^{\mathrm{un}})$ with $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$.

3.3.7. *The pro-unipotent completion of a torsor.*

VARIANT 3.107. We will also use the following variant of the constructions of this section. Let P be a left torsor over Γ, in other words, a set P on which Γ acts freely and transitively on the left. Write $k[P]$ for the k-vector space with basis P. It has the structure of a left $k[\Gamma]$-module. The completion of $k[P]$ is defined as

$$k[P]^{\wedge} = \varprojlim k[P]/J^{N+1}k[P]. \tag{3.108}$$

This is a $k[\Gamma]^{\wedge}$-module equipped with a completed coproduct

$$\nabla^{\vee}\colon k[P]^{\wedge} \longrightarrow k[P]^{\wedge}\hat{\otimes}k[P]^{\wedge},$$

induced by the rule $\nabla^{\vee}a = a \otimes a$ for any $a \in P$. The completed coproducts of $k[\Gamma]^{\wedge}$ and $k[P]^{\wedge}$ are compatible with the module structure in the sense that

$$\nabla^{\vee}(ga) = \nabla^{\vee}(g)\nabla^{\vee}(a).$$

for all $a \in k[P]^{\wedge}$ and $g \in k[\Gamma]^{\wedge}$. Dualizing we obtain a commutative algebra $R = (k[P]^{\wedge})^{\vee}$ with a compatible coproduct

$$\Delta\colon R \longrightarrow A \otimes R, \tag{3.109}$$

where A denotes again $A = (k[P]^{\wedge})^{\vee}$. In other words, R is a Hopf module over A. The unipotent completion of P is

$$P^{\mathrm{un}} = \mathrm{Spec}(R).$$

The coproduct (3.109) induces an action $\Gamma^{\mathrm{un}} \times P^{\mathrm{un}} \to P^{\mathrm{un}}$ that turns P^{un} into a Γ^{un}-torsor.

Mutatis mutandi, the same construction can be made for a right torsor P'. In this case

$$k[P']^{\wedge} = \varprojlim k[P']/k[P']J^{N+1}.$$

Our basic example will be the case when Γ is the fundamental group $\pi_1(M, x)$ and P and P' are the torsor of paths $\pi_1(M; x, y)$ and $\pi_1(M; y, x)$ respectively. In this case, there is also an antipode map

$$S\colon \mathbb{Q}[P]^{\wedge} \to \mathbb{Q}[P']^{\wedge}$$

induced by the rule $S(\gamma) = \gamma^{-1}$ for paths $\gamma \in P$.

⋆⋆⋆

EXERCISE 3.110. Not every representation of the group $G(k)$ thas has "geometric origin" is a representation of G. For instance, consider the $\mathbb{C}$ vector space $V = K(\mathbb{P}^1_{\mathbb{C}})$ of rational functions on the complex projective line. The group $G(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ by Möbius transformations, hence linearly on V.

(1) Let $W \subset V$ be a finite-dimensional vector subspace. Show that the set of poles of the functions belonging to W is finite.

(2) Show that the set of poles that appear in the orbit of the function t is infinite.
(3) Conclude by Lemma 3.87 that the linear representation of $G(\mathbb{C})$ on V does not come from a representation of the algebraic group scheme $G = \mathrm{SL}_2$.

EXERCISE 3.111. Consider the group $\Gamma = \pi_1(S^1, 1) \simeq \mathbb{Z}$. Let γ_0 be a generator of Γ and consider $X_0 = \log(\gamma_0)$ as a power series in $(\gamma_0 - 1) \in J$. Use γ_0 and X_0 to describe explicitly

$$\mathbb{Q}[\pi_1(S^1, 1)]/J^{N+1}, \quad \mathbb{Q}[\pi_1(S^1, 1)]^\wedge, \quad \mathcal{O}(\pi_1(S^1, 1)^{\mathrm{un}}),$$
$$\pi_1(S^1, 1)^{\mathrm{un}}, \quad \mathrm{Lie}(\pi_1(S^1, 1)^{\mathrm{un}}).$$

In particular, deduce that the pro-unipotent completion of $\mathbb{Z}$ is the additive group $\mathbb{G}_a$. Compare this with Exercise 4.23 in the next chapter.

EXERCISE 3.112. Prove that the pro-unipotent completion of the group $\mathbb{Z}/2\mathbb{Z}$ is the trivial group $\mathrm{Spec}(\mathbb{Q})$.

EXERCISE 3.113. Using the diagram for the antipode in Definition 3.45, prove Lemma 3.105.

3.4. The bar complex and Chen's π_1-de Rham theorem. In this section, we make precise the relation between differential forms and the pro-unipotent completion of the fundamental group of a smooth manifold. If one views the latter as the Betti side of a picture, then the de Rham side is given by the cohomology of the bar complex. Both points of view will be related through Chen's π_1-de Rham theorem.

3.4.1. *The reduced bar complex of a connected dg-algebra.* We start by recalling the definition of a differential graded algebra.

DEFINITION 3.114. Let k be a field of characteristic zero. A *differential graded algebra* (*dg-algebra* for short) over k is a graded k-vector space

$$A = \bigoplus_{n \in \mathbb{Z}} A^n,$$

together with the following additional structures:

- a multiplication $A^n \otimes A^m \to A^{n+m}$ for all integers $n, m \in \mathbb{Z}$ which makes A into an associative k-algebra with unit $1 \in A^0$;
- a differential $d\colon A \to A$ such that $d(A^n) \subseteq A^{n+1}$ and

$$d(ab) = da \cdot b + (-1)^n a \cdot db, \qquad a \in A^n.$$

We say that A is *commutative* if, for $a \in A^n$ and $b \in B^m$, the relation $ab = (-1)^{nm} ba$ holds, and *connected* if $A^n = 0$ for $n < 0$ and $A^0 = k$.

An *augmentation* of a dg-algebra is a map of dg-algebras $A \to k$, where k is concentrated in degree zero and has trivial differential.

The example to keep in mind, when $k = \mathbb{R}$ or $\mathbb{C}$, is the algebra $E^*(M, k)$ of smooth k-valued differential forms on a smooth manifold M, together with the wedge product $\wedge$ and the exterior differential d (see Section 2.2.1). A typical augmentation is the evaluation map on an point of M. Note that, a connected dg-algebra has a unique augmentation.

Since we will apply the general constructions to this setting, below we write $\wedge$ for the product in A. Similarly, for k arbitrary, if X is a smooth variety, then $\Omega(X)$ is also a dg-algebra.

Definition 3.115. Let $(A^*, \wedge, d)$ be a connected dg-algebra over k. Set

$$A^+ = \bigoplus_{n>0} A^n.$$

The *reduced bar complex* associated with A, denoted by $B^*(A^*)$, is the total tensor algebra of A^+:

$$B^*(A^*) = k \,\oplus\, A^+ \,\oplus\, (A^+ \otimes A^+) \,\oplus\, (A^+ \otimes A^+ \otimes A^+) \,\oplus\, \dots$$

An element $x_1 \otimes \cdots \otimes x_n$ for $n \geq 1$ will be denoted by the bar notation

$$[x_1|\cdots|x_n],$$

and the element $1 \in k$ by the empty symbol $[\,]$.

The reduced bar complex is provided with the following structure.

Grading: The degree on $B^*(A^*)$ is given by

$$\deg[x_1|\dots|x_n] = \sum_{i=1}^{n} \deg(x_i) - n.$$

Length filtration: It is the increasing filtration where

$$L_m B^*(A^*) \subseteq B^*(A^*)$$

is the subspace generated by elements $[x_1|\cdots|x_n]$ with $n \leq m$.

Differential: The differential takes into account both the differential and the product structures of A^*:

$$\begin{aligned} d[x_1|\cdots|x_n] = &-\sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg[x_j]} [x_1|\cdots|dx_i|\cdots|x_n] \\ &+ \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg[x_j]} [x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n]. \end{aligned} \tag{3.116}$$

Note that, by the previous convention, $\deg[x_j] = \deg(x_j) - 1$. It is easy to check that d is compatible with the grading and that

$d \circ d = 0$. We will write $d = d_I - d_C$, where

$$d_I[x_1|\cdots|x_n] = -\sum_{i=1}^{n}(-1)^{\sum_{j=1}^{i-1}\deg([x_j])}[x_1|\cdots|dx_i|\cdots|x_n] \tag{3.117}$$

$$d_C[x_1|\cdots|x_n] = -\sum_{i=1}^{n-1}(-1)^{\sum_{j=1}^{i}\deg([x_j])}[x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n]. \tag{3.118}$$

Product: It is the shuffle product

$$\nabla([x_1|\cdots|x_r]\otimes[x_{r+1}|\cdots|x_{r+s}]) = \sum_{\sigma\in ш(r,s)}\eta(\sigma)[x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(r+s)}], \tag{3.119}$$

where $\eta(\sigma)$ is determined by the equation

$$a_1 \wedge \cdots \wedge a_{r+s} = \eta(\sigma) a_{\sigma^{-1}(1)} \wedge \cdots \wedge a_{\sigma^{-1}(r+s)}, \tag{3.120}$$

where $\deg(a_i) = \deg(x_i) - 1 = \deg([x_i])$. Although $\eta(\sigma)$ is not determined by σ alone, this abusive notation is the standard one.

Coproduct: The coproduct is the *deconcatenation* coproduct

$$\Delta[x_1|\cdots|x_n] = \sum_{i=0}^{n}[x_1|\cdots|x_i]\otimes[x_{i+1}|\cdots|x_n].$$

Antipode: It is given by

$$S([x_1|\cdots|x_n]) = (-1)^n\eta(\tau_n)[x_n|\cdots|x_1], \tag{3.121}$$

where the sign $\eta(\tau_n)$ is determined by equation (3.120) as before, for the permutation $\tau_n(i) = n - i$.

Remark 3.122. The formula we have written for the differential differs for the classical one that can be found, for instance, in [Tan83] by a sign. The reason is the different definition we have for the iterated integral and the fact that we want the bar construction to be compatible with the iterated integrals. To go from one convention of signs to the other we define the operator $T\colon B^*(A^*) \to B^*(A^*)$ by

$$T([x_1|\cdots|x_n]) = \varepsilon(\tau_n)[x_n|\cdots|x_1] = (-1)^n S([x_1|\cdots|x_n])$$

and $T_\otimes\colon B^*(A^*)\otimes B^*(A^*) \to B^*(A^*)\otimes B^*(A^*)$ by

$$T(a\otimes b) = (-1)^{\deg(a)\deg(b)}T(b)\otimes T(a).$$

If we denote by d', ∇' and Δ', the differential, product and coproduct in [Tan83], then one checks that

$$\nabla = T\circ\nabla'\circ T_\otimes = \nabla'$$
$$\Delta = T_\otimes\circ\Delta'\circ T = \Delta.$$

That is, T is an anti-homomorphism of Hopf algebras, and that

$$d = T\circ d'\circ T.$$

Thus, our differential is just the classical one twisted by T.

Lemma 3.123. *Let $(A^*, d, \wedge)$ be a connected commutative dg-algebra. Then the above operations endow $H^0(B^*(A^*))$, the zeroth cohomology group of the reduced bar complex, with a commutative Hopf algebra structure.*

Proof. As stated e.g. in [Tan83, 0.6], the bar construction $B^*(A^*)$ is a commutative differential graded Hopf Algebra. This means that the product, coproduct and antipode are compatible with the grading and the differential. The latter compatibility is written as

$$\begin{aligned} d \circ \nabla &= \nabla \circ d_{\otimes}, \\ \Delta \circ d &= d_{\otimes} \circ \Delta, \\ S \circ d &= d \circ S, \end{aligned}$$

where $d_{\otimes}$ is the differential induced in $B^*(A^*) \otimes B^*(A^*)$ that carries the usual sign. All these statements can be checked directly. Once we know that all these operations are compatible with the differential, the pass to cohomology. Since they are compatible with the grading, they induce operations on H^0. □

Remarks 3.124.

(1) The commutativity in the graded sense is essential in the previous proof. In fact if the product on A^* is not graded commutative, it is not true that the shuffle product in $B^*(A^*)$ is compatible with the differential.

(2) The complex $B^*(A^*)$ is concentrated in positive degrees, so the cohomology we are interested in is simply

$$H^0(B^*(A^*)) = \operatorname{Ker}(d\colon B^0(A^*) \longrightarrow B^1(A^*)).$$

Note that elements of $B^0(A^*)$ are k-linear combinations of $[\ ]$ and $[x_1|\cdots|x_n]$ with $n \geq 1$ and $\deg(x_i) = 1$ for all $i = 1, \ldots, n$. Also, observe that, restricted to $B^0(A^*)$, the differentials are given by the formulas

$$\begin{aligned} d_I[x_1|\cdots|x_n] &= -\sum_{i=1}^{n} [x_1|\cdots|dx_i|\cdots|x_n], \\ d_C[x_1|\cdots|x_n] &= -\sum_{i=1}^{n-1} [x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n]. \end{aligned}$$

3.4.2. *The (non-reduced) bar construction of a dg-algebra.* When considering non-connected dg-algebras, it is convenient to use the (non-reduced) bar construction. This variant will not be needed for the main example of this text $\mathbb{P}(\mathbb{C})^1 \setminus \{0, 1, \infty\}$.

Definition 3.125. Let $(A^*, \wedge, d)$ be a dg-algebra over k and ε_1 and ε_2 two augmentation (maybe equal). The *bar complex* associated with $A, \varepsilon_1, \varepsilon_2$, denoted by $B^*(A^*, \varepsilon_2, \varepsilon_1)$, is the total tensor algebra of A:

$$B^*(A, \varepsilon_2, \varepsilon_1) = k \oplus A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \dots$$

As in the case of the reduced bar complex, an element $x_1 \otimes \dots \otimes x_n$ for $n \geq 1$ will be denoted by the bar notation

$$[x_1 | \cdots | x_n],$$

and the element $1 \in k$ by the empty symbol $[\ ]$. The *grading* of the bar complex is defined in the same way as in the reduced bar complex. The *differential* takes into account also the augmentations and is given by

$$\begin{aligned} d[x_1|\cdots|x_n] = &- \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} \deg[x_j]} [x_1|\cdots|dx_i|\cdots|x_n] \\ &+ \sum_{i=1}^{n-1} (-1)^{\sum_{j=1}^{i} \deg[x_j]} [x_1|\cdots|x_i \wedge x_{i+1}|\cdots|x_n]. \\ &+ \varepsilon_2(x_1)[x_2|\cdots|x_n] + (-1)^{\sum_{j=1}^{n} \deg[x_j]} \varepsilon_1(x_n)[x_1|\cdots|x_{n-1}]. \end{aligned} \tag{3.126}$$

As before, $\deg[x_j] = \deg(x_j) - 1$, and one checks that d is compatible with the grading and that $d \circ d = 0$.

The *product* is the shuffle product given again by equation (3.119).

If ε_3 is a third augmentation (that may agree with the previous ones) there is a *coproduct*

$$\Delta\colon B^*(A, \varepsilon_3, \varepsilon_2) \otimes B^*(A, \varepsilon_2, \varepsilon_1) \to B^*(A, \varepsilon_3, \varepsilon_1)$$

given by deconcatenation

$$\Delta[x_1|\cdots|x_n] = \sum_{i=0}^{n} [x_1|\cdots|x_i] \otimes [x_{i+1}|\cdots|x_n].$$

.

Finally, the *antipode* is given again by formula (3.121).

REMARKS 3.127.

(1) If A is a connected dg-algebra and ε is the unique augmentation, then the complexes $B^*(A^*)$ and $B^*(A^*, \varepsilon, \varepsilon)$ are homotopically equivalent. This is a consequence of Lemma 3.193.

(2) If $(A, \varepsilon_1, \varepsilon_2) \to (A', \varepsilon_1', \varepsilon_2')$ is a quasi-isomorphism commuting with the augmentations, then it induces a quasi-isomorphism

$$B(A, \varepsilon_2, \varepsilon_1) \to B(A', \varepsilon_2', \varepsilon_1').$$

As a consequence of this remark we have

LEMMA 3.128. *Let $k = \mathbb{R}, \mathbb{C}$ and M a connected differentiable manifold. Let $x, y \in M$ and $A \subset E^*(M, k)$ a connected dg-algebra such that the* *inclusion $A^* \to E^*(M,k)$ is a quasi-isomorphism. Let $\varepsilon_x, \varepsilon_y$ be the augmentations given by evaluation at the points x and y respectively. Then there is a quasi-isomorphism $B^*(A^*) \to B(E^*(M,k), \varepsilon_y, \varepsilon_x)$. In particular*

$$H^0(B^*(A^*)) = H^0(B(E^*(M,k), \varepsilon_y, \varepsilon_x)).$$

3.4.3. *The reduced bar construction and iterated integrals.* Let M be a connected differentiable manifold with the homotopy type of a finite CW complex. Let $E^*(M,\mathbb{C})$ be the differential graded algebra of complex smooth differential forms on M. For simplicity of the exposition, we will assume that there exists a dg- $\mathbb{C}$-algebra A^* provided with an injective morphism of dg-algebras $\varphi\colon A^* \longrightarrow E^*(M)$ such that

(1) A^* is connected, that is $A^0 = \mathbb{C}$ and $A^n = 0$ for $n < 0$.
(2) The induced map in cohomology
$$\varphi\colon H^*(A^*) \to H^*(E^*(M,\mathbb{C}))$$
is an isomorphism.

And we will use the reduced bar complex of A. A similar discussion can be made with the bar complex of the whole dg-algebra $E^*(M,\mathbb{C})$.

The condition of A^* being connected implies that the elements of degree zero of $B^0(A^*)$ are linear combinations of the form

$$\sum [\eta_1|\cdots|\eta_r]$$

with $\eta_i \in A^1 \subset E^1(M)$ one forms. Thus, to any element $x \in B^0(A^*)$ corresponds an iterated integral

$$[\eta_1|\cdots|\eta_r] \longmapsto \left(\gamma \mapsto \int_\gamma \eta_1 \cdots \eta_r\right).$$

In fact, for each pair of points $x, y \in M$, we define a pairing

$$\begin{aligned} \langle\ ,\ \rangle\colon B^0(A^*) \otimes \mathbb{Q}[{}_y\mathcal{P}(M)_x] &\longrightarrow \mathbb{C} \\ [\eta_1|\cdots|\eta_r] \otimes \gamma &\longmapsto \int_\gamma \eta_1 \cdots \eta_r, \end{aligned} \tag{3.129}$$

where ${}_y\mathcal{P}(M)_x$ is the set of piecewise smooth paths as in Section 3.1, and $\mathbb{Q}[{}_y\mathcal{P}(M)_x]$ denotes the $\mathbb{Q}$-vector space with basis ${}_y\mathcal{P}(M)_x$.

We can now translate Theorem 3.19 into the language of the bar construction and the pairing (3.129).

THEOREM 3.130. *Let $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths in M and let $\eta, \eta_1, \eta_2 \in B^0(A^*)$ be degree zero elements of the reduced bar complex of A^*. Then the following three equalities are satisfied:*

$$\langle S(\eta), \gamma\rangle = \langle \eta, S(\gamma)\rangle. \tag{3.131}$$

$$\langle \eta, \gamma_1\gamma_2\rangle = \langle \Delta\eta, \gamma_1 \otimes \gamma_2\rangle. \tag{3.132}$$

$$\langle \eta_1 \otimes \eta_2, \nabla^\vee\gamma\rangle = \langle \eta_1 \sqcup \eta_2, \gamma\rangle. \tag{3.133}$$

A consequence of the previous theorem is the following result that says that the length filtration of the reduced bar complex is dual to the filtration by the augmentation ideal in the group algebra of paths.

Proposition 3.134. *Let x, y be points of M. Let J be the augmentation ideal of $\mathbb{Q}[{}_x\mathcal{P}(M)_x]$, $N \geq 0$ an integer and $\gamma \in J^{N+1}\mathbb{Q}[{}_x\mathcal{P}(M)_y]$ or $\gamma \in \mathbb{Q}[{}_y\mathcal{P}(M)_x]J^{N+1}$. If $\eta \in L_N B^0(A^*)$ has length less than or equal to N, then*

$$\langle \eta, \gamma \rangle = 0.$$

Proof. We only treat the case $\gamma \in J^{N+1}\mathbb{Q}[{}_x\mathcal{P}(M)_y]$ (the other one is completely analogous). The proof proceeds by induction on N.

If $N = 0$, every element of $\gamma \in J\mathbb{Q}[{}_x\mathcal{P}(M)_y]$ can be written as

$$\gamma = \sum_{i=1}^{r} q_i \gamma_i, \quad q_i \in \mathbb{Q}, \quad \sum_{i=1}^{r} q_i = 0, \quad \gamma_i \in {}_x\mathcal{P}(M)_y.$$

If $\eta \in L_0 B^0(A^*)$, then $\eta = \alpha[\,]$ for $\alpha \in \mathbb{C}$. Since

$$\langle [\,], \gamma_i \rangle = 1, \quad \text{for } \gamma_i \in {}_x\mathcal{P}(M)_y,$$

we deduce the result in the case $N = 0$.

Now fix $N > 0$ and assume that the result holds for all $N' < N$. To prove it for N, we may assume that $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in J$, $\gamma_2 \in J^N\mathbb{Q}[{}_x\mathcal{P}(M)_y]$, and $\eta = [\omega_1|\cdots|\omega_N]$. Then the relation (3.132) yields

$$\begin{aligned}
\langle \eta, \gamma \rangle &= \langle \Delta\eta, \gamma_1 \otimes \gamma_2 \rangle \\
&= \sum_{i=0}^{N} \langle [\omega_1|\cdots|\omega_i], \gamma_1 \rangle \langle [\omega_{i+1}|\cdots|\omega_N], \gamma_2 \rangle \\
&= \langle [\,], \gamma_1 \rangle \langle [\omega_1|\cdots|\omega_N], \gamma_2 \rangle + \sum_{i=1}^{N} \langle [\omega_1|\cdots|\omega_i], \gamma_1 \rangle \langle [\omega_{i+1}|\cdots|\omega_N], \gamma_2 \rangle.
\end{aligned}$$

The first summand in the last equality vanishes since $\langle [\,], \gamma_1 \rangle = 0$ and all the factors $\langle [\omega_{i+1}|\cdots|\omega_N], \gamma_2 \rangle$ in the second sum vanishes by the induction hypothesis. Thus, $\langle \eta, \gamma \rangle = 0$, as we wanted to show. □

3.4.4. *The reduced bar complex and the pro-unipotent completion of the fundamental group.* One of the main interests of the reduced bar complex is that it provides us with a criterion to decide whether an iterated integral is a homotopy functional, thus solving the question raised in Section 3.1.

Theorem 3.135. *Let $\eta \in B^0(A^*)$. If $d\eta = 0$, then the iterated integral associated to η is a homotopy functional.*

Proof. Let $x, y \in M$. Consider two homotopic paths γ_1 and γ_2 from x to y and let F be a homotopy between them. Recall from Definition 3.1

that $F\colon [0,1] \to M$ satisfies the conditions

$$F(t,0) = \gamma_1(t), \quad F(t,1) = \gamma_2(t), \quad F(0,s) = x, \quad F(1,s) = y. \tag{3.136}$$

For simplicity, we will assume that F is smooth; the general case follows by taking a polyhedral decomposition, as in the proof of Lemma 3.11. Set

$$\begin{aligned} F_i\colon [0,1]^n \times [0,1] &\longrightarrow M, \\ ((t_1,\dots,t_n),s) &\longmapsto F(t_1,s). \end{aligned}$$

Recall that elements of $B^1(A^*)$ are linear combinations of $\nu = [\nu_1|\cdots|\nu_n]$ with exactly one 2-form among the ω's. Given such a ν, with the 2-form in the i-th position, we define the *integral along* F as

$$\int_F \nu = (-1)^i \int_{[0,1]\times\Delta^n} F_1^*\nu_1 \cdots F_n^*\nu_n,$$

where the second integral is oriented by $ds \wedge dt_1 \wedge \cdots \wedge dt_n$. This definition extends to $B^1(A^*)$ by $\mathbb{C}$-linearity. We claim that

$$\int_{\gamma_2} \omega - \int_{\gamma_1} \omega = \int_F d\omega, \tag{3.137}$$

and the statement of the theorem will of course be an immediate consequence.

The equality (3.137) is proved by a careful application of Stokes's theorem. First observe that

$$d(F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n) = \sum_{i=1}^n (-1)^{i+1} F_1^*\omega_1 \wedge F_i^*(d\omega_i) \wedge \cdots \wedge F_n^*\omega_n$$

by the properties defining the exterior derivative (see Section 2.2.1) and the commutativity of d and F_i^*. Combining this with the definitions of d_I and the integral along F, one gets:

$$\int_F d_I\omega = \int_{[0,1]\times\Delta^n} d(F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n).$$

We now apply Stokes' theorem. Set $\Omega = F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n$.

$$\begin{aligned} \int_F d_I\omega &= \int_{\partial([0,1]\times\Delta^n)} F_1^*\omega_1 \wedge \cdots \wedge F_n^*\omega_n \\ &= \int_{s=1} \Omega - \int_{s=0} \Omega - \int_{t_1=1} \Omega + \sum_{i=1}^{n-1} (-1)^{i+1} \int_{t_i=t_{i+1}} \Omega - (-1)^n \int_{t_n=0} \Omega \end{aligned}$$

By the relations satisfied by F,

$$\begin{aligned} \Omega|_{s=1} &= \gamma_2^*\omega_1 \wedge \cdots \wedge \gamma_2^*\omega_n, \\ \Omega|_{s=0} &= \gamma_1^*\omega_1 \wedge \cdots \wedge \gamma_1^*\omega_n, \\ \Omega|_{t_i=t_{i+1}} &= F_1^*\omega_1 \wedge \cdots \wedge F_i^*(\omega_i \wedge \omega_{i+1}) \wedge F_n^*\omega_n \end{aligned}$$

and $\Omega|_{t_1=1}$ (resp. $\Omega|_{t_n=0}$) vanishes since in that case F_1 (resp. F_n) is a constant function. Besides,

$$\int_F d_C\omega = \sum_{i=1}^{n}(-1)^{i+1}\int_{[0,1]\times\Delta^{n-1}} F_1^*\omega_1\wedge\cdots\wedge F_i^*(\omega_i\wedge\omega_{i+1})\wedge\cdots\wedge F_n^*\omega_n.$$

Putting everything together yields

$$\int_F d_I\omega = \int_{\gamma_2}\omega - \int_{\gamma_1}\omega + \int_F d_C\omega,$$

which is exactly the content of the claim noting that $d = d_I - d_C$. □

Let $x \in M$ and write $\Gamma = \pi_1(M,x)$. The condition that M has the homotopy type of a finite CW complex implies that $H_1(M)$ is finite dimensional. Thus Γ satisfies the hypothesis of Theorem 3.99 and its pro-unipotent completion is given by $\operatorname{Spec}((\mathbb{Q}[\Gamma]^\wedge)^\vee)$.

Since there are no elements of the reduced bar complex of negative degree, that is, $B^n(A^*) = 0$ for $n < 0$, the zero cohomology group of the reduced bar complex of A^* is just the kernel of the differential map,

$$H^0(B^*(A^*)) = \operatorname{Ker}\left(d\colon B^0(A^*) \longrightarrow B^1(A^*)\right),$$

which, by Theorem 3.135, consists of homotopy functionals.

Putting together Theorem 3.135 and Proposition 3.134 we obtain a map

$$H^0(L_NB^*(A^*)) \longrightarrow ((\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})\otimes\mathbb{C})^\vee.$$

THEOREM 3.138 (Chen's π_1-de Rham theorem). *For each integer $N \geq 0$ and points $x, y \in M$, the integration map gives an isomorphism*

$$H^0(L_NB^*(A^*)) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1},\mathbb{C}),$$

and consequently it induces an isomorphism of ind-vector spaces between

$$H^0(B^*(A^*)) = \varinjlim H^0(L_NB^*(A^*))$$

and

$$(\mathbb{C}[\pi_1(M;y,x)]^\wedge)^\vee = \varinjlim(\mathbb{C}[\pi_1(M;y,x)]/\mathbb{C}[\pi_1(M;y,x)]J^{N+1})^\vee.$$

In fact, Theorem 3.130 implies that the last isomorphism of Theorem 3.138 is compatible with the Hopf algebra structures on both sides. We will give a proof of this result in the next section.

COROLLARY 3.139. *For every point $x \in M$, the iterated integral induces an isomorphism of Hopf algebras*

$$H^0(B^*(A^*)) \xrightarrow{\sim} \mathcal{O}(\pi_1(M,x)^{\mathrm{un}})\otimes\mathbb{C}.$$

REMARK 3.140. The isomorphism of Corollary 3.139 depends on the choice of a base point x.

3.4.5. *The case of* $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The main example to which we would like to apply Corollary 3.139 is the manifold $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$. This example will be central for the remainder of the book. The fundamental group of M is the free group in two generators. Thus, its pro-unipotent completion is isomorphic to the spectrum of the Hoffman algebra $\mathfrak{H}$ by Example 3.106. We want to recover this fact as a particular case of Chen's Theorem. For this, we consider the differential forms

$$\omega_0 = \frac{dt}{t}, \quad \omega_1 = \frac{dt}{1-t}. \tag{3.141}$$

Let $A^*_{\mathbb{C}}$ be the dg-algebra over $\mathbb{C}$ given by

$$A^0_{\mathbb{C}} = \mathbb{C}, \qquad A^1_{\mathbb{C}} = \mathbb{C}\,\omega_0 \oplus \mathbb{C}\,\omega_1, \qquad A^{\geq 2}_{\mathbb{C}} = 0,$$

together with the trivial differential and the obvious multiplication. Thus $A^*_{\mathbb{C}} = A^* \otimes \mathbb{C}$, where A is the $\mathbb{Q}$ algebra introduced in Example 2.130. In particular, the inclusion $A^*_{\mathbb{C}} \subset E^*(M,\mathbb{C})$ is a quasi-isomorphism.

Since $d\omega_i = 0$ for $i = 0,1$ and $\omega_0 \wedge \omega_1 = 0$, formula (3.116) shows that the differential in the reduced bar complex $B^*(A^*)$ is identically zero, hence

$$H^0(B^*(A^*)) = B^0(A^*).$$

Moreover, there is an isomorphism of Hopf algebras

$$\begin{array}{ccc} H^0(B^*(A^*)) & \longrightarrow & \mathfrak{H} \\ \omega_0 & \longmapsto & x_0 \\ \omega_1 & \longmapsto & x_1. \end{array}$$

That induces an isomorphism of Hopf algebras

$$H^0(B^*(A^*_{\mathbb{C}})) \longrightarrow \mathfrak{H} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Following Notation 1.153, for a binary sequence α, we will denote by ω_α the element of $H^0(B^*(A^*_{\mathbb{C}}))$ corresponding to x_α.

⋆ ⋆ ⋆

EXERCISE 3.142. Show that the differentials d_I and d_C from equations (3.117) and (3.118) in the definition of the bar complex satisfy

$$d_I^2 = d_C^2 = 0 \qquad d_I d_C + d_C d_I = 0.$$

Deduce that $d = d_I - d_C$ satisfies $d^2 = 0$ as well.

EXERCISE 3.143. Let η_1, η_2 and η_{12} be 1-forms on a differentiable manifold. What conditions should they satisfy for $[\eta_1|\eta_2] - [\eta_{12}]$ to be closed?

3.5. A geometric description of the pro-unipotent completion of the fundamental group. We now explain a proof of Chen's π_1-de Rham Theorem 3.138. This is not the classical proof that one can find in [**Hai87a**, §4], but the strategy we follow will later enable us to exhibit the *motivic* nature of the pro-unipotent completion of the fundamental group of an algebraic variety. The first step in the proof consists of expressing the quotients by powers of the augmentation ideal as relative cohomology groups. This is possible thanks to a beautiful result by Beilinson.

3.5.1. *A Čech complex of sheaves.* Before stating Beilinson's theorem, we introduce a complex of sheaves that will play an important role in what follows.

Sign convention 3.144. Recall that, given a complex (A^*, d) and an integer m, the *shifted complex* $A[m]$ has terms $A[m]^n = A^{n+m}$ and differential $(-1)^m d$. If A is a complex of sheaves on M, there is an isomorphism

$$\mathbb{H}^n(M, A[m]) \xrightarrow{\sim} \mathbb{H}^{n+m}(M, A).$$

Since the differentials of the complexes A and $A[m]$ can have different sign, in claiming that some diagrams commute one needs to be careful about how the above isomorphism is made. We can compute the hypercohomology groups using, for instance, the total complex associated with the Godement resolution from Remark 2.12:

$$\mathbb{H}^n(M, A) = H^n(\mathrm{Tot}(\Gamma(M, \mathcal{C}^*(A^*)))).$$

If an element of $\mathbb{H}^n(M, A)$ is represented by a closed element

$$\omega = \sum_{i+j=n} \omega_{i,j}, \quad \omega_{i,j} \in \Gamma(M, \mathcal{C}^i(A^j)),$$

then ω is sent, by the above identification, to the element

$$\omega[m] = \begin{cases} \sum_{i+j=n} \left(\prod_{\ell=j}^{j-m-1} (-1)^\ell\right) \omega_{i,j}, & \text{if } m < 0, \\ \sum_{i+j=n} \left(\prod_{\ell=j-m}^{j-1} (-1)^\ell\right) \omega_{i,j}, & \text{if } m > 0. \end{cases} \tag{3.145}$$

See Exercise 3.195 for some properties of the operation $\omega \mapsto \omega[m]$.

As in the previous sections, let M be a connected differentiable manifold which has the homotopy type of a finite CW complex, and let $Y_0, \dots, Y_k$ be a finite collection of closed subsets of M. Write

$$Y = Y_0 \cup \cdots \cup Y_k.$$

Notation 3.146. The following notations will be used:

- $[k]$ stands for the index set $\{0, \dots, k\}$. Although it can be confused with the shift $[k]$, it will be clear from the context when we are referring to the set or to the complex.
- For each subset $J \subseteq [k]$, we write Y_J for the intersection $\bigcap_{j \in J} Y_j$.

- Given a topological space T, we denote by $\underline{\mathbb{Q}}_T$ the constant sheaf with stalk $\mathbb{Q}$ on T. We write $\underline{\mathbb{Q}}_J$ for the constant sheaf on Y_J extended by zero to M, that is,
$$\underline{\mathbb{Q}}_J = (\iota_J)_* \underline{\mathbb{Q}}_{Y_J},$$
where $\iota_J \colon Y_J \hookrightarrow M$ is the inclusion. In particular, $\underline{\mathbb{Q}}_\emptyset = \underline{\mathbb{Q}}_M$. If the inclusion is clear from the context and no confusion may arise, we will also denote by $\underline{\mathbb{Q}}_{Y_J}$ the extension by zero to M. For example, if Y_J consists of a single point x, we will rather write $\underline{\mathbb{Q}}_x$ for the skyscraper sheaf with stalk $\mathbb{Q}$ at x.
- Given subsets $J \subseteq K \subseteq [k]$, there is an inclusion $Y_K \subseteq Y_J$. We denote by $d_{K,J} \colon \underline{\mathbb{Q}}_J \to \underline{\mathbb{Q}}_K$ the corresponding restriction map.
- If $K = \{k_0, \dots, k_p\}$ with the indices k_l ordered as $k_0 < \dots < k_p$, and $J = \{k_0, \dots, \hat{k}_i, \dots, k_p\}$, we set $\epsilon(J, K) = (-1)^i$.

For $1 \leq p \leq k+1$, we define a morphism of sheaves
$$d \colon \bigoplus_{|J|=p-1} \underline{\mathbb{Q}}_J \longrightarrow \bigoplus_{|K|=p} \underline{\mathbb{Q}}_K, \quad \text{by} \quad d = \bigoplus_{K,J} \epsilon(J,K) d_{K,J}.$$
We define the complex of sheaves $\widetilde{K}(M; Y_0, \dots, Y_k)$ as
$$0 \to \bigoplus_{|J|=0} \underline{\mathbb{Q}}_J \to \bigoplus_{|J|=1} \underline{\mathbb{Q}}_J \to \dots \to \bigoplus_{|J|=k} \underline{\mathbb{Q}}_J \to \bigoplus_{|J|=k+1} \underline{\mathbb{Q}}_J \to 0, \tag{3.147}$$
and the complex $K(M; Y_0, \dots, Y_k)$ as
$$0 \to \bigoplus_{|J|=0} \underline{\mathbb{Q}}_J \to \bigoplus_{|J|=1} \underline{\mathbb{Q}}_J \to \dots \to \bigoplus_{|J|=k} \underline{\mathbb{Q}}_J \to 0. \tag{3.148}$$
Note that the second complex agrees with the first one except for the last term $\underline{\mathbb{Q}}_{[k]}$ that has been deleted.

LEMMA 3.149. *If all possible intersections of the Y_i are locally contractible, then*
$$\mathbb{H}^n(M, \widetilde{K}(M; Y_0, \dots, Y_k)) = H^n(M, Y, \mathbb{Q}),$$
where the right hand side is relative singular cohomology.

PROOF. The sequence of sheaves
$$0 \to \underline{\mathbb{Q}}_Y \to \bigoplus_{|J|=1} \underline{\mathbb{Q}}_J \to \dots \to \bigoplus_{|J|=p} \underline{\mathbb{Q}}_J \to \dots \to \bigoplus_{|J|=k+1} \underline{\mathbb{Q}}_J \to 0$$
is a resolution *à la Čech* of the constant sheaf $\underline{\mathbb{Q}}_Y$ by the finite closed covering given by the Y_i, extended by zero to the whole M. Therefore it is exact. Hence the complex $\widetilde{K}(M; Y_0, \dots, Y_k)$ is quasi-isomorphic to the complex
$$0 \to \underline{\mathbb{Q}}_M \longrightarrow \iota_* \underline{\mathbb{Q}}_Y \to 0,$$
where $\iota \colon Y \to M$ is the inclusion. The result follows from Example 2.17. □

3.5.2. *Beilinson's theorem.* As in the previous sections, let M be a connected differentiable manifold which has the homotopy type of a finite CW complex, let $x, y \in M$ be base points, and $N \geq 0$ an integer. Let M^N be the N-fold cartesian product of M. Given a point of M^N, we denote by $x_1, \ldots, x_N$ its components. Consider the union $Y = Y_0 \cup \cdots \cup Y_N$ of the closed subspaces $Y_i \subset M^N$ given by:

$$\begin{aligned} Y_0 &= \{x_1 = y\}, \\ Y_i &= \{x_i = x_{i+1}\}, \quad i = 1, \ldots, N-1, \\ Y_N &= \{x_N = x\}. \end{aligned}$$

Sometimes it will useful to introduce extra components $x_0 = y$ and $x_{N+1} = x$ on M^N and write $Y_i = \{x_i = x_{i+1}\}$ for all $i = 0, \ldots, N$.

We define the complexes

$$\begin{aligned} {}_y\mathcal{K}_x\langle N\rangle &= K(M^N; Y_0, \ldots, Y_N), \\ {}_y\widetilde{\mathcal{K}}_x\langle N\rangle &= \widetilde{K}(M^N; Y_0, \ldots, Y_N). \end{aligned}$$

If the base points x and y are distinct, then $Y_0 \cap \cdots \cap Y_N = \emptyset$ and hence the two complexes are equal: ${}_y\mathcal{K}_x\langle N\rangle = {}_y\widetilde{\mathcal{K}}_x\langle N\rangle$. By Lemma 3.149, their hypercohomology computes the relative cohomology group

$$\mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle) = H^*(M^N, Y).$$

In the case where $x = y$, the intersection $Y_0 \cap \cdots \cap Y_N = \{(x, \ldots, x)\}$ consists of a single point and there is a short exact sequence of complexes

$$0 \to \underline{\mathbb{Q}}_{(x,\ldots,x)}[-N-1] \to {}_x\widetilde{\mathcal{K}}_x\langle N\rangle \to {}_x\mathcal{K}_x\langle N\rangle \to 0. \tag{3.150}$$

Note that the leftmost complex has only non-trivial cohomology in degree $N+1$, where it is isomorphic to $H^0(M^N, \underline{\mathbb{Q}}_{(x,\ldots,x)}) = \mathbb{Q}$. Thus, taking hypercohomology from (3.150) yields an exact sequence

$$0 \longrightarrow H^N(M^N, Y, \mathbb{Q}) \longrightarrow \mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle) \xrightarrow{f} \mathbb{Q} \longrightarrow \cdots \tag{3.151}$$

The map f is surjective because it fits into a commutative diagram

$$\begin{tikzcd} \bigoplus_{|I|=N} H^0(Y_I, \mathbb{Q}) \arrow[r] \arrow[dr] & \mathbb{H}^N(M^N, {}_x\mathcal{K}_x\langle N\rangle) \arrow[d, "f"] \\ & \mathbb{H}^0(M^N, \underline{\mathbb{Q}}_{(x,\ldots,x)}) \end{tikzcd}$$

where the diagonal arrow is surjective. The kernel of f is thus $H^N(M^N, Y, \mathbb{Q})$.

We now introduce a relative version of the complex ${}_y\mathcal{K}_x\langle N\rangle$, where we fix x but let y vary. For this, we consider the $(N+1)$–fold product M^{N+1}, with components $x_0, \ldots, x_N$, and the closed subsets

$$Z_i = \{x_i = x_{i+1}\} \subseteq M^{N+1}, \qquad i = 0, \ldots N,$$

where we are still writing $x_{N+1} = x$. We then define

$$
\begin{aligned}
{}_{\bullet}\mathcal{K}_x\langle N\rangle &= K(M^{N+1}; Z_0, \dots, Z_N)\\
{}_{\bullet}\widetilde{\mathcal{K}}_x\langle N\rangle &= \widetilde{K}(M^{N+1}; Z_0, \dots, Z_N).
\end{aligned}
$$

Let $\pi\colon M^{N+1} \to M$ be the projection onto the first factor. We identify the fibre $\pi^{-1}(y)$ over a point $y \in M$ with M^N (with $x_0 = y$). Via this identification, we have $Z_I \cap \pi^{-1}(y) = Y_I$ for all subsets $I \subseteq [N+1]$, hence

$$
{}_{\bullet}\mathcal{K}_x\langle N\rangle|_{\pi^{-1}(y)} = {}_y\mathcal{K}_x\langle N\rangle. \tag{3.152}
$$

It is in this sense that ${}_{\bullet}\mathcal{K}_x\langle N\rangle$ is a relative version of ${}_y\mathcal{K}_x\langle N\rangle$.

The complexes ${}_{\bullet}\mathcal{K}_x\langle N\rangle$ and ${}_y\mathcal{K}_x\langle N\rangle$ satisfy a recurrence relation that will be useful later. The identity morphism between $M^{(N-1)+1}$ and M^N changes the numbering of the components because in the convention we are using, the components of $M^{(N-1)+1}$ start with x_0 while those of M^N start with x_1. This identification sends Z_i, $i = 0, \dots, N-1$ to Y_{i+1}. Let $\iota_y\colon M^{N-1} \to M^N$ be the map

$$
\iota_y(x_1, \dots, x_{N-1}) = (y, x_1, \dots, x_{N-1}).
$$

For each $N > 0$ there is an exact sequence of sheaves of complexes

$$
0 \to (\iota_y)_*{}_y\mathcal{K}_x\langle N-1\rangle[-1] \to {}_y\mathcal{K}_x\langle N\rangle \to {}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle \to 0. \tag{3.153}
$$

To describe this sequence we use the notation that, if $I = (i_1, \dots, i_k)$ is a multiindex, then the multi-index $I+1$ is

$$
I + 1 = (i_1 + 1, \dots, i_k + 1).
$$

Then in degree $0 \leq k \leq N$ is

$$
0 \to \bigoplus_{\substack{I \subset \{0,\dots,N-1\}\\ |I| = k-1}} \underline{\mathbb{Q}}_{\{0\}\cup(I+1)} \to \bigoplus_{\substack{I \subset \{0,\dots,N\}\\ |I| = k}} \underline{\mathbb{Q}}_I \to \bigoplus_{\substack{I \subset \{1,\dots,N\}\\ |I| = k}} \underline{\mathbb{Q}}_I \to 0
$$

The exact sequence (3.150) induces an exact sequence

$$
0 \to \underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \to {}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle \to {}_{\bullet}\mathcal{K}_x\langle N-1\rangle \to 0. \tag{3.154}
$$

In the relative situation, the analogue of the hypercohomology groups of the complex ${}_y\mathcal{K}_x\langle N\rangle$ are the higher direct image sheaves $R^n\pi_*({}_{\bullet}\mathcal{K}_x\langle N\rangle)$. Recall that they are defined as the sheaves of vector spaces associated with the presheaves that, to an open subset $U \subseteq M$, assign the vector space

$$
\mathbb{H}^n(\pi^{-1}(U), {}_{\bullet}\mathcal{K}_x\langle N\rangle).
$$

To understand them, we shall use the following concrete description. Let $S^*(T, \mathbb{Q})$ denote the complex of smooth singular cochains on a differentiable manifold T. Starting from the construction (3.147) applied to M^{N+1} and the subsets $Y_1, \ldots, Y_N$ we obtain a double complex

$$0 \to \bigoplus_{|I|=0} S^*(Y_I \cap \pi^{-1}(U), \mathbb{Q}) \to \bigoplus_{|I|=1} S^*(Y_I \cap \pi^{-1}(U), \mathbb{Q}) \to \ldots$$
$$\cdots \to \bigoplus_{|I|=N} S^*(Y_I \cap \pi^{-1}(U), \mathbb{Q}) \to 0 \quad (3.155)$$

which will be denoted by $S^*(Y_\bullet \cap \pi^{-1}(U), \mathbb{Q})$. The associated total complex computes the hypercohomology

$$\mathbb{H}^n(\pi^{-1}(U), {}_\bullet\mathcal{K}_x\langle N\rangle) = H^n(\mathrm{Tot}(S^*(Y_\bullet \cap \pi^{-1}(U), \mathbb{Q}))).$$

Lemma 3.156. *For every contractible open subset U of M and every point $y \in U$, the inclusion $\pi^{-1}(y) \to \pi^{-1}(U)$ and the identification $\pi^{-1}(y) \simeq M^N$ induce an isomorphism*

$$\mathbb{H}^n(\pi^{-1}(U), {}_\bullet\mathcal{K}_x\langle N\rangle) \to \mathbb{H}^n(M^N, {}_y\mathcal{K}_x\langle N\rangle).$$

In particular, the sheaf $R^n\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$ is a local system on M whose fibre over a point y is given by

$$R^n\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)_y = \mathbb{H}^n(M^N, {}_y\mathcal{K}_x\langle N\rangle).$$

Proof. For every $I \subseteq 1, \ldots, N+1$ the morphism $\pi|_{Y_I} \colon Y_I \to M$ is a fibration. Therefore, given any contractible open subset $U \subseteq M$ and any point $y \in U$, the inclusion $Y_I \cap \pi^{-1}(y) \to Y_I \cap \pi^{-1}(U)$ is a homotopy equivalence. The induced morphism of complexes

$$S^*(Y_I \cap \pi^{-1}(U), \mathbb{Q}) \to S^*(Y_I \cap \pi^{-1}(y), \mathbb{Q})$$

is thus a homotopy equivalence as well. The lemma follows from this. □

Thanks to the lemma, the sheaf $R^n\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$ "glues together" the hypercohomology groups $\mathbb{H}^n(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ for all possible base points y. The map f in the exact sequence (3.151) yields a morphism of sheaves

$$R^n\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle) \to \underline{\mathbb{Q}}_x. \quad (3.157)$$

We also consider the sheaves of vector spaces over M, $\mathbb{Q}[\pi_1(M; \bullet, x)]$ and $\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}$. These sheaves are local systems and the fibre at a point y is given by

local systems of paths

$$\mathbb{Q}[\pi_1(M; \bullet, x)]_y = \mathbb{Q}[\pi_1(M; y, x)].$$

Thus, for every contractible open subset U and point $y \in U$ there is a canonical identification $\mathbb{Q}[\pi_1(M; \bullet, x)](U) = \mathbb{Q}[\pi_1(M; y, x)]$ and a similar identification for $\mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}$. The unit of $\pi_1(M; x)$ induces morphisms of sheaves

$$\underline{\mathbb{Q}}_x \to \mathbb{Q}[\pi_1(M; \bullet, x)]$$
$$\underline{\mathbb{Q}}_x \to \mathbb{Q}[\pi_1(M; \bullet, x)]/\mathbb{Q}[\pi_1(M; \bullet, x)]J^{N+1}. \quad (3.158)$$

We next construct a morphism between the local systems $R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$ and $\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1}$. Let $y\in M$ and let $\gamma\colon[0,1]\to M$ be a smooth path such that $\gamma(0)=x$ and $\gamma(1)=y$. For each subset $I\subseteq\{1,\dots,N+1\}$ of cardinal k, the closed subset $Y_I\subseteq M^{N+1}$ can be identified with M^{N+1-k} by deleting the components which are repeated or equal to x. With each $I\subseteq[N]$ we associate the sign

$$\epsilon(I)=\prod_{i\in I}(-1)^i. \tag{3.159}$$

We denote by $\sigma_\gamma^{N,I}$ the map $\Delta^{N-|I|}\longrightarrow Y_I\cap\pi^{-1}(y)$ given by

$$\sigma_\gamma^{N,I}(t_1,\dots,t_{N-|I|})=\big(y,\gamma(t_1),\dots,\gamma(t_{N-|I|})\big).$$

By linearity, every representative γ of an element in $\mathbb{Q}[\pi_1(M;y,x])]$ gives rise to a smooth singular chain $\sigma_\gamma^{N,I}$ in $Y_I\cap\pi^{-1}(y)$.

For each y we denote by

$$\sigma_y\colon\mathbb{H}^N(M^N,{}_y\mathcal{K}_x\langle N\rangle)\to\mathbb{Q}[\pi_1(M;y,x])]^\vee \tag{3.160}$$

the map that assigns to a closed element

$$\omega=\sum_I\omega_I\in\operatorname{Tot}^N(S^*(Y_\bullet\cap\pi^{-1}(Y),\mathbb{Q}))$$

the linear map $\sigma_y(\omega)\in\mathbb{Q}[\pi_1(M;y,x])]^\vee$ given by

$$\sigma_y(\omega)([\gamma])=\sum_I\epsilon(I)\omega_I(\sigma_\gamma^{N,I}).$$

LEMMA 3.161. *The above defined maps σ_y glue together to a morphism of local systems*

$$\sigma\colon R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)\to(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1})^\vee.$$

PROOF. That σ is well defined amounts to the following facts:

(i) If ω is exact in the complex $\operatorname{Tot}((S^*(Y_\bullet\cap\pi^{-1}(y),\mathbb{Q})))$, then $\sigma(\omega)=0$.

(ii) If γ and γ' are homotopy equivalent paths, then

$$\sum_I\omega_I(\Delta_\gamma^{N,I})=\sum_I\omega_I(\Delta_{\gamma'}^{N,I})$$

(iii) If $[\gamma]\in J^{N+1}$, then

$$\sum_I\omega_I(\Delta_\gamma^{N,I})=0.$$

(iv) For a path γ_0 from y to y', let T_{γ_0} be the parallel transport along γ_0 for both local systems $(\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1})^\vee$ and $R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$. Then

$$T_{\gamma_0}\circ\sigma_y=\sigma_{y'}\circ T_{\gamma_0}.$$

Statements (i) to (iii) show that the morphism σ is well defined at the level of fibres and and statement (iv) shows that it is indeed a morphism of local systems. □

The following result, due to Beilinson, gives a cohomological interpretation of the finite-dimensional pieces in the pro-unipotent completion of the fundamental group. There are two proofs of this theorem in the literature [**Gon01**, §4] and [**DG05**, §3.3].

THEOREM 3.162 (Beilinson).

(1) The sheaf $R^i\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle)$ vanishes for all $i \le N-1$. In particular,

$$\mathbb{H}^i(M^N, {}_y\mathcal{K}_x\langle N\rangle) = 0, \quad i \le N-1.$$

(2) The maps σ_y defined in (3.160) glue together to an isomorphism of local systems

$$\sigma\colon R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle) \to (\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1})^\vee.$$

In particular, there are natural isomorphisms

$$\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow (\mathbb{Q}[\pi_1(M;y,x)]/\mathbb{Q}[\pi_1(M;y,x)]J^{N+1})^\vee.$$

(3) The diagram

$$\begin{tikzcd} R^N\pi_*({}_\bullet\mathcal{K}_x\langle N\rangle) \arrow[r, "\sigma"] \arrow[dr] & (\mathbb{Q}[\pi_1(M;\bullet,x)]/\mathbb{Q}[\pi_1(M;\bullet,x)]J^{N+1})^\vee \arrow[d] \\ & \underline{\mathbb{Q}}_x, \end{tikzcd}$$

where the diagonal arrow is (3.157) and the vertical arrow is the dual of the unit (3.158), is commutative.

PROOF. We first prove statement (3) in the theorem. Since for $y \neq x$ the fibre $((\iota_x)_*\mathbb{Q}_x)_y = 0$ we only need to check what happens at the point x. Then the statement reduces to the commutativity of the diagram

$$\begin{tikzcd} \mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \arrow[r] \arrow[d] & (\mathbb{Q}[\pi_1(M,x)]/J^{N+1})^\vee \arrow[d] \\ \mathbb{H}^N(M^N, \underline{\mathbb{Q}}_{(x,\dots,x)}[-N]) \arrow[r] & \mathbb{Q} \end{tikzcd} \tag{3.163}$$

We can compute $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ as the cohomology of the complex

$$C^* = \operatorname{Tot}(\widetilde{S}^*(Y_I,\mathbb{Q})), \quad |I| \le N$$

where $\widetilde{S}^*$ denotes the normalized complex of smooth cochains. This means that the elements of $\widetilde{S}$ vanish on degenerate chains.

Let γ_x be the constant path x in M. It defines different chains $\sigma_{\gamma_x}^{N,I}$. These chains are degenerate unless $|I| = N$. When $|I| = N$, $\sigma_{\gamma_x}^{N,I}$ is the zero dimensional simplex at the point $(x, \dots, x)$.

Let $\omega = \sum_I \omega_I$ be a closed element of the complex C^*. The left vertical map on the diagram sends ω to

$$\sum_{|I|=N} \epsilon(I, [N])\omega_I(\sigma_{\gamma_x}^{N,I}) \tag{3.164}$$

If we apply the bottom arrow to (3.164) we obtain, taking into account the convention (3.145) to the element

$$\sum_{|I|=N} \epsilon([N])\epsilon(I, [N])\omega_I(\sigma_{\gamma_x}^{N,I}). \tag{3.165}$$

If we apply the top arrow followed by the left vertical arrow we obtain the element

$$\sum_{I} \epsilon(I)\omega_I(\sigma_{\gamma_x}^{N,I}).$$

Since we are using the complex of normalized cochains, this reduces to

$$\sum_{|I|=N} \epsilon(I)\omega_I(\sigma_{\gamma_x}^{N,I}). \tag{3.166}$$

The equality between (3.165) and (3.166) follows from the identity

$$\epsilon([N])\epsilon(I, [N]) = \prod_{i\in[N]} (-1)^i \prod_{i\notin I}(-1)^i = \prod_{i\in I}(-1)^i = \epsilon(I).$$

We now turn to the proof of statements (1) and (2) in the theorem. We proceed by induction on N. The case $N = 0$ is obvious. Since we already now that σ is a morphism of sheaves, it is enough to prove the statements fibrewise. Let $y \in M$. From the exact sequence (3.153), we deduce a long exact sequence

$$\begin{tikzcd}
 & & \mathbb{H}^{N-1}(M^N, {}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle) \arrow[dll, rounded corners] \\
\mathbb{H}^{N-1}(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r, "\iota"] & \mathbb{H}^{N}(M^{N}, {}_y\mathcal{K}_x\langle N\rangle) \arrow[r] & \mathbb{H}^{N}(M^N, {}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle) \arrow[dll, rounded corners, "g"] \\
\mathbb{H}^{N}(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) & &
\end{tikzcd} \tag{3.167}$$

and use it to write down the following diagram with exact rows:

$$\begin{tikzcd}
 & \mathbb{H}^{N-1}(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r, "\iota"] \arrow[d, "\sigma"] & \mathbb{H}^{N}(M^{N}, {}_y\mathcal{K}_x\langle N\rangle) \arrow[r] \arrow[d, "\sigma"] & \mathrm{Ker}(g) \arrow[r] & 0 \\
0 \arrow[r] & (\mathbb{Q}[\pi_1(M; y, x)]/J^N)^{\vee} \arrow[r] & (\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1})^{\vee} \arrow[r] & (J^N/J^{N+1})^{\vee} \arrow[r] & 0.
\end{tikzcd} \tag{3.168}$$

Claim: The left square in the above diagram is commutative.

Indeed, in view of the sign convention 3.144, the first horizontal mal sends a closed form

$$\omega = \sum_{I\subseteq[N-1]} \omega_I$$

representing a class in $\mathbb{H}^{N-1}(M^N, {}_y\mathcal{K}_x\langle N-1\rangle)$ to the form

$$\iota(\omega) = \sum_{I\subseteq[N-1]} (-1)^{|I|}\omega_I,$$

where ω_I is now seen as a cochain in $Y_{\{0\}\cup(I+1)}$. Therefore, one has

$$\sigma(\omega)(\gamma) = \sum_{I\subseteq[N-1]} \epsilon(I)\omega_I(\sigma_\gamma^{N-1,I})$$
$$\sigma(\iota(\omega))(\gamma) = \sum_{I\subseteq[N-1]} \epsilon(\{0\}\cup I)(-1)^I\omega_I(\sigma_\gamma^{N,\{0\}\cup I})$$

for every class $\gamma \in \mathbb{Q}[\pi_1(M;y,x)]$. Since the chains $\sigma_\gamma^{N-1,I}$ and $\sigma_\gamma^{N,\{0\}\cup I}$ are equal and one has $\epsilon(\{0\}\cup I)(-1)^{|I|} = \epsilon(I)$, the square commutes.

Once that we know that the square commutes, an easy diagram chase shows that there exists a map

$$\sigma\colon \operatorname{Ker}(g) \longrightarrow (J^N/J^{N+1})^\vee,$$

that slightly abusively will still be denoted by σ, completing (3.167) to a commutative diagram.

Lemma 3.169.

(1) *The equality* $\mathbb{H}^i(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = 0$ *holds for all* $i \leq N-1$.

(2) *The map* $\sigma\colon \operatorname{Ker}(g) \to (J^N/J^{N+1})^\vee$ *is an isomorphism.*

Proof. Let $\pi\colon M^N \to M$ be the projection onto the first factor (note that, in what precedes, the same symbol π was used to denote the projection onto the first factor of M^{N+1}). We shall compute $\mathbb{H}^i(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)$ using the Leray spectral sequence associated with π:

$$E_2^{p,q} = H^p(M, R^q\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \Longrightarrow \mathbb{H}^{p+q}(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle). \quad (3.170)$$

Taking higher direct images with respect to π from the exact sequence of complexes (3.154) yields isomorphisms

$$R^i\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \cong R^i\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle), \quad i \leq N-2,$$

and an exact sequence of sheaves

$$0 \to R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \to R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle) \to \underline{\mathbb{Q}}_x \to 0. \quad (3.171)$$

The exactness on the right follows, after passing to the fibre at x, from the surjectivity of the map f in the sequence (3.151).

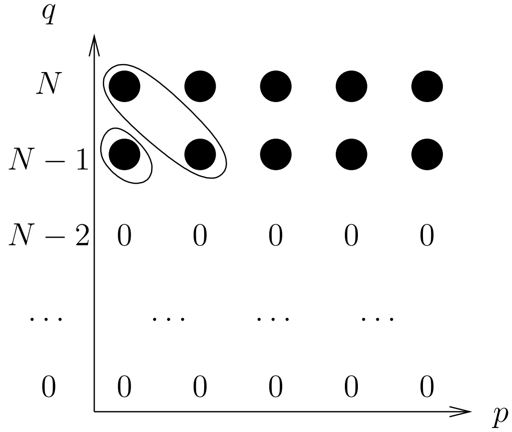


FIGURE 14. The Leray spectral sequence for ${}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle$.

Now recall that the induction hypothesis in the proof of the theorem is that $R^i\pi_*({}_{\bullet}\mathcal{K}_x\langle N-1\rangle)$ vanishes for all $i \leq N-2$, hence

$$R^i\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle) = 0 \text{ for all } i \leq N-2.$$

Therefore, the Leray spectral sequence (3.170) looks as depicted in Figure 14. From this we deduce the equality

$$\mathbb{H}^i(M^N, {}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle) = \begin{cases} 0 & i \leq N-2 \\ H^0(M, R^{N-1}\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle)) & i = N-1 \end{cases} \tag{3.172}$$

and a short exact sequence of vector spaces

$$\begin{aligned} 0 \to H^1(M, R^{N-1}\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \to \mathbb{H}^N(M^N, {}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle) \to \\ \to H^0(M, R^N\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \to 0. \end{aligned} \tag{3.173}$$

To prove statement (1) in the lemma, it remains to show that

$$H^0(M, R^{N-1}\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle)) = 0.$$

The long exact sequence of cohomology associated with the short exact sequence of sheaves (3.171) yields

$$\begin{aligned} 0 \to H^0(R^{N-1}\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \to H^0(R^{N-1}\pi_*({}_{\bullet}\mathcal{K}_x\langle N-1\rangle)) \xrightarrow{a} \mathbb{Q} \to \\ \to H^1(R^{N-1}\pi_*({}_{\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \xrightarrow{b} H^1(R^{N-1}\pi_*({}_{\bullet}\mathcal{K}_x\langle N-1\rangle)) \to 0. \end{aligned} \tag{3.174}$$

We shall prove that the map a is an isomorphism, hence the connecting morphism above is zero and the map b is an isomorphism as well. For this we need to compute the cohomology of the sheaf $R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$. By the induction hypothesis in the theorem, the map

$$\sigma\colon R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle) \longrightarrow (\mathbb{Q}[\pi_1(M;\bullet,x)]/J^N)^\vee \tag{3.175}$$

is an isomorphism and in particular the sheaf $R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$ is a local system on M with fibre

$$R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)_x \cong (\mathbb{Q}[\pi_1(M,x)]/J^N)^\vee. \tag{3.176}$$

Setting $\Gamma = \pi_1(M,x)$, the cohomology of $R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)$ can be computed as the group cohomology of Γ acting on (3.176):

$$H^i(M, R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)) = H^i(\Gamma, (\mathbb{Q}[\Gamma]/J^N)^\vee).$$

Consider the short exact sequence of Γ-modules

$$0 \to (\mathbb{Q}[\Gamma]/J^N)^\vee \to \mathbb{Q}[\Gamma]^\vee \to (J^N)^\vee \to 0. \tag{3.177}$$

The Γ-module $\mathbb{Q}[\Gamma]^\vee$ being injective, its cohomology is concentrated in degree zero and there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(\Gamma,(\mathbb{Q}[\Gamma]/J^N)^\vee) \arrow[r] & H^0(\Gamma,\mathbb{Q}[\Gamma]^\vee) \arrow[r] & H^0(\Gamma,(J^N)^\vee) \arrow[dll, out=-10, in=170] \\ & H^1(\Gamma,(\mathbb{Q}[\Gamma]/J^N)^\vee) \arrow[rr] & & 0. \end{tikzcd} \tag{3.178}$$

Recall that, if A is a Γ-module, then $H^0(\Gamma, A)$ is the group of invariants A^Γ. From this one easily checks:

- The cohomology $H^0(\Gamma,\mathbb{Q}[\Gamma]^\vee)$ is the one-dimensional $\mathbb{Q}$-vector space generated by the function
$$\sum_{\gamma\in\Gamma} a_\gamma[\gamma] \mapsto \sum_{\gamma\in\Gamma} a_\gamma$$
and the dual of the unit (3.158) induces an isomorphism
$$H^0(\Gamma,\mathbb{Q}[\Gamma]^\vee) \xrightarrow{\sim} \mathbb{Q}.$$
- The cohomology $H^0(\Gamma,(J^N)^\vee)$ is equal to $(J^N/J^{N+1})^\vee$ and the map
$$H^0(\Gamma,\mathbb{Q}[\Gamma]^\vee) \to (J^N/J^{N+1})^\vee$$
in the long exact sequence (3.178) is the zero map.

Putting together the above facts, the isomorphism (3.175), and the long exact sequence (3.178), we deduce

$$H^0(M, R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)) \cong \mathbb{Q} \tag{3.179}$$
$$H^1(M, R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)) \cong (J^N/J^{N+1})^\vee. \tag{3.180}$$

Besides, the map a in (3.174) agrees with the isomorphism (3.179) by statement (3) in the theorem. From this and (3.172), we derive

$$\mathbb{H}^{N-1}(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) = H^0(M, R^{N-1}\pi_{*\bullet}\widetilde{\mathcal{K}}_x\langle N-1\rangle) = 0,$$

thus concluding the proof of statement (1) in the theorem.

We now turn to the proof of statement (2) in the lemma. Combining the fact that the map b in (3.174) is an isomorphism with (3.180), we get

$$H^1(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \cong (J^N/J^{N+1})^\vee.$$

Besides, by the exact sequence (3.173), there is an inclusion

$$H^1(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \subseteq \mathbb{H}^N(M^N, {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle).$$

Claim: This subspace is equal to $\mathrm{Ker}(g)$.

To prove the claim, we consider the long exact sequence of sheaves obtained by taking higher direct images from (3.153):

$$\begin{aligned}\cdots \to R^{N-1}\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle) &\xrightarrow{\varphi} R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)\\ &\to R^N\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \to R^N\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle) \to \cdots \qquad (3.181)\end{aligned}$$

Note that the sheaves $R^q\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)$ are all skyscraper sheaves supported at the point $y \in M$, hence have only cohomology in degree zero. Therefore, in the exact sequence

$$\begin{aligned}0 \to H^1(M, R^{N-1}\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)) &\to \mathbb{H}^N(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle)\\ &\to H^0(M, R^N\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)) \to 0 \qquad (3.182)\end{aligned}$$

obtained by applying the Leray spectral sequence to $(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)$, the leftmost term vanishes and the last but one map is an isomorphism.

Let us introduce the sheaf $\mathcal{F} = \mathrm{Coker}(\varphi)$ and consider the commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & H^1(R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \mathbb{H}^N(M^N, {}_\bullet\mathcal{K}_x\langle N-1\rangle) \arrow[r, "g"] \arrow[d] & \mathbb{H}^N(M^{N-1}, {}_y\mathcal{K}_x\langle N-1\rangle) \arrow[d] \\
H^0(\mathcal{F}) \arrow[r] & H^0(R^N\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \arrow[r] \arrow[d] & H^0(R^N\pi_*(\iota_y)_*({}_y\mathcal{K}_x\langle N-1\rangle)) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

where the first column is (3.173), the second column is (3.182), and the last row is part of the exact sequence obtained by taking cohomology from (3.181). The above diagram immediately implies that

$$H^1(M, R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)) \subseteq \mathrm{Ker}(g)$$

and to prove the reverse inclusion it is enough to show that $H^0(M, \mathcal{F}) = 0$. To get this vanishing we will show that $\mathcal{F}$ is the extension by zero of a local system on $M \setminus \{y\}$. We need to distinguish whether the base points x and y are distinct or equal.

Case $x \neq y$. Write $U = M^N \setminus \pi^{-1}(y)$. Since the complex $(\iota_y)_{*\,y}\mathcal{K}_x\langle N-1\rangle$ is supported at $\pi^{-1}(y)$, one first sees from (3.153) that

$$ {}_y\mathcal{K}_x\langle N\rangle|_U \cong {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle|_U $$

and combining this with (3.154) one obtains a short exact sequence

$$ 0 \to \underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \to {}_y\mathcal{K}_x\langle N\rangle|_U \to {}_\bullet\mathcal{K}_x\langle N-1\rangle|_U \to 0. $$

In the associated long exact sequence

$$ \begin{aligned} R^{N-1}\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle|_U) \xrightarrow{h} R^N\pi_*\underline{\mathbb{Q}}_{(x,\dots,x)}[-N] \to \\ R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle|_U) \to R^N\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle|_U) \to 0, \end{aligned} $$

the map h is surjective. Thus we get an isomorphism

$$ R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)|_{M\setminus\{y\}} \to R^N\pi_*({}_\bullet\mathcal{K}_x\langle N-1\rangle)|_{M\setminus\{y\}}. $$

Since the right-hand side is a local system by Lemma 3.156, the same is true for the left-hand side. Let now $V \subseteq M$ be a contractible open subset containing y. From the exact sequence (3.153) we obtain a long exact sequence of hypercohomology

$$ \begin{aligned} \cdots \to \mathbb{H}^i(\pi^{-1}(V), {}_y\mathcal{K}_x\langle N\rangle) \to \mathbb{H}^i(\pi^{-1}(V), {}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle) \xrightarrow{j} \\ \mathbb{H}^i(\pi^{-1}(y), {}_y\mathcal{K}_x\langle N-1\rangle) \to \cdots \end{aligned} $$

The fact that V is contractible implies that, for all $i \geq 0$, the map j is an isomorphism. Therefore $\mathbb{H}^N(\pi^{-1}(V), {}_y\mathcal{K}_x\langle N\rangle) = 0$, hence

$$ R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)_y = 0. $$

Finally, since the source of the map

$$ \varphi\colon R^{N-1}\pi_*(\iota_y)_*({}_y\widetilde{\mathcal{K}}_x\langle N-1\rangle) \to R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle) $$

is a skyscraper sheaf supported at y, it follows that this map is identically zero. We have thus shown that $\mathcal{F} = R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)$ is the extension by zero of a local system on $M \setminus \{y\}$.

Case $x = y$. On $U = M^N \setminus \pi^{-1}(x)$, the exact sequence (3.154) yields an isomorphism

$$ {}_x\mathcal{K}_x\langle N\rangle|_U \cong {}_\bullet\mathcal{K}_x\langle N-1\rangle|_U, $$

that implies that $\mathcal{F}|_{M\setminus\{x\}} = R^N\pi_*({}_x\mathcal{K}_x\langle N\rangle)|_{M\setminus\{x\}}$ is a local system. Let $V \subseteq M$ be a contractible open subset containing x. In this case it is no longer true that ${}_y\mathcal{K}_x\langle N\rangle|_{\pi^{-1}(V)}$ has vanishing hypercohomology. Identifying $(x,\dots,x)$ with $Y_{\{1,\dots,N\}}$ there is a map

$$\underline{\mathbb{Q}}_{(x,\dots,x)}[-N]\to {}_y\mathcal{K}_x\langle N\rangle|_{\pi^{-1}(V)}.$$

Using that V is contractible, this map induces an isomorphism in hypercohomology

$$\mathbb{Q}=\mathbb{H}^N(\pi^{-1}(V),\underline{\mathbb{Q}}_{(x,\dots,x)}[-N])\cong\mathbb{H}^N(\pi^{-1}(V),{}_y\mathcal{K}_x\langle N\rangle).$$

Therefore

$$R^N\pi_*({}_y\mathcal{K}_x\langle N\rangle)_x=\mathbb{Q}\neq 0.$$

In this case the map

$$R^{N-1}\pi_*(\iota_x)_*({}_x\widetilde{\mathcal{K}}_x\langle N-1\rangle)_x\xrightarrow{\varphi}R^N\pi_*({}_x\mathcal{K}_x\langle N\rangle)_x$$

is surjective and we again deduce that $\mathcal{F}_x=0$. Therefore $H^0(M,\mathcal{F})=0$, the map labeled e is injective and

$$\mathrm{Ker}(g)=H^1(M,R^{N-1}\pi_*({}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle))\cong(J^N/J^{N+1})^\vee.$$

To finish the proof of the lemma one needs to check that the above isomorphism is compatible with the map σ. We leave this verification to the reader. □

We can now finish the proof of Beilinson's Theorem 3.162.

Recall that statement (1) is the vanishing $\mathbb{H}^i(M^N,{}_y\mathcal{K}_x\langle N\rangle)=0$ in all degrees $i\le N-1$. By (3.167), this group fits into a long exact sequence

$$\to\mathbb{H}^{i-1}(M^{N-1},{}_y\mathcal{K}_x\langle N-1\rangle)\to\mathbb{H}^i(M^N,{}_y\mathcal{K}_x\langle N\rangle)\to\mathbb{H}^i(M^N,{}_\bullet\widetilde{\mathcal{K}}_x\langle N-1\rangle)$$

For $i\le N-1$, the leftmost term vanishes by the induction hypothesis and the rightmost term vanishes by the first part of Lemma 3.169, hence the middle term vanishes as well.

Finally, to prove statement (2) we observe that, thanks to the long exact sequence (3.167), Lemma 3.169, and the induction hypothesis, in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^{N-1}(M^{N-1},{}_y\mathcal{K}_x\langle N-1\rangle) \arrow[r, "\iota"] \arrow[d, "\sigma"'] & \mathbb{H}^N(M^N,{}_y\mathcal{K}_x\langle N\rangle) \arrow[r] \arrow[d, "\sigma"'] & \mathrm{Ker}(g) \arrow[r] \arrow[d, "\sigma"'] & 0 \\
0 \arrow[r] & (\mathbb{Q}[\pi_1(M;y,x)]/J^N)^\vee \arrow[r] & (\mathbb{Q}[\pi_1(M;y,x)]/J^{N+1})^\vee \arrow[r] & (J^N/J^{N+1})^\vee \arrow[r] & 0
\end{tikzcd} \tag{3.183}$$

the rows are exact and the first and third vertical maps are isomorphisms. By the short five lemma, the second vertical arrow is also an isomorphism. This concludes the proof. □

3.5.3. *Simplicial and cosimplicial objects.* In order to relate the cohomology groups appearing in Beilinson's theorem to the bar construction, we shall use the language of simplicial and cosimplicial objects.

Let $\mathbf{\Delta}$ denote the category with objects the finite sets

$$\Delta_n = \{0, \dots, n\}, \qquad n \geq 0,$$

and morphisms the increasing maps between the various Δ_n with respect to the natural order. Any morphism in $\mathbf{\Delta}$ can be written as a composition of *faces* $\delta^i \colon \Delta_n \to \Delta_{n+1}$, $i = 0, \dots, n+1$, and *degeneracies* $\sigma^i \colon \Delta_{n+1} \to \Delta_n$, $i = 0, \dots, n$, which are defined as follows:

$$\delta^i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases} \qquad \sigma^i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

In other words, δ^i is the map that skips i, while σ^i repeats i

Definition 3.184. Let $\mathcal{C}$ be a category. A *simplicial* (resp. *cosimplicial*) object in $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ (resp. $\mathbf{\Delta} \to \mathcal{C}$).

Using the above characterization of morphisms in $\mathbf{\Delta}$, simplicial and cosimplicial objects admit a very concrete description. For instance, a *cosimplicial object* $X^\bullet$ becomes a collection $(X^n)_{n \geq 0}$ of objects of $\mathcal{C}$, together with morphisms

$$\begin{aligned} \delta^i \colon X^n &\longrightarrow X^{n+1}, \qquad & i &= 0, \dots, n+1 \\ \sigma^i \colon X^{n+1} &\longrightarrow X^n, \qquad & i &= 0, \dots, n, \end{aligned}$$

satisfying the commutativity relations

$$\begin{array}{lll} (a) & \delta^j \delta^i = \delta^i \delta^{j-1}, & \text{for } i < j, \\ (b) & \sigma^j \sigma^i = \sigma^i \sigma^{j+1}, & \text{for } i \leq j, \\ (c) & \sigma^j \delta^i = \delta^i \sigma^{j-1}, & \text{for } i < j, \\ (d) & \sigma^j \delta^i = \mathrm{Id}, & \text{for } i = j, j+1, \\ (e) & \sigma^j \delta^i = \delta^{i-1} \sigma^j, & \text{for } i > j+1. \end{array} \tag{3.185}$$

The maps δ^i and σ^i are again called faces and degeneracies, and one usually represents these data by a diagram of the form

$$X^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X^2 \cdots .$$

The description of a simplicial object is the dual one. It is thus given by a collection of objects $(X_n)_{n \geq 0}$, together with morphisms

$$\begin{aligned} \delta_i \colon X_{n+1} &\longrightarrow X_n, \quad & i &= 0, \dots, n+1 \\ \sigma_i \colon X_n &\longrightarrow X_{n+1}, \quad & i &= 0, \dots, n \end{aligned}$$

satisfying the commutativity relations dual to (3.185).

REMARK 3.186. The category $\mathbf{\Delta}$ is equivalent to the category of totally ordered non-empty finite sets, denoted by **FOS**. Therefore, we can also view a simplicial object $X_\bullet$ as a functor $\mathbf{FOS}^{\mathrm{op}} \to \mathcal{C}$ by setting $X_I = X_{|I|-1}$, where $|I|$ denotes the cardinal of I.

3.5.4. *Simplicial objects and chain complexes.* Simplicial and cosimplicial objects in an abelian category are very close to chain and cochain complexes. In this paragraph, we review some constructions relating them.

Given a simplicial object $X_\bullet$ in an abelian category, the associated chain complex is the complex CX_* with

$$CX_n = X_n, \qquad d = \sum_{i=0}^{n} (-1)^i \delta_i \colon CX_n \to CX_{n-1},$$

and the normalized chain complex is

$$\mathcal{N}X_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} \delta_i \simeq X_n / \sum_{i=0}^{n-1} \operatorname{Im} \sigma_i, \qquad d = \sum_{i=0}^{n} (-1)^i \delta_i.$$

Similarly, if $X^\bullet$ is a cosimplicial object in an abelian category, the associated cochain complex is the complex CX^* with

$$CX^n = X^n, \qquad d = \sum_{i=0}^{n+1} (-1)^i \delta^i \colon CX^n \to CX^{n+1},$$

and the normalized cochain complex is

$$\mathcal{N}X^n = X^n / \sum_{i=0}^{n-1} \operatorname{Im} \delta^i \simeq \bigcap_{i=0}^{n-1} \operatorname{Ker} \sigma^i, \qquad d = \sum_{i=0}^{n+1} (-1)^i \delta^i.$$

Now for each $N \geq 0$ and $X_\bullet$ in an abelian category, we introduce a new complex $C_*(\Delta_N, X_\bullet)$. For each $\emptyset \neq J \subset \Delta_N$, using the convention of Remark 3.186, we have the object $X_J = X_{|J|-1}$. If $K = \{k_0, \dots, k_p\}$ with the indices k_l in increasing order, and $J = \{k_0, \dots, \hat{k}_i, \dots, k_p\}$, we set as before $\epsilon(J, K) = (-1)^i$ and

$$d_{K,J} = \delta_i \colon X_K \to X_J.$$

Then we define

$$C_p(\Delta_N, X_\bullet) = \bigoplus_{\substack{\emptyset \neq I \subset \Delta_N \\ |I| = p+1}} X_I \tag{3.187}$$

with differential $d \colon C_p(\Delta_N, X_\bullet) \to C_{p-1}(\Delta_N, X_\bullet)$ given by

$$d = \bigoplus_{J \subset K} \epsilon(J, K) d_{K,J}.$$

For a chain complex C_*, let $\sigma_{\leq N}$ denote the *bête* filtration

$$\sigma_{\leq N} C_n = \begin{cases} C_n, & \text{if } n \leq N, \\ 0, & \text{if } n > N. \end{cases}$$

For a proof of the following result see [**DG05**, Proposition 3.10].

PROPOSITION 3.188. *Given a simplicial object $X_\bullet$ in an abelian category, the complexes $C_*(\Delta_N, X_\bullet)$ and $\sigma_{\leq N}\mathcal{N}X_*$ are functorially homotopically equivalent.*

3.5.5. *A cosimplicial manifold.* The key ingredient in the proof of Chen's Theorem 3.138 is the following cosimplicial manifold. We keep the notation from the previous paragraphs: M is a connected differentiable manifold having the homotopy type of a finite CW complex, and $x, y \in M$ are base points.

CONSTRUCTION 3.189. We denote by ${}_yM_x^\bullet$ the cosimplicial manifold with components

$${}_yM_x^n = M \times \overset{n}{\cdots} \times M,$$

coface maps

$$\delta^i \colon {}_yM_x^n \to {}_yM_x^{n+1}, \quad i = 0, \dots, n+1,$$

given by

$$\delta^i(x_1, \dots, x_n) = \begin{cases} (y, x_1, \dots, x_n), & \text{if } i = 0, \\ (x_1, \dots, x_i, x_i, \dots, x_n), & \text{if } 0 < i < n+1, \\ (x_1, \dots, x_n, x), & \text{if } i = n+1, \end{cases}$$

and codegeneracy maps

$$\sigma^i \colon {}_yM_x^{n+1} \to {}_yM_x^n, \quad i = 0, \dots, n,$$

given by

$$\sigma^i(x_1, \dots, x_{n+1}) = (x_1, \dots, x_i, x_{i+2}, \dots, x_{n+1}). \tag{3.190}$$

3.5.6. *An isomorphism of cohomology groups.* Given a differentiable manifold X, we denote by $S^*(X, \mathbb{Q})$ the complex of smooth singular cochains on X with rational coefficients and we write

$$S_\bullet^* = S^*({}_yM_x^\bullet, \mathbb{Q}).$$

Since $S_\bullet^*$ is a simplicial object in the category of complexes of $\mathbb{Q}$-vector spaces, we can apply to it the functor $C_*(\Delta_N, \cdot)$ to obtain a chain complex of cochain complexes $C_*(\Delta_N, S_\bullet^*)$. We convert it into a double cochain complex by changing the sign of the chain index,

$$C^p(\Delta_N, S_\bullet^q) \coloneqq C_{-p}(\Delta_N, S_\bullet^q),$$

we obtain a second quadrant complex of complexes. Let $\operatorname{Tot} C^*(\Delta_N, S_\bullet^*)$ denote the corresponding total complex.

REMARK 3.191. We do not need to worry about convergence issues because $C_p(\Delta_N, \cdot) = 0$ for $p < 0$ or $p > N$.

LEMMA 3.192. *There is a functorial isomorphism*

$$\mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow H^*(\operatorname{Tot} C^*(\Delta_N, S^*_\bullet)[-N]).$$

PROOF. We use smooth singular cochains to compute the cohomology groups $\mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle)$, so consider the double complex $S_Y^{*,*}$ given by

$$S_Y^{p,q} = \bigoplus_{|I|=q} S^p(Y_I, \mathbb{Q}), \quad p \geq 0, \quad 0 \leq q \leq N,$$

with differential in the q direction

$$d' \colon \bigoplus_{|J|=q-1} S^p(Y_J, \mathbb{Q}) \longrightarrow \bigoplus_{|K|=q} S^p(Y_K, \mathbb{Q}), \quad d' = \bigoplus_{J \subset K} \epsilon(J, K) d_{K,J}.$$

Let $\operatorname{Tot} S_Y^*$ be the associated total complex. By construction

$$\mathbb{H}^*(M^N, {}_y\mathcal{K}_x\langle N\rangle) = H^*(\operatorname{Tot} S_Y^*).$$

Let $\emptyset \neq I \subset \Delta_N$ and $J = \Delta_N \setminus I$. The inclusion $I \subset \Delta_N$ induces a map

$$M^{|I|-1} = {}_yM_x^I \to {}_yM_x^{\Delta_N} = M^N$$

which is an isomorphism of ${}_yM_x^I$ with Y_J. Let

$$f_J \colon S^*(Y_J, \mathbb{Q}) \to S^*({}_yM_x^I, \mathbb{Q})$$

be the corresponding map of cochains. Let $\epsilon(J) = \prod_{j\in J}(-1)^j$ the sign introduced in (3.159). Since J is the complement of I in Δ_N, then $|J| = N + 1 - |I|$, and

$$\epsilon(J, K)\epsilon(I, \Delta_n \setminus K) = \epsilon(J)\epsilon(K).$$

In consequence, the map

$$\operatorname{Tot} S_Y^* \to \operatorname{Tot} C^*(\Delta_N, S)[-N]$$

that sends $S^p(Y_J, \mathbb{Q})$ to $S^p({}_yM_x^I, \mathbb{Q})$ through the map $\epsilon(J)f_J$ is an isomorphism of complexes, thus proving the lemma. $\square$

3.5.7. *The normalized cochain complex and the reduced bar complex.* As in the previous section, let $E^*(M, \mathbb{C})$ denote the de Rham algebra of smooth complex valued differential forms on M. For simplicity we will assume that we have $A^*(M) \subseteq E^*(M, \mathbb{C})$ a connected differential graded $\mathbb{C}$-algebra such that $A(M) \to E^*(M, \mathbb{C})$ is a quasi-isomorphism. We set

$$A^*({}_yM_x^n) = A^*(M) \otimes \overset{n}{\cdots} \otimes A^*(M).$$

In particular, $A^*({}_yM_x^0) = \mathbb{C}$. These complexes are functorial on $n \in \mathbf{\Delta}$, hence define a simplicial dg-algebra $A^*({}_yM_x^\bullet)$. Thus $\mathcal{N}A^*({}_yM_x^\bullet)$ is a chain complex of cochain complexes. We denote by $\operatorname{Tot} \mathcal{N}A^*({}_yM_x^\bullet)$ the associated total complex.

The subcomplexes $\sigma_{\leq N}\mathcal{N}A^*({}_yM_x^\bullet)$ define a filtration of $\operatorname{Tot}\mathcal{N}A^*({}_yM_x^\bullet)$. Be aware that the index N in the bête filtration refers only to the chain degree and not to the total degree.

LEMMA 3.193. *The map*

$$\begin{aligned}\psi\colon B^*(A^*(M)) &\longrightarrow \operatorname{Tot}\mathcal{N}A^*({}_yM_x^\bullet)\\ [\omega_1|\cdots|\omega_n] &\longmapsto (-1)^{\sum_{i=1}^n i\deg\omega_i}\omega_1\otimes\cdots\otimes\omega_n\end{aligned}$$

is an isomorphism of complexes that sends the N-th step of the length filtration $L_NB^*(A^*(M))$ to $\operatorname{Tot}\sigma_{\leq N}\mathcal{N}A^*({}_yM_x^\bullet)$. Similarly, if ε is the unique augmentation of $A^*(M)$, then the same formula gives us an isomorphism

$$B^*(A^*(M),\varepsilon,\varepsilon)\longrightarrow \operatorname{Tot}\mathcal{C}A^*({}_yM_x^\bullet).$$

PROOF. This lemma is an easy verification. We will only prove the first statement, the second one being analogous. We set

$$A^+(M)=\bigoplus_{n>0}A^n(M)=A^*(M)/\mathbb{C}.$$

By the shape (3.190) of the codegeneracy maps σ^i we deduce that, for $i=0,\dots,n-1$,

$$\operatorname{Im}(\sigma_i)=\operatorname{Im}((\sigma_i)^*)=A^*(M)\otimes\cdots\otimes\underset{i+1}{\mathbb{C}}\otimes\cdots\otimes A^*(M).$$

Therefore

$$\mathcal{N}_nA^*({}_yM_x^\bullet)=A^*({}_yM_x^n)/\sum_{i=0}^{n-1}\operatorname{Im}(\sigma_i)=A^+(M)\otimes\cdots\otimes A^+(M).$$

It follows that the map ψ is an isomorphism of graded vector spaces that respects the filtrations. We next compute the differential in the complex $\operatorname{Tot}\sigma_{\leq N}\mathcal{N}A^*({}_yM_x^\bullet)$. Let

$$\omega=\omega_1\otimes\cdots\otimes\omega_n\in\mathcal{N}^{-n}A^m({}_yM_x^\bullet)\subset\operatorname{Tot}\mathcal{N}^{m-n}A^*({}_yM_x^\bullet).$$

Then $d\omega=d_1\omega+(-1)^md_2\omega$, where d_1 is the differential in the de Rham complex and d_2 is the differential in the normalized complex. Therefore

$$\begin{aligned}d\omega=\sum_{i=1}^n(-1)^{\sum_{j=1}^{i-1}\deg(\omega_j)}\omega_1\otimes\cdots\otimes d\omega_i\otimes\cdots\otimes\omega_n\\ +(-1)^m\sum_{i=1}^{n-1}(-1)^i\omega_1\otimes\cdots\otimes\omega_i\wedge\omega_{i+1}\otimes\cdots\otimes\omega_n.\end{aligned}\tag{3.194}$$

Comparing this formula with the differential in Definition 3.115 and noting that $m=\sum\deg(\omega_i)$, one sees that $\psi\circ d=d\circ\psi$. This finishes the proof. $\square$

3.5.8. *Proof of Theorem 3.138.*

PROOF. If $N=0$, then $L_0B^*(A^*(M))=\mathbb{C}$ given by the constant functions, while

$$\mathbb{Q}[\pi_1(M;y,x)]/J\mathbb{Q}[\pi_1(M;y,x)]=\mathbb{Q}.$$

Moreover, the map in Theorem 3.138 sends the constant function $a\in\mathbb{C}$ to the map that sends $1\in\mathbb{Q}$ to a, that is clearly an isomorphism.

Fix now $N>0$. Applying Lemma 3.193 and Proposition 3.188 we obtain a quasi-isomorphism

$$L_NB^*(A^*(M))\xrightarrow{\sim}\operatorname{Tot}C_*(\Delta_N,A^*({}_yM_x^\bullet)).$$

For each n, the composition

$$A^*(M)^{\otimes n}\otimes\mathbb{C}\longrightarrow E^*({}_yM_x^n,\mathbb{C})\longrightarrow S^*({}_yM_x^n,\mathbb{Q})\otimes\mathbb{C}$$

is a quasi-isomorphism, functorial in n, from which we deduce a quasi-isomorphism

$$L_NB^*(A^*(M))\otimes\mathbb{C}\xrightarrow{\sim}\operatorname{Tot}C_*(\Delta_N,S_\bullet^*)\otimes\mathbb{C}.$$

Combining this quasi-isomorphism with Lemma 3.192 and Theorem 3.162 we deduce the isomorphism

$$H^0(L_NB^*(A^*(M))\otimes\mathbb{C})\longrightarrow(\mathbb{C}[\pi(M;y,x)]/J^{N+1}\mathbb{C}[\pi(M;y,x)])^\vee.$$

Therefore, we get an isomorphism

$$\begin{aligned}H^0(B^*(A^*(M))\otimes\mathbb{C})&=\varinjlim_N H^0(L_NB^*(A^*(M))\otimes\mathbb{C})\longrightarrow\\&(\varprojlim_N\mathbb{C}[\pi(M;y,x)]/J^{N+1}\mathbb{C}[\pi(M;y,x)])^\vee=(\mathbb{C}[\pi(M;y,x)]^\wedge)^\vee,\end{aligned}$$

as we wanted to prove. □

⋆ ⋆ ⋆

EXERCISE 3.195. Let M be a topological space and A^* a complex of abelian groups. Consider the Godement resolution $\mathcal{C}^*$ from Remark 2.12. Let $\omega\in\operatorname{Tot}^n(\Gamma(M,\mathcal{C}^*(A^*)))$ and let $\omega[k]$ be defined as in formula (3.145).

(1) Show that, if ω is closed, then $\omega[k]$ is closed in the complex

$$\operatorname{Tot}^n(\Gamma(M,\mathcal{C}^*(A[k]^*))).$$

(2) Show that $\omega[k][-k]=\omega$.

3.6. A mixed Hodge structure on the pro-unipotent completion of the fundamental group (after Hain).

3.6.1. *Construction of the mixed Hodge structure.* Hain [**Hai87b**] and Morgan [**Mor78**] have shown that, if $M = X(\mathbb{C})$ is the set of complex points of a smooth algebraic variety, then each of the quotients of the pro-unipotent completion of the fundamental group of M carries a natural mixed Hodge structure. Using the geometric interpretation of such quotients provided by Beilinson's Theorem 3.162, one can improve this result a little bit, showing that, in fact, if a variety is defined over a subfield $k \subset \mathbb{C}$ we obtain a mixed Hodge structure over k. Later we will see that Beilinson's theorem allow us to upgrade these quotients to motives. For now, the precise statement is the following.

THEOREM 3.196. *Let k be a subfield of $\mathbb{C}$, X a smooth algebraic variety over k, $M = X(\mathbb{C})$ the set of complex points of X viewed as a differentiable manifold, and $x, y \in X(k) \subseteq M$ two k-rational base points. For each $N \geq 0$, the finite-dimensional $\mathbb{Q}$-vector space*

$$\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1}\mathbb{Q}[\pi_1(M; y, x)]$$

carries a mixed Hodge structure over k which is functorial with respect to morphisms of pointed varieties. Moreover, given integers $N_1 \geq N_2 \geq 0$, the quotient map

$$\mathbb{Q}[\pi_1(X(\mathbb{C}); y, x)]/J^{N_1+1} \longrightarrow \mathbb{Q}[\pi_1(X(\mathbb{C}); y, x)]/J^{N_2+1}$$

is a morphism of mixed Hodge structures over k.

PROOF. We will prove that the dual

$$(\mathbb{Q}[\pi_1(M; y, x)]/J^{N+1}\mathbb{Q}[\pi_1(M; y, x)])^\vee$$

carries a mixed Hodge structure. By Beilinson's theorem 3.162 we know that

$$\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle) \longrightarrow (\mathbb{Q}[\pi(M; y, x)]/J^{N+1}\mathbb{Q}[\pi(M; y, x)])^\vee$$

and the groups $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ can be interpreted as certain relative singular cohomology groups of algebraic varieties over k, thus can be endowed with a mixed Hodge structure over k.

We can also use Lemma 3.192 and Proposition 3.188 to identify the groups $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ with certain singular cohomology groups of a simplicial manifold ${}_yM_x^\bullet$. All the maps involved in ${}_yM_x^\bullet$ are algebraic and defined over k, therefore ${}_yM_x^\bullet$ is the simplicial manifold obtained by taking complex points of a simplicial smooth variety over k. Using a variant over k of the main construction of [**Del74**], we endow $\mathbb{H}^N(M^N, {}_y\mathcal{K}_x\langle N\rangle)$ with a mixed Hodge structure over k.

The claimed functoriality properties follow from the functorial properties of the mixed Hodge structures on the cohomology of simplicial varieties. □

We have constructed a pro-mixed Hodge structure on the pro-unipotent completion of the fundamental group by abstract means. Following Hain [**Hai87b**], Chen's theorem provides us with a very clear and transparent way to construct such mixed Hodge structure. We now sketch this construction when X is smooth and defined over $\mathbb{C}$. We will just show how to define the Hodge and weight filtrations. Consider the dg-algebra $E^*_{\overline{X}^{\mathrm{an}}}(\log D)$ as in Section 2.6.1. It has two augmentations ε_1 and ε_2 given by evaluating at x and y respectively. The Hodge and weight filtrations of $E^*_{\overline{X}^{\mathrm{an}}}(\log D)$ determine the Hodge and weight filtration on $B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))$, by saying that, if $\omega_i \in F^{p_i}$ for $i = 1, \dots, r$, then

$$[\omega_1|\cdots|\omega_r] \in F^{p_1+\cdots+p_r},$$

while, if $\omega_i \in W_{n_i}$, then

$$[\omega_1|\cdots|\omega_r] \in W_{n_1+\cdots+n_r+r},$$

that is, the Hodge type is the sum of Hodge types, while the weight is the sum of weights plus the length of the element. Then

$$\begin{aligned} F^p H^0(B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))) &= \operatorname{Im}(H^0(F^p B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))), \\ W_m H^0(B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))) &= \operatorname{Im}(H^0(W_m B^*(E^*_{\overline{X}^{\mathrm{an}}}(\log D))). \end{aligned}$$

3.6.2. *The case of* $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Let us now specialize to the case where $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ and $M = X(\mathbb{C})$, as in paragraph 3.4.5 and $x, y \in X(\mathbb{Q})$. As we have seen in Example 2.130, we do not need to work with the whole infinite dimensional dg-algebra $E^*_{\overline{X}^{\mathrm{an}}}(\log D)$, but we can work with the smaller $\mathbb{Q}$-algebra

$$A = \mathbb{Q} \oplus \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1.$$

In this case both augmentations ε_1 and ε_2 given by evaluating at x and y respectively agree with the trivial augmentation

$$\begin{array}{rccc} \varepsilon\colon & A & \to & \mathbb{Q} \\ & 1 & \mapsto & 1 \\ & \omega_0 & \mapsto & 0 \\ & \omega_1 & \mapsto & 0. \end{array} \tag{3.197}$$

This has the added advantage to give us already a mixed Hodge structure over $\mathbb{Q}$. Since A is connected we can use the reduced bar construction. Arguing as in paragraph 3.4.5, the Hopf algebra $H^0(B^*(A))$ is isomorphic to the Hoffman algebra. In each finite dimensional subspace $H^0(L_N B^*(A_{\mathbb{C}}))$, the Hodge filtration is the decreasing filtration determined by

$$[\omega_{i_1}|\cdots|\omega_{i_p}] \in F^p$$

and the weight filtration is the increasing filtration determined by

$$[\omega_{i_1}|\cdots|\omega_{i_n}] \in W_{2n}.$$

We now describe an ind-mixed Hodge structure $\{{}_yA_x^{\mathrm{H},N}\}_{N\geq 0}$ that corresponds to the algebra of functions over the pro-unipotent completion of the fundamental group and a dual pro-mixed Hodge structure $\{{}_yU_x^{\mathrm{H},N}\}_{N\geq 0}$ that corresponds to the universal enveloping algebra of the Lie algebra of the pro-unipotent completion of the fundmental group.

For the Betti part of ${}_yA_x^{\mathrm{H},N}$, we write

$$ {}_yA_x^{\mathrm{B},N} = (\mathbb{Q}[\pi(M;y,x)]/J^{N+1}\mathbb{Q}[\pi(M;y,x)])^\vee $$

with the weight filtration given, for $-1 \le k \le N$, by

$$ \begin{aligned} W_{2k}({}_yA_x^{\mathrm{B},N}) &= W_{2k+1}({}_yA_x^{\mathrm{B},N}) \\ &= (J^{k+1}\mathbb{Q}[\pi(M;y,x)]/J^{N+1}\mathbb{Q}[\pi(M;y,x)])^\perp. \end{aligned} $$

For the de Rham side, we have

$$ {}_yA_x^{\mathrm{dR},N} = L_N H^0(B^*(A^*)) $$

with the weight filtration given, for $-1 \le k \le N$, by

$$ \begin{aligned} W_{2k}({}_yA_x^{\mathrm{dR},N}) &= W_{2k+1}({}_yA_x^{\mathrm{dR},N}) \\ &= (L_k H^0(B^*(A^*)))^\perp. \end{aligned} $$

The Hodge filtration is given by defining

$$ F^p({}_yA_x^{\mathrm{dR},N}) $$

as the subspace generated by words of length ℓ with $p \le \ell \le N$. Note that only the Betti part depends on the points x, y.

By duality we write

$$ {}_yU_x^{\mathrm{B},N} = \mathbb{Q}[\pi(M;y,x)]/J^{N+1}\mathbb{Q}[\pi(M;y,x)] $$

and

$$ {}_yU_x^{\mathrm{dR},N} = L_N H^0(B^*(A^*))^\vee $$

with the dual filtrations.

We denote by $\mathrm{comp}_{\mathrm{dR,B}}$ the isomorphism of Theorem 3.138 and by $\mathrm{comp}_{\mathrm{B,dR}}$ its dual. Then the mixed Hodge structures

$$ {}_yA_x^{\mathrm{H},N} := (({}_yA_x^{\mathrm{B},N}, W), ({}_yA_x^{\mathrm{dR},N}, W, F), \mathrm{comp}_{\mathrm{dR,B}}^{-1}) $$

form an inductive system of mixed Hodge structures over $\mathbb{Q}$ and

$$ {}_yU_x^{\mathrm{H},N} := (({}_yU_x^{\mathrm{B},N}, W), ({}_yU_x^{\mathrm{dR},N}, W, F), \mathrm{comp}_{\mathrm{B,dR}}) $$

form a projective system of mixed Hodge structures over $\mathbb{Q}$.

3.6.3. *Iterated integrals as periods of the fundamental group.* We now show that iterated integrals along paths between x and y are periods of the mixed Hodge structure ${}_yA_x^{\mathrm{H},N}$. We keep the notation $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $M = X(\mathbb{C})$.

EXAMPLE 3.198. Let $\boldsymbol{s} = (s_1, \dots, s_n)$ be a positive multi-index of weight N and write $\mathrm{bs}(\boldsymbol{s}) = (\varepsilon_1, \dots, \varepsilon_N)$ for the associated binary sequence. We consider the algebraic differential form on X^N given by

$$ \omega = \mathrm{pr}_1^*\omega_{\varepsilon_1} \wedge \cdots \wedge \mathrm{pr}_N^*\omega_{\varepsilon_N}, $$

where $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{1-t}$, as usual, and $\mathrm{pr}_i \colon X^N \to X$ denote the various projections. Since ω has maximal degree, it defines a class $[(\omega, 0)]$

in the relative de Rham cohomology $H^N_{\mathrm{dR}}(X^N, Y)$, where Y is as in Section 3.5.2. From lemmas 3.192 and 3.193 and Proposition 3.188 we derive

$$H^N_{\mathrm{dR}}(X^N, Y) = {}_y{A_x}^{\mathrm{dR},N}.$$

On the other hand, every path $\gamma\colon [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, determines a singular simplex

$$\begin{array}{cccc} \sigma\colon & \Delta^N & \longrightarrow & M^N \\ & (t_1, \dots, t_N) & \longmapsto & (\gamma(t_1), \dots, \gamma(t_N)), \end{array}$$

where Δ^N is the simplex of Notation 1.107. Clearly, the support of the chain $\partial\sigma$ is contained in Y. Therefore σ determines a class $[\sigma]$ in the relative singular homology group $H_N(M^N, Y, \mathbb{Q})$. By Theorem 3.162,

$$H_N(M^N, Y) = {}_y{A_x}^{\mathrm{B},N}.$$

The period associated with these two classes is the iterated integral

$$\langle [(\omega, 0)], [\sigma] \rangle = \int_\sigma \omega = \int_\gamma \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_N}.$$

Here we have used two points $x, y \in X(\mathbb{Q})$. In order to obtain multiple zeta values we need to consider the case $x = 0$ and $x = 1$, but these points do not belong to $X(\mathbb{Q})$. For this reason we will need to consider tangential base points in the next section.

⋆ ⋆ ⋆

Exercise 3.199 (The nerve of a category). Let $\mathcal{C}$ be a small category. Let $N(\mathcal{C})_0$ be the set of objects and $N(\mathcal{C})_1$ the set of morphisms. For each $n \geq 2$, define $N(\mathcal{C})_n$ as the set of n-tuples of composable morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} C_n. \tag{3.200}$$

On the one hand, we have maps

$$\delta_i\colon N(C)_n \to N(C)_{n-1} \qquad i = 0, \dots, n,$$

given by composing at the i-th object or removing it whenever $i = 0$ or n. In other words, δ_i sends an n-tuple as in (3.200) to the $(n-1)$-tuple

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n.$$

On the other hand, there are maps

$$\sigma_i\colon N(C)_n \to N(C)_{n+1} \qquad i = 0, \dots, n,$$

obtained by inserting an identity morphism at the i-th object, that is,

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} C_i \xrightarrow{\mathrm{Id}} C_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} C_n.$$

(a) Prove that $N(\mathcal{C})_\bullet$, together with the maps δ_i as faces and the maps σ_i as degeneracies, has the structure of a simplicial set. This construction is called the *nerve* of the category $\mathcal{C}$.

(b) In particular, identify the simplicial identity which corresponds to the associativity of the composition of morphisms.

EXERCISE 3.201. Describe explicitly the mixed Hodge structure on the pro-unipotent completion of the fundamental group of $\mathbb{G}_m$.

3.7. Tangential base points. In this section, we continue working with the manifold $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, the differential forms ω_0 and ω_1 and the dg-algebra $A^*_{\mathbb{C}}$ of paragraph 3.4.5.

Theorems 1.108 and 1.117 show that multiple zeta values and polylogarithms can be seen as iterated integrals. Nevertheless we face a technical problem. The differential forms ω_0 and ω_1 that appear in these theorems have singularities at the points 0, 1 and ∞. Hence they are differential forms on M, but to obtain multiple zeta values we need to integrate along the straight path **dch**

$$\mathbf{dch}\colon \quad \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ t & \longmapsto & t \end{array} \tag{3.202}$$

which is not contained in M because the end points are 0 and 1. Since **dch** is not a path in M, the formulas in theorems 1.108 and 1.117 are not strictly speaking iterated integrals.

Thus, to see multiple zeta values and polylogarithms as iterated integrals we have to consider *tangential base points*. As we will see, these are related to the regularization discussed in Section 1.7. Tangential base points will also play an important role later when we consider the algebraic structure of $\mathbb{P}^1$: the variety $\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$ does not contain any smooth integral point, thus we will need tangential base points to have a motivic version of the fundamental group of $\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}$ defined over $\mathbb{Z}$.

3.7.1. *Paths with tangential base points.* For simplicity, we will introduce tangential base points only in the case of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, the only one we need, but the reader should be aware that the constructions extend easily to any smooth projective curve minus a finite number of points.

DEFINITION 3.203. Let $x \in \{0, 1\}$ be either the point zero or the point one in $\mathbb{P}^1(\mathbb{C})$. A *tangential base point* is a pair (x, v), where v is a non-zero tangent vector to $\mathbb{P}^1(\mathbb{C})$ at x.

Intuitively, a path has an end point at a tangential base point (x, v) if the end point is x and the tangent vector at the end point is v. However, the presence of tangential base points causes a nuisance. On the one hand, in order to be able to compose paths we need to allow tangential points to be reached by the paths at intermediate points. On the other hand, to define homotopy between paths in an easy way it is better to avoid tangential points at intermediate points along the path. To remedy this problem we define two kind of paths, the ones that allow tangential points at intermediate steps (hence can be composed) and the ones that avoid tangential points. The former will be called cuspidal paths because of the shape we will impose at the tangential points, while the latter will be called clean paths. Then we define a homotopy equivalence of clean paths and a map from the space of cuspidal paths to the space of homotopy classes of clean paths.

DEFINITION 3.204. Let $\boldsymbol{x} = (x, v)$ and $\boldsymbol{y} = (y, w)$ be two tangential base points. A *cuspidal path from* $\boldsymbol{x}$ *to* $\boldsymbol{y}$ is a piecewise smooth map $\gamma\colon [0,1] \to M \cup \{0, 1\}$ satisfying the following conditions

(1) the end points of the path are

$$\gamma(0) = x, \quad \frac{d\gamma}{dt}(0) = v,$$
$$\gamma(1) = y, \quad \frac{d\gamma}{dt}(1) = -w;$$

(2) the set $\{t \in (0,1) \mid \gamma(t) \in \{0,1\}\}$ is finite. Moreover, if t_0 belongs to this set, the left and right tangent vector to γ at t_0 are non-zero and opposed:

$$0 \neq \frac{d^+\gamma}{dt}(t_0) = -\frac{d^-\gamma}{dt}(t_0)$$

This set is called the *set of cusps* of γ.

When the set of cusps is empty, γ is called a *clean path from* $\boldsymbol{x}$ *to* $\boldsymbol{y}$.

The space of cuspidal paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ is denoted by ${}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ while the subspace of clean paths is denoted ${}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$.

3.7.2. *Composition of paths with tangential base points.* The composition of paths (3.3) cannot be applied directly to define

$${}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{y}} \otimes {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}} \longrightarrow {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{x}}$$

for tangential base points $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ because condition (1) imposes that the derivative of the path at zero and one is a fixed vector, while the parametrization used in (3.3) would multiply this vector by 2. Thus to define the composition of paths we consider the functions

$$\phi_1(t) = t + 2t^2, \quad \phi_2(t) = 5t - 2 - 2t^2.$$

These functions are smooth and satisfy the properties

$$\begin{aligned} &\phi_1(0) = 0, && \phi_1(1/2) = 1, && \phi_1'(0) = 1,\\ &\phi_2(1/2) = 0, && \phi_2(1) = 1, && \phi_2'(1) = 1,\\ &\phi_1'(t) > 0,\ t \in [0, 1/2], && \phi_2'(t) > 0,\ t \in [1/2, 1], &&\\ &\phi_1'(1/2) = \phi_2'(1/2). &&&& \end{aligned}$$

In fact, any pair of smooth functions satisfying all the above properties would serve for our purposes.

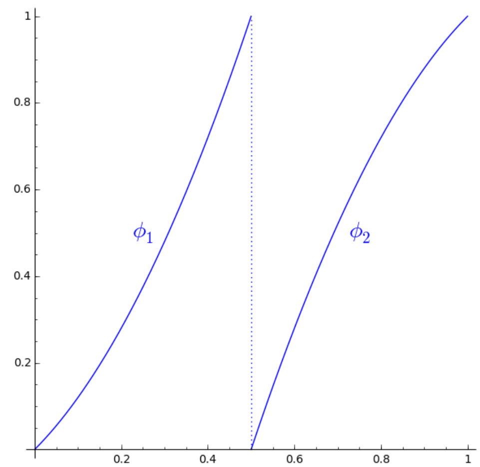


FIGURE 15. The functions ϕ_1 and ϕ_2

We define the composition of paths as

$$\begin{array}{ccc} {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{y}} \otimes {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}} & \longrightarrow & {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{x}} \\ (\gamma_1, \gamma_2) & \longmapsto & \gamma_1\gamma_2 \end{array}$$

where

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_2(\phi_1(t)), & 0 \le t \le \frac{1}{2}, \\ \gamma_1(\phi_2(t)), & \frac{1}{2} \le t \le 1. \end{cases} \tag{3.205}$$

3.7.3. *Homotopy of paths.* Let $\gamma_1, \gamma_2 \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$ be two clean paths. A homotopy between γ_1 and γ_2 is a map

$$F\colon [0,1] \times [0,1] \longrightarrow M \cup \{0,1\}$$

such that

$$\begin{array}{lll} F(t,0) = \gamma_1(t), & F(t,1) = \gamma_2(t), & t \in [0,1] \\ F(0,s) = x, & F(1,s) = y, & s \in [0,1] \\ \dfrac{\partial F}{\partial t}(0,s) = v, & \dfrac{\partial F}{\partial t}(1,s) = -w, & s \in [0,1] \\ F(t,s) \in M, & 0 < t < 1, & 0 \le s \le 1. \end{array}$$

The space $\pi(M; \boldsymbol{y}, \boldsymbol{x})$ is the set of homotopy classes of clean paths from $\boldsymbol{x}$ to $\boldsymbol{y}$. Similar notation will be used when only one of the base points is tangential.

We next construct a map ψ from ${}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ to $\pi(M; \boldsymbol{y}, \boldsymbol{x})$. Let $d(x,y)$ be the standard Euclidean distance in $\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$. Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$. For

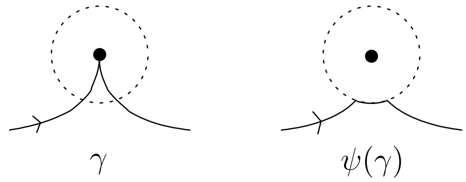


FIGURE 16. Retraction at a cusp

each t_i in the set of cusps of γ, we can find real numbers $0 < \varepsilon_i, \eta_i, \eta'_i < \frac{1}{2}$ satisfying the conditions

(1) t_i is the only cusp in the interval $[t_i - \eta'_i, t_i + \eta_i]$ and γ is smooth in the intervals $[t_i - \eta'_i, t_i)$ $(t_i, t_i + \eta_i]$;
(2) the intervals $[t_i - \eta'_i, t_i + \eta_i]$ are disjoint and do not contain 0 or 1;
(3) the image of $[t_i - \eta'_i, t_i + \eta_i]$ satisfies

$$d(\gamma(t_i + \eta_i), \gamma(t_i)) = \varepsilon_i, \quad d(\gamma(t), \gamma(t_i)) < \varepsilon_i, \text{ for } t_i < t < t_i + \eta_i$$
$$d(\gamma(t_i - \eta'_i), \gamma(t_i)) = \varepsilon_i, \quad d(\gamma(t), \gamma(t_i)) < \varepsilon_i, \text{ for } t_i - \eta'_i < t < t_i;$$

(4) the tangent vector to γ does not change very much

$$\left\| \frac{d\gamma}{dt}(t) - \frac{d^-\gamma}{dt}(t_i) \right\| \leq \frac{1}{2} \left\| \frac{d^-\gamma}{dt}(t_i) \right\|, \text{ for } t \in [t_i - \eta'_i, t_i)$$
$$\left\| \frac{d\gamma}{dt}(t) - \frac{d^+\gamma}{dt}(t_i) \right\| \leq \frac{1}{2} \left\| \frac{d^+\gamma}{dt}(t_i) \right\|, \text{ for } t \in (t_i, t_i + \eta'_i].$$

Note that condition (4) implies that the path γ cannot turn around the point $\gamma(t_i)$ between $t_i - \eta'_i$ and $t_i + \eta_i$.

For each cusp t_i let $B(\gamma(t_i), \varepsilon_i)$ be the open ball of centre $\gamma(t_i)$ and radius ε_i and let $r_i \colon \mathbb{C} \setminus \{\gamma(t_i)\} \to \mathbb{C} \setminus B(\gamma(t_i), \varepsilon_i)$ be the radial retraction. Then we define a new path γ° defined outside the cusps by

$$\gamma^\circ(s) = \begin{cases} \gamma(s) & \text{if for all } i, \ s \notin [t_i - \eta'_i, t_i + \eta_i], \\ r_i(\gamma(s)) & \text{if } s \in [t_i - \eta'_i, t_i + \eta_i], \ s \neq t_i. \end{cases} \tag{3.206}$$

Condition (2) in the Definition 3.204 implies that γ° can be extended continuously to the cusps t_i defining a clean path also denoted γ°. The retraction at a cusp is represented in figure 16.

The following proposition is clear.

PROPOSITION-DEFINITION 3.207. *The homotopy class of clean paths of γ° does not depend on the choice of the numbers $\varepsilon_i, \eta_i, \eta'_i$. The homotopy class of γ° in $\pi(M; \boldsymbol{y}, \boldsymbol{x})$ is denoted by $\psi(\gamma)$.*

Using the map ψ we can define a composition of clean paths

$$\pi(M; \boldsymbol{z}, \boldsymbol{y}) \times \pi(M; \boldsymbol{y}, \boldsymbol{x}) \longrightarrow \pi(M; \boldsymbol{z}, \boldsymbol{x}).$$

DEFINITION 3.208. Let $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ be base points, tangential or not. Given classes $\gamma_1 \in \pi(M;\boldsymbol{z},\boldsymbol{y})$ and $\gamma_2 \in \pi(M;\boldsymbol{y},\boldsymbol{x})$, we choose representatives $\widetilde{\gamma}_1 \in {}_{\boldsymbol{z}}\mathcal{P}(M)^0_{\boldsymbol{y}}$ and $\widetilde{\gamma}_2 \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$. Then $\widetilde{\gamma}_1\widetilde{\gamma}_2 \in {}_{\boldsymbol{z}}\mathcal{P}(M)_{\boldsymbol{x}}$ and we define

$$\gamma_1\gamma_2 = \psi(\widetilde{\gamma}_1\widetilde{\gamma}_2).$$

PROPOSITION 3.209. *The composition of clean paths given in Definition 3.208 does not depend on the choice of liftings and turns $\pi(M;\boldsymbol{x},\boldsymbol{x})$ into a group and $\pi(M;\boldsymbol{y},\boldsymbol{x})$ (resp. $\pi(M;\boldsymbol{x},\boldsymbol{y})$) into a right (resp. left) $\pi(M;\boldsymbol{x},\boldsymbol{x})$-torsor.*

The fact that the fundamental groups with different base points are isomorphic can be easily extended to tangential base points. The next proposition is proved like the classical one.

PROPOSITION 3.210. *Let $\boldsymbol{x}_i$, $1 = 1,\dots,4$ be any base points of M (tangential or not). Let $\gamma_1 \in {}_{\boldsymbol{x}_4}\mathcal{P}(M)_{\boldsymbol{x}_3}$ and $\gamma_2 \in {}_{\boldsymbol{x}_2}\mathcal{P}(M)_{\boldsymbol{x}_1}$. Then the following map is an isomorphism:*

$$\begin{array}{ccc} \pi(M;\boldsymbol{x}_3,\boldsymbol{x}_2) & \longrightarrow & \pi(M;\boldsymbol{x}_4,\boldsymbol{x}_1) \\ \gamma & \longmapsto & \gamma_1\gamma\gamma_2. \end{array}$$

3.7.4. *Logarithmic asymptotic developments.* We would like to extend the notion of iterated integral to tangential base points. The main problem is that the integral may diverge, so one needs to regularize it. We start by discussing some preliminaries about asymptotic developments.

DEFINITION 3.211. Let $0 < \tau \leq 1$ be a real number and $f\colon (0,\tau) \to \mathbb{C}$ a continuous function. We say that f admits a *logarithmic asymptotic development (of degree less than or equal to r)* if it can be written as

$$f(t) = f_0(t) + \sum_{k=0}^{r} a_k \log(t)^k.$$

with $|f_0(t)| = O(t^{1-\delta})$ for some $\delta < 1$ and $a_k \in \mathbb{C}$.

LEMMA 3.212. *Let $0 < \tau \leq 1$ be a real number and $f\colon (0,\tau) \to \mathbb{C}$ a continuous function. If it admits a logarithmic asymptotic development then the development is unique.*

PROOF. Let $f\colon (0,\tau) \to \mathbb{C}$ be a continuous function that admits an assymptotic development

$$f(t) = f_0(t) + \sum_{k=0}^{r} a_k \log(t)^k.$$

We can recover a_r as

$$a_r = \lim_{t\to 0} \frac{f(t)}{\log(t)^r}.$$

Once we know $a_{s+1}, \dots, a_r$ we can recover a_s as

$$a_s = \lim_{t \to 0} \frac{f(t) - \sum_{k=s+1}^{r} a_k \log(t)^k}{\log(t)^s}.$$

Finally, $f_0 = f(t) - \sum_{k=0}^{r} a_k \log(t)^k$. Hence the development is unique. $\square$

3.7.5. *Asymptotic developments of iterated integrals.* We now fix the two tangential base points $\mathbf{0} = (0, 1)$ and $\mathbf{1} = (1, -1)$, that is, the tangent vector 1 at the point 0 and the tangent vector -1 at the point 1. For instance, the path $\mathbf{dch}(t) = t$ belongs to ${}_{\mathbf{1}}\mathcal{P}(M)_{\mathbf{0}}$.

Let $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\} \cup M$ be base points (tangential or not), $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ a piecewise smooth clean path, and $(\varepsilon_1, \dots, \varepsilon_r)$ a binary sequence with $\varepsilon_i \in \{0, 1\}$. We consider the iterated integral

$$\int_\gamma \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.$$

Since the form ω_0 has a pole at 0 and the form ω_1 has a pole at 1, this integral may diverge. For instance

$$\int_{\mathbf{dch}} \omega_0 = \infty.$$

However, if the form ω_{ε_1} has no pole at the point $\boldsymbol{y}$ and the form ω_{ε_r} has no pole at the point $\boldsymbol{x}$, then the above integral is convergent. For instance, if $\gamma = \mathbf{dch}$, the integral will be convergent when $\varepsilon_1 = 0$ and $\varepsilon_r = 1$, that is, when the binary sequence is admissible.

We now describe the regularization process. Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$ be a clean path. For $0 < \eta < \frac{1}{2}$, we write

$$\gamma_\eta(t) = \gamma(t(1-\eta) + (1-t)\eta).$$

This is a path from $\gamma(\eta)$ to $\gamma(1-\eta)$, hence completely contained in M.

Lemma 3.213. *Let $(\varepsilon_1, \dots, \varepsilon_r)$ be a binary sequence. Then the function $(0, 1/2) \to \mathbb{C}$ given by*

$$\eta \mapsto \int_{\gamma_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}$$

admits a logarithmic asymptotic development of degree $\leq r$.

Proof. We write

$$\gamma_{\eta,1}(t) = \gamma(t(1-\eta) + (1-t)/2),$$
$$\gamma_{\eta,2}(t) = \gamma(t/2 + (1-t)\eta).$$

The path $\gamma_{\eta,2}$ goes from $\gamma(\eta)$ to $\gamma(1/2)$ and $\gamma_{\eta,1}$ is a path from $\gamma(1/2)$ to $\gamma(1-\eta)$. Moreover, $\gamma_\eta = \gamma_{\eta,1}\gamma_{\eta,2}$ (recall that, according to our convention for the composition of paths (3.3), this means that we first walk along $\gamma_{\eta,2}$, then along $\gamma_{\eta,1}$). Using equations (3.20) and (3.21) in Theorem 3.19, it is enough to show that the functions

$$\eta \mapsto \int_{\gamma_{\eta,i}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}, \quad i = 1, 2$$

admit a logarithmic asymptotic development of degree less than or equal to r. Since both cases are analogous, we will only consider $i = 2$. We prove the existence of a logarithmic asymptotic development by induction on r. The result is clear for $r = 0$. Let us assume that it holds for a binary sequence of length less than r. If $\gamma_{\eta,2}^* \omega_{\varepsilon_i} = g_{\varepsilon_i}(t)dt$ and $\gamma^* \omega_{\varepsilon_i} = h_{\varepsilon_i}(t)dt$, then:

$$\begin{aligned}\int_{\gamma_{\eta,2}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} &= \int\limits_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} g_{\varepsilon_1}(t_1) \cdots g_{\varepsilon_r}(t_r) dt_1 \cdots dt_r \\ &= \int\limits_{1/2 \geq t_1 \geq \cdots \geq t_r \geq \eta} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_r}(t_r) dt_1 \cdots dt_r.\end{aligned}$$

Now we compute

$$\begin{aligned}I(\eta) &:= \int\limits_{1/2 \geq t_1 \geq \cdots \geq t_r \geq \eta} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_r}(t_r) dt_1 \cdots dt_r \\ &= \int\limits_{1/2 \geq t_r \geq \eta} h_{\varepsilon_r}(t_r) \Big(\int\limits_{1/2 \geq t_1 \geq \cdots \geq t_{r-1} \geq t_r} h_{\varepsilon_1}(t_1) \cdots h_{\varepsilon_{r-1}}(t_{r-1}) dt_1 \cdots dt_{r-1} \Big) dt_r.\end{aligned}$$

By the shape of ω_{ε_r} we deduce that

$$h_{\varepsilon_r}(t_r) = \frac{\alpha}{t_r} + O(1),$$

where α is non-zero if ω_{ε_r} has a pole at the point $\boldsymbol{x}$ and is zero otherwise. We also apply the induction hypothesis to the inner integral to get

$$I(\eta) = \int\limits_{1/2 \geq t_r \geq \eta} \Big(\frac{\alpha}{t_r} + O(t_r^0) \Big) \Big(O(t_r^{1-\delta}) + \sum_{k=0}^{r-1} b_k \log(t_r)^k \Big) dt_r.$$

Estimating this integral, we deduce that $I(\eta)$ admits a logarithmic asymptotic development of the sought shape, proving the result. $\square$

3.7.6. *Regularized iterated integrals.*

Definition 3.214. Let $(\varepsilon_1, \ldots, \varepsilon_r)$ be a binary sequence and let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)^0_{\boldsymbol{x}}$ be a clean path. Let

$$\int_{\gamma_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = f_0(\eta) + \sum_{k=0}^{r} a_k \log(\eta)^k$$

be the logarithmic asymptotic development provided by Lemma 3.213. Then the *regularized iterated integral* along γ is defined as

$$\int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = a_0.$$

PROPOSITION 3.215. *Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ be a cuspidal path and γ° a representative of the class $\psi(\gamma)$ obtained as in (3.206). The regularized integral*

$$\int_{\gamma^\circ}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}$$

does not depend on the choice of γ°.

PROOF. Let γ_1° and γ_2° be two choices. Since γ_1° and γ_2° only differ from γ in a small neighborhood of the cusps, for small enough η,

$$\gamma_1^\circ(\eta) = \gamma_2^\circ(\eta),\ \gamma_1^\circ(1-\eta) = \gamma_2^\circ(1-\eta).$$

Moreover $\gamma_{1,\eta}^\circ$ and $\gamma_{2,\eta}^\circ$ are homotopically equivalent. As seen in paragraph 3.4.5 $H^0(B^*(A^*)) = B^0(A)$, thus all the iterated integrals that can be constructed from ω_0 and ω_1 are homotopy functionals. Therefore

$$\int_{\gamma_{1,\eta}^\circ} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \int_{\gamma_{2,\eta}^\circ} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}$$

from which the result follows. □

DEFINITION 3.216. Let $\gamma \in {}_{\boldsymbol{y}}\mathcal{P}(M)_{\boldsymbol{x}}$ be a cuspidal path. Let γ° be a representative of the class $\psi(\gamma)$ obtained as in (3.206). We define

$$\int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \int_{\gamma^\circ}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r}.$$

Clearly, when the iterated integral is convergent, the value of the regularized integral agrees with the value of the integral.

Regularized iterated integrals share many of the properties of iterated integrals. In particular, Theorem 3.19 can be extended to the new setting.

THEOREM 3.217. *Let $\gamma, \gamma_1, \gamma_2$ be cuspidal in M whose end points are either $\mathbf{0}$, $\mathbf{1}$ or belong to M and such that $\gamma_2(1) = \gamma_1(0)$. Let $(\varepsilon_1, \dots, \varepsilon_{r+s})$ be a binary sequence. Then*

(1)
$$\int_{\gamma}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = (-1)^r \int_{\gamma^{-1}}^{\mathrm{reg}} \omega_{\varepsilon_r} \cdots \omega_{\varepsilon_1}.$$

(2)
$$\int_{\gamma_1\gamma_2}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} = \sum_{i=0}^{r} \int_{\gamma_1}^{\mathrm{reg}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_i} \int_{\gamma_2}^{\mathrm{reg}} \omega_{\varepsilon_{i+1}} \cdots \omega_{\varepsilon_r}.$$

(3)

$$\int_\gamma^{\mathrm{reg}} \omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r}\int_\gamma^{\mathrm{reg}} \omega_{\varepsilon_{r+1}}\cdots\omega_{\varepsilon_{r+s}} = \sum_{\sigma\in ш(r,s)}\int_\gamma^{\mathrm{reg}} \omega_{\varepsilon_{\sigma^{-1}(1)}}\cdots\omega_{\varepsilon_{\sigma^{-1}(r+s)}}$$

PROOF. We first prove (1). If γ is cuspidal and γ° is a clean path in the homotopy class $\psi(\gamma)$ obtained as in (3.206), then $(\gamma^\circ)^{-1}$ is a clean path in the homotopy class $\psi(\gamma^{-1})$ obtained as in (3.206). Therefore, we can assume that γ is a clean path. By construction, $(\gamma^{-1})_\varepsilon = (\gamma_\varepsilon)^{-1}$. By Theorem 3.19 the asymptotic expansions of

$$\int_{\gamma_\varepsilon}\omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r},\ \text{and}\ (-1)^r\int_{\gamma_\varepsilon^{-1}}^{\mathrm{reg}}\omega_{\varepsilon_r}\cdots\omega_{\varepsilon_1}$$

agree. Thus we have the equality of regularized integrals.

Statement (3) also follows from the corresponding statement in Theorem 3.19.

Statement (2) is slightly more difficult due to the posibility that the joining point is a tangential base point. The proof goes as follows.

Assume that γ_1 and γ_2 are clean paths. Let $\gamma = \gamma_1\gamma_2$ be their composition and γ^0 a clean path representing γ as in (3.206). For sufficiently small η, the path $(\gamma^0)_\eta$ is homotopic to $\gamma_{1,\eta}\gamma_{0,\eta}\gamma_{2,\eta}$, where $\gamma_{0,\eta}$ denotes the straight path form $\gamma_2(1-\eta)$ to $\gamma_1(\eta)$ (see Figure 17 below). By the usual formula for the composition of paths

$$\int_{(\gamma^0)_\eta}\omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r} = \sum_{j=0}^{r}\sum_{k=j}^{r}\int_{\gamma_{1,\eta}}\omega_{\varepsilon_1}\cdots\omega_{\varepsilon_j}\int_{\gamma_{0,\eta}}\omega_{\varepsilon_{j+1}}\cdots\omega_{\varepsilon_k}\int_{\gamma_{2,\eta}}\omega_{\varepsilon_{k+1}}\cdots\omega_{\varepsilon_r}. \tag{3.218}$$

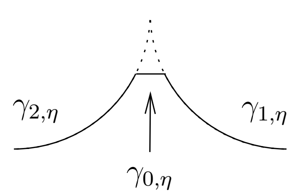


FIGURE 17. $(\gamma^0)_\eta \sim \gamma_{2,\eta}\gamma_{0,\eta}\gamma_{1,\eta}$

LEMMA 3.219. *One has* $\int_{\gamma_{0,\eta}}\omega_{\varepsilon_{j+1}}\cdots\omega_{\varepsilon_k} = O(\eta^{k-j})$.

PROOF. The key point is that we have power series expansions

$$\gamma_2(1-\eta) = \gamma_2(1) - \gamma_2'(1)\eta + O(\eta^2)$$
$$\gamma_1(\eta) = \gamma_1(0) + \gamma_1'(0)\eta + O(\eta^2).$$

Since $\gamma_2(1) = \gamma_1(0)$ and $\gamma_2'(1) = -\gamma_1'(0)$, it follows that

$$|\gamma_2(1-\eta) - \gamma_1(\eta)| = O(\eta^2).$$

Using the equation $\gamma_{0,\eta} = t\gamma_1(\eta) + (1-t)\gamma_2(1-\eta)$, one sees that

$$\gamma_{0,\eta}^* \frac{dz}{z} = \frac{(\gamma_1(\eta) - \gamma_2(1-\eta))dt}{t\gamma_2(\eta) + (1-t)\gamma_2(1-\eta)}$$

Since the numerator is $O(\eta^2)$ and the denominator is $O(\eta)$, it follows that $\gamma_{0,\eta}^*\omega = O(\eta)dt$, hence

$$\int_{\gamma_{0,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_k = O(\eta^{k-j}),$$

proving the lemma. □

To conclude the proof of the theorem, we observe that, by the lemma, the integral $\int_{\gamma_{0,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_k}$ does not contribute to the constant term in the logarithmic asymptotic development of (3.218) when $k > j$. Therefore,

$$\begin{aligned} &\underline{\mathrm{const}} \int_{(\gamma^0)_\eta} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_r} \\ &\qquad = \sum_{j=0}^{r} \left(\underline{\mathrm{const}} \int_{\gamma_{1,\eta}} \omega_{\varepsilon_1} \cdots \omega_{\varepsilon_j} \right) \left(\underline{\mathrm{const}} \int_{\gamma_{2,\eta}} \omega_{\varepsilon_{j+1}} \cdots \omega_{\varepsilon_r} \right), \end{aligned}$$

from which the result follows. Here $\underline{\mathrm{const}}$ means the constant term a_0 in the logarithmic asymptotic expansion. □

As we did before for "honest" base points, the properties of iterated integrals can be concisely rephrased in terms of the bracket. If γ is a piecewise smooth path and $\eta \in B^0(A^*)$, we denote

$$\langle \eta, \gamma \rangle^{\mathrm{reg}} = \int_\gamma^{\mathrm{reg}} \eta.$$

THEOREM 3.220. *Let $\gamma, \gamma_1, \gamma_2$ be piecewise smooth paths with any base points and let $\eta, \eta_1, \eta_2 \in B^0(A^*)$ be elements of the bar complex of A^* of degree zero. Then*

(1) $\langle \eta, \gamma \rangle^{\mathrm{reg}} = \langle S(\eta), \gamma^{-1} \rangle^{\mathrm{reg}}$.

(2) $\langle \eta, \gamma_1\gamma_2 \rangle^{\mathrm{reg}} = \langle \Delta\eta, \gamma_1 \otimes \gamma_2 \rangle^{\mathrm{reg}}$.

(3) $\langle \eta_1, \gamma \rangle^{\mathrm{reg}} \cdot \langle \eta_2, \gamma \rangle^{\mathrm{reg}} = \langle \eta_1 \sqcup \eta_2, \gamma \rangle^{\mathrm{reg}}$.

3.7.7. *regularized iterated integrals and regularized zeta values.*

EXAMPLE 3.221. Let us compute an example of a regularized iterated integral in length 3:

$$\zeta(1,2)^{\mathrm{reg}} = \int_{\mathbf{dch}}^{\mathrm{reg}} \omega_1\omega_0\omega_1$$

By definition, this is the constant term in the asymptotic logarithmic development of the function

$$\eta \longmapsto \int\limits_{1-\eta\geq t_1\geq t_2\geq t_3\geq 0} \frac{dt_1dt_2dt_3}{(1-t_1)t_2(1-t_3)}.$$

To be completely precise, according to the above recipe we should have required $t_3 \geq \eta$ as well. Note, however, that the last form ω_1 has no pole at 0, so the two asymptotic logarithmic developments agree.

We first compute the integral following the method of examples 1.103 and 1.105. We obtain

$$\int\limits_{1-\eta\geq t_1\geq t_2\geq t_3\geq 0} \frac{dt_1dt_2dt_3}{(1-t_1)t_2(1-t_3)} = \sum_{m>n>0} \frac{(1-\eta)^m}{n^2m}. \tag{3.222}$$

This power series converges for $0<\eta<1$ but diverges for $\eta=0$ and we have to find an asymptotic expansion in $\log\eta$. To this end, we use the equality

$$\begin{aligned}\int\limits_{1-\eta\geq t_1\geq t_2\geq t_3\geq 0} \frac{dt_1dt_2dt_3}{(1-t_1)t_2(1-t_3)} &= \int\limits_{\substack{1-\eta\geq t_2\geq t_3\geq 0\\ 1-\eta\geq t_1\geq 0}} \frac{dt_1dt_2dt_3}{(1-t_1)t_2(1-t_3)}\\ &\quad -2\int\limits_{1-\eta\geq t_2\geq t_1\geq t_3\geq 0} \frac{dt_1dt_2dt_3}{(1-t_1)t_2(1-t_3)},\end{aligned} \tag{3.223}$$

which is a simple consequence of the decomposition of the integration domain, together with the fact that the integrand is symmetric in t_1 and t_3 (this explains why the last term appears twice). Observe that

$$\int_{1-\eta\geq t_1\geq 0} \frac{dt_1}{1-t_1} = \sum_{k\geq 1} \frac{(1-\eta)^k}{k} = -\log(\eta).$$

Combining this with the power series expansions as in Example 1.105, one sees that the right-hand side of (3.223) is equal to

$$-\log(\eta)\sum_{n\geq 1}\frac{(1-\eta)^n}{n^2} - 2\sum_{m>n\geq 1}\frac{(1-\eta)^m}{m^2n}. \tag{3.224}$$

One can check (Exercise 3.230) directly that this expansion agrees with the right-hand side of (3.222).

To see that the power expansion (3.224) is useful we have to prove that the series appearing in that expansion define continuous functions of η.

LEMMA 3.225. *The following estimates hold when η goes to 0^+:*

$$\sum_{n\geq 1}\frac{(1-\eta)^n}{n^2}=\zeta(2)+O(\eta\log\eta), \tag{3.226}$$

$$\sum_{m>n\geq 1}\frac{(1-\eta)^m}{m^2n}=\zeta(2,1)+O(\eta\log^2\eta). \tag{3.227}$$

PROOF. To prove the estimate (3.226), we need to study

$$\zeta(2)-\sum_{n\geq 0}\frac{(1-\eta)^n}{n^2}=\sum_{n\geq 0}\frac{1-(1-\eta)^n}{n^2}.$$

For $0<\eta<1$, we have the inequalities

$$0<1-(1-\eta)^n<1, \qquad 0<1-(1-\eta)^n<n\eta.$$

Therefore

$$0<\sum_{n\geq 1}\frac{1-(1-\eta)^n}{n^2}<\sum_{n=1}^{\lfloor\frac{1}{\eta}\rfloor}\frac{\eta}{n}+\sum_{n>\lfloor\frac{1}{\eta}\rfloor}\frac{1}{n^2}.$$

Since the first sum is $O(\eta\log\eta)$ and the second is $O(\eta)$, the first estimate follows. The second one is obtained in a similar way. □

From Lemma 3.225 we obtain

$$\int\limits_{1-\eta\geq t_1\geq t_2\geq t_3\geq 0}\frac{dt_1dt_2dt_3}{(1-t_1)t_2(1-t_3)}=-2\zeta(2,1)-\zeta(2)\log\eta+O(\eta\log^2\eta),$$

from which it follows that

$$\zeta(1,2)^{\mathrm{reg}}=-2\zeta(2,1).$$

The value of $\zeta(1,2)^{\mathrm{reg}}$ is equal to the one obtained by *shuffle regularization* in Example 1.178. This is of course no coincidence, as we now prove:

THEOREM 3.228. *Let $(\varepsilon_1,\dots,\varepsilon_r)$ be a binary sequence and consider the corresponding word $w=x_{\varepsilon_1}\cdots x_{\varepsilon_r}$. Then:*

$$\zeta_{\sqcup\!\sqcup}(w)=\int_\gamma^{\mathrm{reg}}\omega_{\varepsilon_1}\cdots\omega_{\varepsilon_r}.$$

PROOF. By Proposition 1.173, we need to show that the integral on the right hand side satisfies the conditions determining $\zeta_{\sqcup\!\sqcup}(w)$. Condition (1.174) follows from Theorem 1.108 combined with the observation that, when the binary sequence is admissible, then the regularized integral agrees with the usual integral. Condition (1.175) is checked by a direct computation. Finally condition (1.176) is Theorem 3.217 3. □

3.7.8. *Chen's theorem for tangential base points.* We finish this section by stating a version of Chen's theorem with tangential base points. Recall that we are writing $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, and $A^*_{\mathbb{C}}$ is the dg-algebra of parapraph 3.4.5.

THEOREM 3.229 (Chen's π_1 theorem for tangential base points). *For each integer $N \geq 0$ and each pair of points $\boldsymbol{x}, \boldsymbol{y}$ (tangential or not), the regularized iterated integrals induce an isomorphism*

$$L_N H^0(B^*(A^*_{\mathbb{C}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]/J^{N+1}\mathbb{Q}[\pi(M; \boldsymbol{x})], \mathbb{C}).$$

Passing to the limit, we deduce an isomorphism between $H^0(B^0(A^*_{\mathbb{C}}))$ and the topological dual $(\mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge)^\vee$.

PROOF. We need to show that the pairing between $L_N H^0(B^*(A^*_{\mathbb{C}}))$ and $\pi_1(M; \boldsymbol{y}, \boldsymbol{x})/J^{N+1}$ is non-degenerate. Since both spaces are finite dimensional, it suffices to prove that there is no non-zero $\gamma \in \pi_1(M; \boldsymbol{x}, \boldsymbol{y})/J^{N+1}$ such that $\langle \omega, \gamma \rangle = 0$ for all ω. Indeed, assume that such a γ exists. Choose usual base points x' and y' and paths γ_1, γ_2 going from $\boldsymbol{x}$ to x' and from $\boldsymbol{y}$ to y'. Then, by Theorem 3.220 (2), for $\omega \in L_N H^0(B^*(A^*))$

$$\langle \omega, \gamma_1 \gamma \gamma_2 \rangle = \sum \langle \omega_1, \gamma_1 \rangle \langle \omega_2, \gamma \rangle \langle \omega_3, \gamma_2 \rangle,$$

where all the elements $\omega_1, \omega_2, \omega_3$ are of length $\leq N$. Thus $\langle \omega, \gamma_1 \gamma \gamma_2 \rangle = 0$ for all $\omega \in L_N H^0(B^*(A^*))$. By Chen's Theorem 3.138, $\gamma_1 \gamma \gamma_2 = 0$ and hence the same is true for γ. □

⋆ ⋆ ⋆

EXERCISE 3.230. By expanding $\log(\eta)$ as a power series in $(1-\eta)$, prove the following equality of functions for $0 < \eta < 1$:

$$\sum_{m>n>0} \frac{(1-\eta)^m}{n^2 m} = -\log(\eta) \sum_{n \geq 0} \frac{(1-\eta)^n}{n^2} - 2 \sum_{m>n\geq 1} \frac{(1-\eta)^m}{m^2 n}.$$

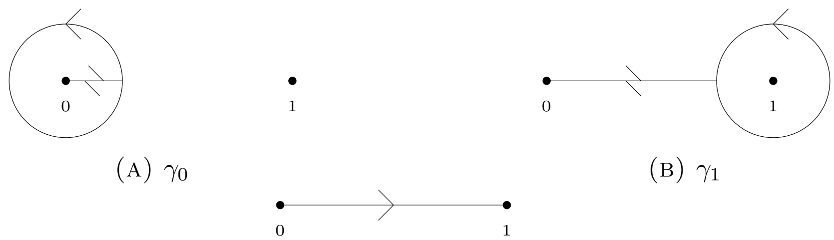
EXERCISE 3.231. Let $n \geq 2$ be an integer. Adapt Example 3.221 to compute the regularized iterated integral

$$\int_{\mathbf{dch}}^{\mathrm{reg}} \omega_1 \omega_0^{n-1} \omega_1$$

and show that the result coincides with $\zeta_{\sqcup\!\sqcup}(1, n)$.

3.8. Polylogarithms and their monodromy. In this section, we explain how to make the isomorphism of Chen's Theorem 3.229 more explicit in the case of $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ by using polylogarithms.

3.8.1. *Generators of the fundamental group of M.* In the previous section, we have introduced the tangential base points $\mathbf{0}$ and $\mathbf{1}$. The fundamental group $\pi_1(M, \mathbf{0})$ is generated by the paths γ_0 and γ_1 of Figure 18. The space of paths $\pi(M; \mathbf{1}, \mathbf{0})$ is generated as a right $\pi_1(M, \mathbf{0})$-module by the straight path $\mathbf{dch}$ also represented in Figure 18.



(A) γ_0 (B) γ_1 (C) $\mathbf{dch}$

FIGURE 18. Generators

The fundamental group $\pi_1(M, \mathbf{1})$ is generated by the paths

$$\gamma_0' = \mathbf{dch} \cdot \gamma_0 \cdot \mathbf{dch}^{-1}, \quad \gamma_1' = \mathbf{dch} \cdot \gamma_1 \cdot \mathbf{dch}^{-1},$$

and the space $\pi(M; \mathbf{0}, \mathbf{1})$ is generated as a right $\pi_1(M, \mathbf{1})$-module or as a left $\pi_1(M, \mathbf{0})$-module by the path $\mathbf{dch}^{-1}$.

3.8.2. *The dual of Chen's map.* We saw in paragraph 3.4.5 that the cohomology in degree zero of the reduced bar complex associated with $A^*_{\mathbb{C}}$ is isomorphic, as a Hopf algebra, to the complex Hoffman algebra $\mathfrak{H} \otimes \mathbb{C}$. In Example 3.62 we identified the dual $\mathfrak{H}^\vee$ with the algebra $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. We extend Notation 1.153 as follows.

NOTATION 3.232. If α is a binary sequence, we will denote by x_α the corresponding word in the Hoffman algebra $\mathfrak{H}$, by ω_α the differential form ω_α in $B^0(A^*) \simeq \mathfrak{H}$ and by e_α the dual element to x_α in $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$.

Let $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, together with two base points $\boldsymbol{x}$ and $\boldsymbol{y}$ (tangential or not). Given a path γ from $\boldsymbol{x}$ to $\boldsymbol{y}$ and $\omega \in B^0(A^*_{\mathbb{C}})$, we define

$$L_\omega(\gamma) = \int_\gamma^{\mathrm{reg}} \omega \in \mathbb{C}.$$

Or, in the notation of Theorem 3.220

$$L_\omega(\gamma) = \langle \omega, \gamma \rangle^{\mathrm{reg}}.$$

For a binary sequence α, we set $L_\alpha(\gamma) = L_{\omega_\alpha}(\gamma)$. Consider the generating series

$$L(\gamma) = \sum_\alpha L_\alpha(\gamma) e_\alpha \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle.$$

Therefore, if $\omega \in B^0(A^*_{\mathbb{C}}) \simeq \mathfrak{H} \otimes \mathbb{C} \simeq \mathbb{C}\langle\langle e_0, e_1\rangle\rangle^\vee$, we have

$$L(\gamma)(\omega) = L_\omega(\gamma).$$

3.8.3. *The map L and polylogarithms.* Recall that in Definition 1.110 we attached to a positive multi-index $\boldsymbol{s}$ a complex-valued function $\mathrm{Li}_{\boldsymbol{s}}$, the polylogarithm, defined on the open unit disc $|z| < 1$. The relation between these notions is explained by the following lemma, whose proof is parallel to that of Theorem 1.117. We left the details to the reader.

LEMMA 3.233. *Let z be a complex number such that $0 < |z| < 1$, γ any path from $\mathbf{0}$ to z contained in the unit disc, and $\mathbf{s}$ a positive multi-index. Let $\mathrm{bs}(\boldsymbol{s})$ denote the associated binary sequence. Then:*

$$\mathrm{Li}_{\mathbf{s}}(z) = L_{\mathrm{bs}(\mathbf{s})}(\gamma).$$

3.8.4. *Computation of $L(\gamma_0)$.* For any $z \in \mathbb{C} \setminus \{0, 1\}$, any path γ from $\mathbf{0}$ to z and any binary sequence α, $L_\alpha(\gamma)$ is defined. By abuse of notation, we will write $L_\alpha(z)$ and think of it as a *multivalued function.*

EXAMPLE 3.234. Let $z \in \mathbb{C} \setminus \{0, 1\}$. Let us show that, for each $n \geq 1$, the following equality of multivalued functions holds:

$$L_{0^n}(z) = \frac{1}{n!}(\log z)^n \tag{3.235}$$

Let γ be any path from $\mathbf{0}$ to z. We argue by induction on n. First, for $n = 1$, to compute the value

$$L_0(\gamma) = \int_\gamma^{\mathrm{reg}} \frac{dt}{t}$$

one needs to find a logarithmic asymptotic development for

$$\begin{aligned}\eta \mapsto \int_\eta^{1-\eta} \gamma^*(\tfrac{dt}{t}) &= \int_\eta^{1-\eta} \frac{\gamma'}{\gamma} dt \\ &= \log \gamma(1-\eta) - \log \gamma(\eta).\end{aligned}$$

Since $\gamma(0) = 0$ and $\gamma'(0) = 1$, one has $\gamma(\eta) = \eta(1 - O(\eta))$ as η goes to zero. On the other hand, $\gamma(1 - \eta) = z + O(\eta)$. Thus,

$$\log \gamma(1-\eta) - \log \gamma(\eta) = \log z + O(\eta) - \log \eta$$

and the regularization assigns the value

$$L_0(z) = \log z.$$

Assume now that the identity (3.235) holds for $n - 1$. Since the number of shuffles of type $(1, n-1)$ is n (cf. Exercise 1.134), relation (3) of Theorem 3.217 gives the result we wanted:

$$nL_{0^n}(z) = \int_\gamma^{\mathrm{reg}} \omega_0 \int_\gamma \omega_0 \overset{n-1}{\cdots} \omega_0 = \frac{1}{(n-1)!}(\log z)^n.$$

EXAMPLE 3.236. We are now ready to compute $L(\gamma_0)$. Arguing as in Example 3.234, one gets

$$L_{0^n}(\gamma_0) = \frac{1}{n!}(2\pi i)^n.$$

If α is a non-empty positive binary sequence, Lemma 3.233 implies that

$$L_\alpha(\gamma_0) = 0.$$

In fact, it follows from the compatibility with the shuffle product, part (3) of Theorem 3.220, that $L_{\alpha 0^k}(\gamma_0) = 0$ for all $\alpha \neq \emptyset$ and all $k \geq 0$. Summing up, we deduce that

$$L(\gamma_0) = \sum_\alpha L_\alpha(\gamma_0)e_\alpha = \sum_{n\geq 0} \frac{(2\pi i)^n}{n!} e_0^n = \exp(2\pi i e_0). \tag{3.237}$$

Thanks to the symmetry $z \mapsto 1 - z$, it follows that

$$L(\gamma_1') = \exp(2\pi i e_1). \tag{3.238}$$

3.8.5. *L evaluated at* **dch** *and the Drinfeld associator.*

EXAMPLE 3.239. Theorem 3.228 implies that, for each binary sequence α, the equality $L_\alpha(\mathbf{dch}) = \zeta_\sqcup(x_\alpha)$ holds. Therefore

$$L(\mathbf{dch}) = \sum_\alpha \zeta_\sqcup(x_\alpha)e_\alpha. \tag{3.240}$$

We write $\Phi(e_0, e_1)$ for this power series with real coefficients. We also write

$$\Phi_{KZ}(e_0, e_1) = \Phi(e_0, -e_1) = \sum_\alpha (-1)^{l(\alpha)}\zeta_\sqcup(x_\alpha)e_\alpha, \tag{3.241}$$

where $l(\alpha)$ is the number of entries equal to 1 in α as in Definition 1.124.

DEFINITION 3.242. The power series $\Phi_{KZ}(e_0, e_1) \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$ is called the *Drinfeld associator.*

3.8.6. *Chen's theorem revisited.*

THEOREM 3.243. *For any two base points $\boldsymbol{x}$ and $\boldsymbol{y}$, the map L can be extended to a continuous $\mathbb{C}$-linear isomorphism*

$$L\colon \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge \longrightarrow \mathbb{C}\langle\langle e_0, e_1\rangle\rangle = \operatorname{Hom}(\mathfrak{H}, \mathbb{C}).$$

The following properties hold:

(1) *If $u \in \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge$, then*
$$S^\vee(L(u)) = L(S(u)).$$
In particular, if $\gamma \in \pi_1(M; \boldsymbol{y}, \boldsymbol{x})$ is a path, $S^\vee(L(\gamma)) = L(\gamma^{-1})$.

(2) *Given three points $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, and elements $v \in \mathbb{C}[\pi_1(M; \boldsymbol{y}, \boldsymbol{x})]^\wedge$, $u \in \mathbb{C}[\pi_1(M; \boldsymbol{z}, \boldsymbol{y})]^\wedge$, one has*
$$L(uv) = L(u)L(v).$$

(3) If $u \in \mathbb{C}[\pi_1(M;\boldsymbol{y},\boldsymbol{x})]^\wedge$, then

$$\nabla^\vee(L(u)) = (L \otimes L)(\Delta(u)).$$

In particular, if $\gamma \in \pi_1(M;\boldsymbol{y},\boldsymbol{x})$ is a path, then $L(\gamma)$ is a group-like element.

PROOF. We first extend L by linearity to $\mathbb{C}[\pi_1(M;\boldsymbol{y},\boldsymbol{x})]$. By construction, for any path γ, the series $L(\gamma)$ starts by one. Therefore, any element in the augmentation ideal of $\mathbb{C}[\pi_1(M;\boldsymbol{y},\boldsymbol{x})]$ is sent to an element of the ideal generated by e_0 and e_1. Thus, it can be extended uniquely to a morphism

$$L\colon \mathbb{C}[\pi_1(M;\boldsymbol{y},\boldsymbol{x})]^\wedge \longrightarrow \mathbb{C}\langle\langle e_0,e_1\rangle\rangle = \operatorname{Hom}(\mathfrak{H},\mathbb{C}).$$

That this yields an isomorphism is simply a reformulation of Theorem 3.229. Clearly, it is enough to check properties (1) to (3) on paths. All of them follow from Theorem 3.217.

We start proving (1) using Theorem 3.220 (1).

$$\begin{aligned} L(\gamma^{-1}) = \sum_\alpha \langle \omega_\alpha, \gamma^{-1}\rangle^{\mathrm{reg}} e_\alpha &= \sum_\alpha \langle S(\omega_\alpha), \gamma\rangle^{\mathrm{reg}} e_\alpha \\ &= \sum_\alpha \langle \omega_\alpha, \gamma\rangle^{\mathrm{reg}} S^\vee(e_\alpha) = S^\vee(L(\gamma)). \end{aligned}$$

We next prove (2) using 3.220 (2)

$$\begin{aligned} L(\gamma_1\gamma_2) = \sum_\alpha \langle \omega_\alpha, \gamma_1\gamma_2\rangle^{\mathrm{reg}} e_\alpha &= \sum_\alpha \langle \Delta\omega_\alpha, \gamma_1\otimes\gamma_2\rangle^{\mathrm{reg}} e_\alpha \\ &= \sum_{\alpha',\alpha''} \langle \omega_{\alpha'}\otimes\omega_{\alpha''}, \gamma_1\otimes\gamma_2\rangle^{\mathrm{reg}} e_{\alpha'}e_{\alpha''} = L(\gamma_1)L(\gamma_2). \end{aligned}$$

Finally we prove (3) using 3.220 (3).

$$\begin{aligned} \nabla^\vee(L(\gamma)) &= \sum_\alpha \langle \omega_\alpha, \gamma\rangle^{\mathrm{reg}} \nabla^\vee e_\alpha \\ &= \sum_\alpha \langle \omega_\alpha, \gamma\rangle^{\mathrm{reg}} \sum_{\alpha',\alpha''} ш(\alpha',\alpha'';\alpha) e_{\alpha'}\otimes e_{\alpha''} = \sum_{\alpha',\alpha''} \langle \omega_{\alpha'} ш \omega_{\alpha''}, \gamma\rangle^{\mathrm{reg}} e_{\alpha'}\otimes e_{\alpha''} \\ &= \sum_{\alpha',\alpha''} \langle \omega_{\alpha'}, \gamma\rangle^{\mathrm{reg}} \langle \omega_{\alpha''}, \gamma\rangle^{\mathrm{reg}} e_{\alpha'}\otimes e_{\alpha''} = L(\gamma)\otimes L(\gamma). \end{aligned}$$

This concludes the proof. □

EXAMPLE 3.244. From Theorem 3.243 (3) we deduce that $\Phi(e_0,e_1) = L(\mathbf{dch})$ is a group-like element. In particular, it is the exponential of a Lie-like element and its inverse as power series is given by its antipode

$$L(\mathbf{dch}^{-1}) = \Phi(e_0,e_1)^{-1} = S^\vee(\Phi(e_0,e_1)). \tag{3.245}$$

From examples 3.236 and 3.239 and the compatibility of L with the composition of paths in 3.243 (2) we can compute L on the remaining generators of $\pi_1(M,\mathbf{0})$ and $\pi_1(M,\mathbf{1})$.

$$\begin{aligned}L(\gamma_1) &= \Phi(e_0,e_1)^{-1}\exp(2\pi i e_1)\Phi(e_0,e_1),\\ L(\gamma_0') &= \Phi(e_0,e_1)\exp(2\pi i e_o)\Phi(e_0,e_1)^{-1}.\end{aligned}$$

3.8.7. *The Knizhnik-Zamolodchikov equation.* Theorem 3.243 encodes all the properties of the series L, hence of polylogarithms. The first property is that it satisfies the so-called Knizhnik-Zamolodchikov equation :

PROPOSITION 3.246. *$L(z)$ satisfies the differential equation*

$$\frac{d}{dz}L(z) = \left(\frac{e_0}{z}+\frac{e_1}{1-z}\right)L(z). \tag{3.247}$$

PROOF. Fix $z\in M$, let γ be a path with end point z and let $\gamma_\varepsilon(t) = z+t\varepsilon$. To compute the derivative of $L(z)$ we need to evaluate the limit

$$\lim_{\varepsilon\to 0}\frac{L(\gamma_\varepsilon\gamma)-L(\gamma)}{\varepsilon}.$$

By Theorem 3.243 (2)

$$L(\gamma_\varepsilon\gamma)-L(\gamma) = (L(\gamma_\varepsilon)-1)L(\gamma).$$

Moreover,

$$L(\gamma_\varepsilon)-1 = \int_{\gamma_\varepsilon}^{\mathrm{reg}}\omega_0 e_0 + \int_{\gamma_\varepsilon}^{\mathrm{reg}}\omega_1 e_1 + O(\varepsilon^2).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\gamma_\varepsilon}\omega_0 = \frac{1}{z} \quad\text{and}\quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\gamma_\varepsilon}\omega_1 = \frac{1}{1-z},$$

we conclude

$$\frac{d}{dz}L(z) = \left(\frac{e_0}{z}+\frac{e_1}{1-z}\right)L(z).$$

Finishing the proof. □

3.8.8. *The monodromy of L.* The second property we want to derive is an explicit description of the monodromy of L as a multivalued function.

THEOREM 3.248. *Let $z\in M$ and γ a path from $\mathbf{0}$ to z. Then*

$$\begin{aligned}L(\gamma\cdot\gamma_0) &= L(\gamma)\exp(2\pi i e_0),\\ L(\gamma\cdot\gamma_1) &= L(\gamma)\Phi(e_0,e_1)^{-1}\exp(2\pi i e_1)\Phi(e_0,e_1).\end{aligned}$$

PROOF. Follows immediately from Theorem 3.243 (2) and examples 3.236 and 3.244. □

3.8.9. *Further properties of the Drinfeld associator.* We next derive the basic properties of Drinfeld associator Φ_{KZ}. Let $U\mathfrak{a}_4$ be the universal enveloping algebra of the Lie algebra of the pro-unipotent completion of the pure braid group on 4 strings. It is the algebra of power series in letters $t_{i,j}$, $1 \le i,j \le 4$ with the relations

$$\begin{aligned} t_{i,i} = 0, &\qquad t_{i,j} = t_{j,i}, \\ [t_{i,j}, t_{i,k} + t_{j,k}] = 0, &\ \text{ for } i,j,k \text{ different}, \\ [t_{i,j}, t_{k,l}] = 0, &\ \text{ for } i,j,k,l \text{ different}. \end{aligned}$$

THEOREM 3.249 (Drinfeld [**Dri90**]). *The Drinfeld associator satisfies the following relations.*

(1) Symmetry relation: $\Phi_{KZ}(e_0, e_1)\Phi_{KZ}(e_1, e_0) = 1$.

(2) Hexagon relation: Write $e_\infty = -e_0 - e_1$, *then*

$$e^{i\pi e_0}\Phi_{KZ}(e_\infty, e_0)e^{i\pi e_\infty}\Phi_{KZ}(e_1, e_\infty)e^{i\pi e_1}\Phi_{KZ}(e_0, e_1) = 1.$$

(3) Pentagon relation: For $t_{i,j} \in U\mathfrak{a}_4$ *we have*

$$\begin{aligned} &\Phi_{KZ}(t_{1,2}, t_{2,3} + t_{2,4})\Phi_{KZ}(t_{1,3} + t_{2,3}, t_{3,4}) \\ &\qquad = \Phi_{KZ}(t_{2,3}, t_{3,4})\Phi_{KZ}(t_{1,2} + t_{2,3}, t_{2,4} + t_{3,4})\Phi_{KZ}(t_{1,2}, t_{2,3}). \end{aligned}$$

PROOF. We start proving (1). Consider the automorphism of M given by $z \mapsto 1 - z$. This automorphism sends the form ω_i to $-\omega_{1-i}$ for $i = 0, 1$, hence it sends e_0 to $-e_1$ and e_1 to $-e_0$. Moreover it sends $\mathbf{dch}$ to $\mathbf{dch}^{-1}$. Therefore we deduce that $L(\mathbf{dch}^{-1}) = \Phi(-e_1, -e_0)$. Therefore

$$1 = L(\mathbf{dch})L(\mathbf{dch}^{-1}) = \Phi(e_0, e_1)\Phi(-e_1, -e_0),$$

which is equivalent to (1).

To prove (2) we need to introduce more tangential points and paths. Let $\mathbf{0}^- = (0, -1)$ be the tangent vector -1 at 0 and $\mathbf{1}^- = (1, 1)$ be the tangent vector 1 at 1. We consider the point ∞ with local coordinate $u = 1/z$ and denote $\boldsymbol{\infty} = (\infty, 1)$ the tangent point 1 at ∞ with respect to this coordinate and $\boldsymbol{\infty}^- = (\infty, -1)$. We denote by $\delta_0 \in \pi(M; \mathbf{0}, \mathbf{0}^-)$ the path that starts in $\mathbf{0}^-$, gives half a turn around zero in the counterclockwise direction and ends in $\mathbf{0}$. Similarly, $\delta_1 \in \pi(M; \mathbf{1}^-, \mathbf{1})$ is the path that starts in $\mathbf{1}$, gives half a turn in the counterclockwise direction and ends in $\mathbf{1}^-$ and $\delta_\infty \in \pi(M; \boldsymbol{\infty}^-, \boldsymbol{\infty})$ is the path that starts in $\boldsymbol{\infty}$, gives half a turn in the counterclockwise direction and ends in $\boldsymbol{\infty}^-$. Finally we denote by $\mathbf{dch}_{\infty,1} \in \pi(M; \boldsymbol{\infty}, \mathbf{1}^-)$ the straight path that starts in $\mathbf{1}^-$ and ends in $\boldsymbol{\infty}$ through the real numbers greater than one and by $\mathbf{dch}_{0,\infty} \in \pi(M; \mathbf{0}^-, \boldsymbol{\infty}^-)$ the straight path that starts in $\boldsymbol{\infty}^-$ and ends in $\mathbf{0}^-$ through the negative real numbers. All these paths are represented in Figure 19.

Clearly, the composition

$$\delta_0 \cdot \mathbf{dch}_{0,\infty} \cdot \delta_\infty \cdot \mathbf{dch}_{\infty,1} \cdot \delta_1 \cdot \mathbf{dch}$$

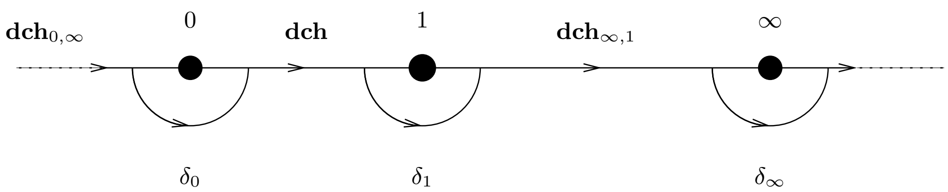


FIGURE 19. paths

is homotopically equivalent to the trivial path. Therefore, by Theorem 3.243 (2),

$$L(\delta_0)L(\mathbf{dch}_{0,\infty})L(\delta_\infty)L(\mathbf{dch}_{\infty,1})L(\delta_1)L(\mathbf{dch}) = 1. \tag{3.250}$$

Arguing as in Example 3.236, we can see that

$$L(\delta_0) = \exp(\pi i e_0).$$

We now consider the automorphism of M given by $z \mapsto 1/(1-z)$. This map sends δ_0 to δ_1 and δ_1 to δ_∞. It also sends $\mathbf{dch}$ to $\mathbf{dch}_{\infty,1}$ and $\mathbf{dch}_{\infty,1}$ to $\mathbf{dch}_{0,\infty}$.

Moreover the pull back by this isomorphism sends the form ω_0 to the form ω_1 and the form ω_1 to the form $-\omega_0 - \omega_1$. Dualizing we deduce that this automorphism sends e_0 to $-e_1$ and e_1 to $e_0 - e_1$. We deduce that

$$L(\delta_1) = \exp(-\pi i e_1), \qquad L(\delta_\infty) = \exp(\pi i(e_1 - e_0)),$$
$$L(\mathbf{dch}_{\infty,1}) = \Phi(-e_1, e_0 - e_1), \quad L(\mathbf{dch}_{0,\infty}) = \Phi(e_1 - e_0, -e_0).$$

Thus equation (3.250) reads

$$e^{i\pi e_0}\Phi(e_1 - e_0, -e_0)e^{i\pi(e_1-e_0)}\Phi(-e_1, e_0 - e_1)e^{-i\pi e_1}\Phi(e_0, e_1) = 1,$$

which is equivalent to

$$e^{i\pi e_0}\Phi_{KZ}(e_1 - e_0, e_0)e^{i\pi(e_1-e_0)}\Phi_{KZ}(-e_1, e_1 - e_0)e^{-i\pi e_1}\Phi_{KZ}(e_0, -e_1) = 1.$$

The hexagon relation is obtained by replacing e_1 by $-e_1$.

The proof of (3) involves considering a path in the moduli space $M_{0,5}$ which is a complex surface. To write it properly, we would need to discuss tangential base points and local monodromy in higher dimensions, so we will omit it. See for instance [**Had**] for an outline. □

3.8.10. *The associator relations and the extended double shuffle relations.* We close this section by quoting

THEOREM 3.251 (Furusho [**Fur10**], [**Fur11**]).

a) Let $(\zeta^s(\alpha))_\alpha$ be a collection of real numbers, one for each binary sequence. Denote by $\zeta^s\colon \mathfrak{H}^0 \to \mathbb{R}$ the map obtained from these numbers by linearity. If the power series

$$\sum_\alpha (-1)^\alpha \zeta^s(\alpha) e_\alpha$$

is group-like and satisfies the associator relations of Theorem 3.249, then $(\mathbb{R}, \zeta^s)$ satisfies the extended double shuffle relations (Definition 1.192).

b) *Let $\varphi \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$ be a group-like element with the coefficient of e_0e_1 equal to $-\zeta(2) = -\pi^2/6$. If φ satisfies the pentagon relation 3.249 (3), then it satisfies the symmetry relation 3.249 (1) and the hexagon relation 3.249 (2).*

⋆⋆⋆

Exercise 3.252. Compute explicitly the terms up to degree 5 of the Drinfeld associator $\Phi_{KZ}(e_0, e_1)$. Show that, with the exception of the unit in degree 0, they can be all written as commutators.

Exercise 3.253. In this exercise, we show how Theorem 3.248 encodes the monodromy of multiple polylogarithms in one variable. We start with Li_3, which is the coefficient of $e_0e_0e_1$ in L. Let $z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and let γ be a path between $\mathbf{0}$ and z.

(1) Find the coefficient of $e_0e_0e_1$ in $L(\gamma\cdot\gamma_0)$ and $L(\gamma\cdot\gamma_1)$. The obtained expressions give us the monodromy of Li_3.
(2) Compute the monodromy through γ_0 and γ_1 of the functions
$$L_\emptyset = 1, \quad L_0, \quad L_1, \quad L_{0001}, \quad L_{01001}.$$

3.9. The fundamental groupoid of $\mathbb{P}^1\setminus\{0, 1, \infty\}$. We continue studying the manifold $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, but we view it as the set of complex points of the variety $X = \mathbb{P}^1_{\mathbb{Q}}\setminus\{0, 1, \infty\}$ defined over $\mathbb{Q}$. Recall from Example 2.130 that the dg-algebra A^* computes the algebraic de Rham cohomology of X.

3.9.1. *Summary of structures.* For convenience and to fix notations, we start by summarizing some results of the previous sections.

Summary 3.254. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{\mathbf{0}, \mathbf{1}\} \cup X(\mathbb{Q})$ be base points, tangential or not. We have at our disposal the following structures.

(1) *(Betti side.)* An affine pro-algebraic scheme over $\mathbb{Q}$
$${}_{\boldsymbol{y}}\Pi^{\mathrm{B}}_{\boldsymbol{x}} \coloneqq \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}; \boldsymbol{y}, \boldsymbol{x})^{\mathrm{un}},$$
a pro-$\mathbb{Q}$-vector space
$${}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}} \coloneqq \mathbb{Q}\big[\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}; \boldsymbol{y}, \boldsymbol{x})\big]^\wedge,$$
the subspace of Lie-like elements
$${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{B}}_{\boldsymbol{x}} \coloneqq \{x \in {}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}} \mid \nabla^\vee x = 1 \otimes x + x \otimes 1\},$$
and an ind-$\mathbb{Q}$-algebra
$${}_{\boldsymbol{y}}A^{\mathrm{B}}_{\boldsymbol{x}} \coloneqq \mathcal{O}({}_{\boldsymbol{y}}\Pi^{\mathrm{B}}_{\boldsymbol{x}}) = ({}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}})^\vee.$$

(2) *(De Rham side.)* An affine pro-algebraic scheme over $\mathbb{Q}$

$$ {}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}} := \operatorname{Spec}(\mathfrak{H}), $$

a pro-$\mathbb{Q}$-vector space

$$ {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{dR}} := \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle, $$

the subspace of Lie-like elements

$$ {}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{dR}} := \{x \in {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{dR}} \mid \nabla^{\vee} x = 1 \otimes x + x \otimes 1\}, $$

and an ind-$\mathbb{Q}$-algebra

$$ {}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{dR}} = \mathfrak{H}. $$

(3) *(Comparison.)* Comparison isomorphisms[7]

$$ \begin{aligned} \operatorname{comp}_{\mathrm{dR,B}}^{\Pi}&\colon {}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{B}} \times_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}} \times_{\mathbb{Q}} \mathbb{C}, \\ \operatorname{comp}_{\mathrm{dR,B}}^{U}&\colon {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{B}} \hat{\otimes}_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}} \mathbb{C}, \\ \operatorname{comp}_{\mathrm{dR,B}}^{\mathcal{L}}&\colon {}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{B}} \hat{\otimes}_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}\mathcal{L}_{\boldsymbol{x}}^{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}} \mathbb{C}, \\ \operatorname{comp}_{\mathrm{B,dR}}^{A}&\colon {}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} {}_{\boldsymbol{y}}A_{\boldsymbol{x}}^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}. \end{aligned} $$

Observe that the de Rham side on Summary 3.254 is independent of the base points. In fact, there is a canonical de Rham path ${}_{\boldsymbol{y}}1_{\boldsymbol{x}}^{\mathrm{dR}}$ in ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}}$ (it is the unit element in the group scheme $\operatorname{Spec}(\mathfrak{H})$ and corresponds to the kernel of the counit $\varepsilon\colon \mathfrak{H} \to \mathbb{Q}$). Since the pro-algebraic scheme ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}}$ is independent of the base points, we will suppress them from the notation and we will write $\Pi^{\mathrm{dR}} = \operatorname{Spec}\mathfrak{H}$.

Moreover, for $? = \mathrm{B}, \mathrm{dR}$, the pro-algebraic schemes come together with morphisms

$$ {}_{\boldsymbol{z}}\Pi_{\boldsymbol{y}}^{?} \times {}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{z}}\Pi_{\boldsymbol{x}}^{?} \tag{3.255} $$

induced from the composition of paths on the Betti side and the coproduct of $\mathfrak{H}$ on the de Rham side. These maps turn ${}_{\boldsymbol{x}}\Pi_{\boldsymbol{x}}^{?}$ into a pro-unipotent group scheme and ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{?}$ into a right ${}_{\boldsymbol{x}}\Pi_{\boldsymbol{x}}^{?}$ torsor and a left ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{y}}^{?}$ torsor.

Therefore, the pro-$\mathbb{Q}$-vector spaces come equipped with the following structures:

(1) a composition of paths

$$ \Delta^{\vee}\colon {}_{\boldsymbol{z}}U_{\boldsymbol{y}}^{?} \otimes {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{z}}U_{\boldsymbol{x}}^{?}; $$

(2) units

$$ \eta_{\boldsymbol{x}}^{\vee}\colon \mathbb{Q} \longrightarrow {}_{\boldsymbol{x}}U_{\boldsymbol{x}}^{?}; $$

(3) a completed coproduct

$$ \nabla^{\vee}\colon {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \longrightarrow {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?} \hat{\otimes} {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{?}; $$

[7]Recall that ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{B}}$ and ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}}$ are affine schemes over $\mathbb{Q}$. Below, the notation $\times_{\mathbb{Q}}\mathbb{C}$ is a shorthand for $\times_{\operatorname{Spec}(\mathbb{Q})} \operatorname{Spec}(\mathbb{C})$.

(4) counits

$$\epsilon^{\vee} \colon {}_{\boldsymbol{y}}U^{?}_{\boldsymbol{x}} \longrightarrow \mathbb{Q};$$

(5) a dual antipode

$$S^{\vee} \colon {}_{\boldsymbol{y}}U^{?}_{\boldsymbol{x}} \longrightarrow {}_{\boldsymbol{x}}U^{?}_{\boldsymbol{y}}.$$

And the ind-algebras ${}_{\boldsymbol{y}}A^{?}_{\boldsymbol{x}}$ come equipped with the dual structures.

All the comparison isomorphisms $\mathrm{comp}^{?}_{\mathrm{B,dR}}$ are given by the regularized iterated integrals. For instance

$$\mathrm{comp}^{U}_{dR,B} \colon {}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}} \longrightarrow {}_{\boldsymbol{y}}U^{\mathrm{dR}}_{\boldsymbol{x}}$$

agrees with the map L of Theorem 3.243. Moreover they are compatible with all the structures: The group and torsor structures on Π, the product, unit, coproduct, counit and antipode for A and the completed coproduct, counits, the composition of paths, units and the dual antipode for U.

It is immediate to extend the construction of Section 3.6.2 to tangential base points. Therefore the spaces ${}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}}$ and ${}_{\boldsymbol{y}}A^{\mathrm{B}}_{\boldsymbol{x}}$ come equipped with a weight filtration W and the spaces ${}_{\boldsymbol{y}}U^{\mathrm{dR}}_{\boldsymbol{x}}$ and ${}_{\boldsymbol{y}}A^{\mathrm{dR}}_{\boldsymbol{x}}$ with a weight filtration W and a Hodge filtration F in such a way that

$${}_{\boldsymbol{y}}A^{\mathrm{H}}_{\boldsymbol{x}} \coloneqq (({}_{\boldsymbol{y}}A^{\mathrm{B}}_{\boldsymbol{x}}, W), ({}_{\boldsymbol{y}}A^{\mathrm{dR}}_{\boldsymbol{x}}, W, F), \mathrm{comp}_{\mathrm{B,dR}})$$

is in ind-$\mathbf{MHS}(\mathbb{Q})$ and

$${}_{\boldsymbol{y}}U^{\mathrm{H}}_{\boldsymbol{x}} \coloneqq (({}_{\boldsymbol{y}}U^{\mathrm{B}}_{\boldsymbol{x}}, W), ({}_{\boldsymbol{y}}U^{\mathrm{dR}}_{\boldsymbol{x}}, W, F), \mathrm{comp}^{-1}_{\mathrm{dR,B}})$$

is in pro-$\mathbf{MHS}(\mathbb{Q})$.

The filtrations in ${}_{\boldsymbol{y}}U^{?}_{\boldsymbol{x}}$ induce a weight filtration on ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{B}}_{\boldsymbol{x}}$ and weight and Hodge filtrations on ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{dR}}_{\boldsymbol{x}}$, so that

$${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}} \coloneqq (({}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{B}}_{\boldsymbol{x}}, W), ({}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{dR}}_{\boldsymbol{x}}, W, F), \mathrm{comp}^{-1}_{\mathrm{dR,B}})$$

is also in pro-$\mathbf{MHS}(\mathbb{Q})$.

Moreover, it is easy to check that all the previous structures of ${}_{\bullet}A^{\mathrm{H}}_{\bullet}$ are morphisms of ind-$\mathbf{MHS}(\mathbb{Q})$ and the corresponding structures of ${}_{\bullet}U^{\mathrm{H}}_{\bullet}$ are morphisms of pro-$\mathbf{MHS}(\mathbb{Q})$.

Variant 3.256. The same structures are available for other varieties. For instance, everything can be easily generalized to any variety of the form $X' = \mathbb{P}^1_{\mathbb{Q}} \setminus S$ for $S \subset \mathbb{P}^1(\mathbb{Q})$ a finite set. In this case we will use the notation ${}_{\boldsymbol{y}}\Pi(X')^{\mathrm{B}}_{\boldsymbol{x}}$ for the pro-algebraic scheme in the Betti side and similar notation for the other structures. In the sequel, we will only need the case $X' = \mathbb{G}_m$.

In this particular case we have

$$\begin{aligned}
{}_{\boldsymbol{y}}\Pi(\mathbb{G}_m)_{\boldsymbol{x}}^{\mathrm{dR}} &= \mathbb{A}^1_{\mathbb{Q}} = \mathbb{G}_a \\
{}_{\boldsymbol{y}}A(\mathbb{G}_m)_{\boldsymbol{x}}^{\mathrm{dR}} &= \mathbb{Q}[x_0] \\
{}_{\boldsymbol{y}}U(\mathbb{G}_m)_{\boldsymbol{x}}^{\mathrm{dR}} &= \mathbb{Q}[\![e_0]\!] \\
{}_{\boldsymbol{y}}\mathcal{L}(\mathbb{G}_m)_{\boldsymbol{x}}^{\mathrm{dR}} &= \mathbb{Q}e_0
\end{aligned}$$

and the map

$$\mathrm{comp}^{U}_{\mathrm{dR,B}} \colon {}_{\boldsymbol{y}}U(\mathbb{G}_m)_{\boldsymbol{x}}^{\mathrm{B}} \longrightarrow {}_{\boldsymbol{y}}U(\mathbb{G}_m)_{\boldsymbol{x}}^{\mathrm{dR}}$$

will also be denoted by L.

3.9.2. *The fundamental groupoid and the local monodromy.* From now on, we focus our attention on the pro-unipotent group picture ${}_{\boldsymbol{y}}\Pi^{?}_{\boldsymbol{x}}$. The reader will have no difficulty writing the analogous statements for $U^?$, $\mathcal{L}^?$ and $A^?$.

DEFINITION 3.257. The diagram consisting of the four schemes ${}_{\boldsymbol{y}}\Pi^{?}_{\boldsymbol{x}}$, $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$ with the composition of paths will be called the *tangential fundamental groupoid of* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. It is represented schematically in Figure 20.

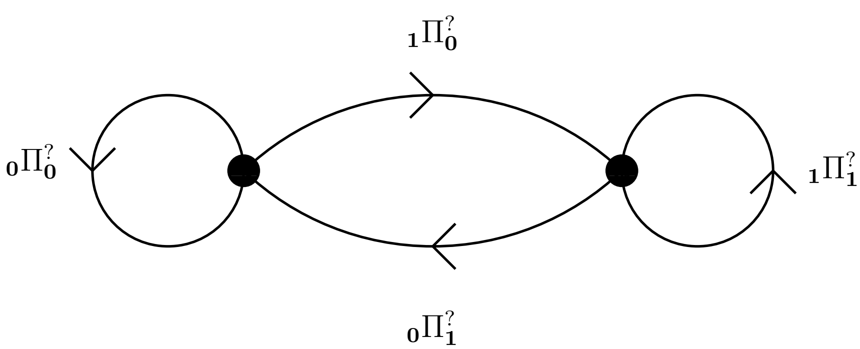


FIGURE 20. The fundamental groupoid

To the tangential fundamental groupoid we want to add the *local monodromy* around 0 and 1.

We start with the local monodromy around 0 in the de Rham side. There is a morphism of Hopf algebras $\mathfrak{H} \to \mathbb{Q}[x]$ that sends any word containing x_1 to zero, and $x_0 \overset{n}{\cdots} x_0$ to $x^n/n!$. We can see this as a map

$${}_{\mathbf{0}}A^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}}$$

that induces maps

$$\mathbb{G}_a = {}_{\mathbf{0}}\Pi(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}, \text{ and } {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\mathbf{0}} \longrightarrow {}_{\mathbf{0}}U^{\mathrm{dR}}_{\mathbf{0}}.$$

The local monodromy around 0 in the Betti side is obtained topologically as follows. Let Δ^* be a small punctured disc around zero in $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$. The local monodromy is the composition of the inverse of the isomorphism

$$\pi_1(\Delta^*,\mathbf{0})^{\mathrm{un}} \to \pi_1(\mathbb{G}_m,\mathbf{0})^{\mathrm{un}} = \mathbb{G}_a$$

with the natural map

$$\pi_1(\Delta^*,\mathbf{0})^{\mathrm{un}} \to \pi_1(\mathbb{P}^1\setminus\{0,1,\infty\},\mathbf{0})^{\mathrm{un}}.$$

Similarly, the de Rham side of the local monodromy around 1 is induced by the map of Hopf algebras $\mathfrak{H}\to\mathbb{Q}[x]$ that sends any word containing x_0 to zero, and $x_1.\overset{n}{\cdots}.x_1$ to $x^n/n!$. While the Betti side is obtained from a small punctured disc around 1.

The local monodromy maps are morphisms of ind-$\mathbf{MHS}(\mathbb{Q})$ in the case of A and morphism of pro-$\mathbf{MHS}(\mathbb{Q})$ in the case of U. This means that the pair of maps

$$_{\mathbf{0}}A_{\mathbf{0}}^{\mathrm{dR}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{dR}} \qquad {}_{\mathbf{0}}A_{\mathbf{0}}^{\mathrm{B}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{B}}$$

is a morphism of ind-$\mathbf{MHS}(\mathbb{Q})$

$$_{\mathbf{0}}A_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}A(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}},$$

while the pair of maps

$$_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{dR}} \longrightarrow {}_{\mathbf{0}}U_{\mathbf{0}}^{\mathrm{dR}} \qquad {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{B}} \longrightarrow {}_{\mathbf{0}}U_{\mathbf{0}}^{\mathrm{B}}$$

is a morphism of pro-$\mathbf{MHS}(\mathbb{Q})$

$$_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}U_{\mathbf{0}}^{\mathrm{H}},$$

and the same is true for the local monodromy maps around 1.

Definition 3.258. We will denote by D^{dR} the diagram consisting of the four schemes ${}_{\boldsymbol{y}}\Pi_{\boldsymbol{x}}^{\mathrm{dR}}$, $\boldsymbol{x},\boldsymbol{y}\in\{\mathbf{0},\mathbf{1}\}$, the morphisms given by the composition of paths, the scheme $\mathbb{G}_a$ and the two local monodromies

$$\mathbb{G}_a \longrightarrow {}_{\mathbf{0}}\Pi_{\mathbf{0}}^{\mathrm{dR}}, \qquad \mathbb{G}_a \longrightarrow {}_{\mathbf{1}}\Pi_{\mathbf{1}}^{\mathrm{dR}}.$$

Similarly, we write D_U^{dR}, D_A^{dR} for the corresponding diagram for the vector spaces U and the algebras A. Similarly we will denote $D_{\cdot}^{\mathrm{B}}$ for the corresponding diagrams on the Betti side. Finally we will denote D_U^{H} for the pair of diagrams D_U^{B} and D_U^{dR} together as a diagram of pro-$\mathbf{MHS}(\mathbb{Q})$.

We will see in chapter 4.6 that the diagram D_U^{H} is "motivic".

3.9.3. *The automorphisms of D^{dR}.* We denote by $\mathrm{Aut}(D^{\mathrm{dR}})$ the group of automorphisms of D^{dR} in the following sense: to give an element of $\mathrm{Aut}(D^{\mathrm{dR}})$ amounts to giving an automorphism of pro-algebraic schemes of each ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$ and an automorphism of $\mathbb{G}_a$ that are compatible with the composition of paths (3.255) and the local monodromy maps. The group $\mathrm{Aut}(D^{\mathrm{dR}})$ is a pro-algebraic group.

We denote by $\mathrm{Aut}^0(D^{\mathrm{dR}})$ the subgroup of $\mathrm{Aut}(D^{\mathrm{dR}})$ that acts as the identity on $\mathbb{G}_a$. There is an exact sequence

$$0 \to \mathrm{Aut}^0(D^{\mathrm{dR}}) \to \mathrm{Aut}(D^{\mathrm{dR}}) \to \mathbb{G}_m \to 0.$$

Lemma 3.259. *There is an isomorphism of schemes*

$$\begin{array}{ccc} \mathrm{Aut}^0(D^{\mathrm{dR}}) & \longrightarrow & \Pi^{\mathrm{dR}} \\ f & \longmapsto & \gamma_f, \end{array}$$

where γ_f is determined by the equation

$$f({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \gamma_f.$$

Proof. Recall that the dual of $\mathfrak{H}$, that agrees with the completed universal enveloping algebra of $\mathrm{Lie}({}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}})$, is the algebra $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. Let R be a $\mathbb{Q}$-algebra. The elements of ${}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R)$ are the group-like elements of $R\langle\langle e_0, e_1\rangle\rangle$. Moreover we have identities

$$\begin{aligned} {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) &= {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R), \\ {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{1}}(R) &= {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}, \\ {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{1}}(R) &= {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}. \end{aligned} \tag{3.260}$$

Let $f \in \mathrm{Aut}^0(D^{\mathrm{dR}})(R)$. Since f is the identity in $\mathbb{G}_a$ we deduce that

$$\begin{aligned} f(\exp(e_0)) &= \exp(e_0), \\ f({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \exp(e_1) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}) &= {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \exp(e_1) \cdot {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}. \end{aligned}$$

We also have $f({}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{0}}$ and ${}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}} \cdot {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} = {}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{0}}$. Therefore the fact that f is compatible with the composition of paths implies that it is determined by the image of ${}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}$. We write

$$f({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}} \cdot \gamma_f$$

for an element $\gamma_f \in {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) \subset R\langle\langle e_0, e_1\rangle\rangle$.

Conversely, let $\gamma \in {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(R) = \mathrm{Spec}(\mathfrak{H})(R)$. It is a group-like element of the algebra $R\langle\langle e_0, e_1\rangle\rangle$. To give an element of $\mathrm{Aut}({}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}})(R)$ is equivalent to give a continuous automorphism of $R\langle\langle e_0, e_1\rangle\rangle$ that is compatible with the completed coproduct and the antipode. We define

$$f_\gamma(e_0) = e_0, \qquad f_\gamma(e_1) = \gamma^{-1} \cdot e_1 \cdot \gamma.$$

This determines a continuous automorphism of $R\langle\langle e_0, e_1\rangle\rangle$. To show that it is compatible with the completed coproduct, it is enough to check it for the generator e_1. On the one hand,

$$\begin{aligned} f_\gamma(\nabla^\vee(e_1)) &= f_\gamma(1\otimes e_1 + e_1\otimes 1) \\ &= 1\otimes(\gamma^{-1}\cdot e_1\cdot\gamma) + (\gamma^{-1}\cdot e_1\cdot\gamma)\otimes 1. \end{aligned}$$

On the other hand, using that γ is group-like,

$$\begin{aligned} \nabla^\vee(f_\gamma(e_1)) &= \nabla^\vee(\gamma^{-1}\cdot e_1\cdot\gamma) \\ &= \gamma^{-1}\otimes\gamma^{-1}\cdot(1\otimes e_1 + e_1\otimes 1)\cdot\gamma\otimes\gamma \\ &= 1\otimes(\gamma^{-1}\cdot e_1\cdot\gamma) + (\gamma^{-1}\cdot e_1\cdot\gamma)\otimes 1. \end{aligned}$$

The fact that f_γ is compatible with the dual antipode follows from the fact that, by Lemma 3.105, since γ is group-like, then $S^\vee(\gamma) = \gamma^{-1}$.

In consequence f_γ determines an element of $\mathrm{Aut}({}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}})(R)$ that we also denote f_γ. Writing

$$f_\gamma({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}\cdot\gamma, \qquad f_\gamma({}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}) = \gamma^{-1}\cdot{}_{\mathbf{0}}1^{\mathrm{dR}}_{\mathbf{1}}$$

and using the identities (3.260), we obtain R-automorphisms of the four schemes ${}_{\boldsymbol{y}}\Pi^{\mathrm{dR}}_{\boldsymbol{x}}$, $\boldsymbol{x},\boldsymbol{y}\in\{\mathbf{0},\mathbf{1}\}$. By construction, these automorphisms are compatible with the composition of paths. Moreover they are compatible with the identity automorphism of $\mathbb{G}_a$ through any of the two local monodromies. Thus we obtain an element $f_\gamma\in\mathrm{Aut}^0(D^{\mathrm{dR}})(R)$.

Clearly, the assignments $f\mapsto\gamma_f$ and $\gamma\mapsto f_\gamma$ are inverse to each other, and this concludes the proof of the lemma. $\square$

3.9.4. *A new product structure.* The isomorphism of schemes of Lemma 3.259 is not a morphism of groups. Therefore, it induces a new group structure on $\mathrm{Spec}(\mathfrak{H})$.

DEFINITION 3.261. We denote by $(\Pi,\circ)$ the scheme $\Pi = \mathrm{Spec}(\mathfrak{H})$ with the product structure induced by the isomorphism of Lemma 3.259.

As schemes $\Pi = \Pi^{\mathrm{dR}} = {}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$ but the product structure is different. Therefore we obtain a new Lie bracket on the Lie algebra of Π which is still the set of Lie-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ that is called the *Ihara bracket* and a new coproduct on $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$ that is called the *Goncharov coproduct*. We now make all these structures explicit.

We start by computing the new product structure of Π that we denote by $\circ$. This product is determined by the equation

$$f_\gamma(f_\mu({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}})) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}\cdot(\gamma\circ\mu).$$

For a group-like element γ, we will denote by $\langle\gamma\rangle_0$ the restriction of f_γ to ${}_{\mathbf{0}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$ and also the corresponding continuous automorphism of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$.

Recall that it is given by

$$\langle\gamma\rangle_0(e_0) = e_0, \qquad \langle\gamma\rangle_0(e_1) = \gamma^{-1}\cdot e_1 \cdot \gamma.$$

Since f_γ is compatible with the composition of paths, then

$$f_\gamma(f_\mu({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}})) = f_\gamma({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}\cdot\mu) = f_\gamma({}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}})\cdot f_\gamma(\mu) = {}_{\mathbf{1}}1^{\mathrm{dR}}_{\mathbf{0}}\cdot\gamma\cdot\langle\gamma\rangle_0(\mu),$$

hence

$$\gamma\circ\mu = \gamma\cdot\langle\gamma\rangle_0(\mu). \tag{3.262}$$

3.9.5. *The Ihara bracket.* We now compute the new bracket induced in the set of Lie-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. Recall the notion of derivation from Definition 2.25. Given a Lie-like element $x \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, we define a continuous derivation $\partial_x \colon \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \to \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ as follows:

$$\partial_x(y) = \frac{d}{dt}\big(\langle\exp(tx)\rangle_0(y)\big)\Big|_{t=0}.$$

Explicitly, this derivation is determined by

$$\partial_x e_0 = 0, \qquad \partial_x e_1 = -x\cdot e_1 + e_1\cdot x.$$

Let now x and y be two Lie-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. We denote by $[x,y] = x\cdot y - y\cdot x$ the Lie bracket corresponding to the composition of paths. The Lie bracket induced by $\circ$ will be denoted by $\{x,y\}$. It is determined by

$$\{x,y\} = \frac{d}{du}\frac{d}{dv}\big(\exp(ux)\circ\exp(vy)\circ\exp(-ux)\circ\exp(-vy)\big)\Big|_{\substack{u=0\\ v=0}}.$$

Explicitly, it is given by

$$\{x,y\} = [x,y] + \partial_x y - \partial_y x. \tag{3.263}$$

3.9.6. *Goncharov coproduct.* Let us now turn to the computation of the new coproduct on the algebra $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$.

Following Notation 3.232, if α is a binary sequence, then $x_\alpha \in \mathfrak{H}$ is the corresponding word in the alphabet $\{x_0, x_1\}$, while $e_\alpha \in \mathfrak{H}$ is the corresponding word in the alphabet $\{e_0, e_1\}$. As a function $x_\alpha \in \mathfrak{H} = \mathcal{O}(\Pi)$, the word x_α sends a group-like element of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ to the coefficient of the word e_α.

Recall that, by Lemma 3.105, the dual antipode of a group-like element γ is given by $S^\vee(\gamma) = \gamma^{-1}$, while for a word $w = e_{\varepsilon_0}\dots e_{\varepsilon_n}$ the dual antipode is given in Example 3.62 by

$$S^\vee(w) = w^* = (-1)^n e_{\varepsilon_n}\dots e_{\varepsilon_1}.$$

We deduce that, if $\gamma = \sum_w \gamma_w w$ is a group-like element, then

$$\gamma^{-1} = \sum_w \gamma_w w^*. \tag{3.264}$$

The Goncharov coproduct, denoted by Δ^Γ, is the coproduct induced in $\mathfrak{H}$ by $\circ$ and is determined by the equation

$$\Delta^\Gamma(x)(\gamma\otimes\mu) = x(\gamma\circ\mu) = x(\gamma\cdot\langle\gamma\rangle_0(\mu)). \tag{3.265}$$

Note that the product $\circ$ can be defined, for a group-like element γ and an arbitrary element $e \in \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ by

$$\gamma\circ e = \gamma\cdot\langle\gamma\rangle_0(e). \tag{3.266}$$

This product is linear in the variable e. In particular, for a word w in the alphabet $\{e_0, e_1\}$, the product $\gamma\circ w$ is described as follows:

(1) if the word starts with e_0 add γ at the beginning, while if the word starts with e_1 add nothing at the beginning;
(2) if the word ends with e_1 add γ at the end, while if the word ends with e_0 add nothing at the end;
(3) between e_0 and e_1 insert γ^{-1} and between e_1 and e_0 insert γ;
(4) between two consecutive occurrences of e_0 or two consecutive occurrences of e_1 insert nothing.

For instance

$$\gamma\circ(e_0e_0e_1e_0e_1e_1) = \gamma e_0e_0\gamma^{-1}e_1\gamma e_0\gamma^{-1}e_1e_1\gamma.$$

To give a more compact description of this product we introduce the following notation

$${}_1\gamma_0 = \gamma,\ {}_0\gamma_1 = \gamma^{-1},\ {}_0\gamma_0 = 1,\ {}_1\gamma_1 = 1.$$

For a binary sequence $\alpha = (\varepsilon_1, \ldots, \varepsilon_n)$, we have

$$\gamma\circ e_\alpha = {}_1\gamma_{\varepsilon_1}\cdot e_{\varepsilon_1}\cdot {}_{\varepsilon_1}\gamma_{\varepsilon_2}\cdot e_{\varepsilon_2}\cdots e_{\varepsilon_n}\cdot {}_{\varepsilon_n}\gamma_0. \tag{3.267}$$

Given the shape (3.267) of the product $\circ$ and the inversion formula (3.264), for any binary sequence α, we introduce the following symbols

$$\begin{aligned} &I(1;\alpha;0) = x_\alpha, \qquad I(0;\alpha;1) = x^*_\alpha,\\ &I(0;\alpha;0) = I(1;\alpha;1) = 1,\ \text{if } \alpha = \emptyset,\\ &I(0;\alpha;0) = I(1;\alpha;1) = 0,\ \text{if } \alpha \neq \emptyset.\end{aligned} \tag{3.268}$$

All of them are elements of $\mathfrak{H}$, hence functions on Π. Then, for a binary sequence α, a group-like element $\gamma\in\Pi(\mathbb{Q})$ and elements $\varepsilon,\varepsilon'\in\{0,1\}$, we have the duality

$$x_\alpha({}_{\varepsilon'}\gamma_\varepsilon) = I(\varepsilon';\alpha;\varepsilon)(\gamma). \tag{3.269}$$

Armed with this notation, we can compute Goncharov's coproduct. Let α be a binary sequence and γ, μ group-like elements of $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. Write

$\mu = \sum_w \mu_w w$. Then, by equation (3.265),

$$\begin{aligned}(\Delta^\Gamma x_\alpha)(\gamma \otimes \mu) &= x_\alpha(\gamma \circ \mu)\\ &= x_\alpha(\sum_w \mu_w \gamma \circ w)\\ &= x_\alpha\big[\sum_w \mu_w ({}_1\gamma_{\varepsilon_1(w)} \cdot e_{\varepsilon_1(w)} \cdots e_{\varepsilon_{\mathrm{wt}(w)}} \cdot {}_{\varepsilon_{\mathrm{wt}(w)}(w)}\gamma_0)\big],\end{aligned}$$

where $\mathrm{wt}(w)$ denotes the weight of w as in Definition 1.124, and $\varepsilon_i(w)$ is defined to be 0 or 1 depending wether the i-th letter appearing in w is e_0 or e_1. Let us write $\alpha = \varepsilon_1 \cdots \varepsilon_n$ and set $\varepsilon_0 = 1$ and $\varepsilon_{n+1} = 0$.

We need to compute the coefficient of the word e_α in the above bracketed expression. We will get a contribution for each subword of e_α corresponding to a binary subsequence $\varepsilon_{i_1} \cdots \varepsilon_{i_k}$ of α. It is easy to see that the coefficient we are looking for is given by:

$$\sum_{\substack{0=i_0<i_1<\cdots\\ <i_k<i_{k+1}=n+1}} I(\varepsilon_0; \varepsilon_{i_1} \cdots \varepsilon_{i_k}; \varepsilon_{n+1})(\mu) \prod_{p=0}^{k} I(\varepsilon_{i_p}; \varepsilon_{i_p+1} \cdots \varepsilon_{i_{p+1}-1}; \varepsilon_{i_{p+1}})(\gamma).$$

The upshot of these computations is the following result, which was first obtained by Goncharov [**Gon05**, Thm. 1.2].

PROPOSITION 3.270. *Let $\varepsilon_0 \cdots \varepsilon_{n+1}$ be a binary sequence. The isomorphism of Lemma 3.259 induces, by transport of structure, the following coproduct on the algebra $\mathfrak{H}$:*

$$\Delta^\Gamma I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_n; \varepsilon_{n+1}) =$$
$$\sum_{\substack{0=i_0<i_1<\cdots\\ <i_k<i_{k+1}=n+1}} \prod_{p=0}^{k} I(\varepsilon_{i_p}; \varepsilon_{i_p+1} \cdots \varepsilon_{i_{p+1}-1}; \varepsilon_{i_{p+1}}) \otimes I(\varepsilon_0; \varepsilon_{i_1} \cdots \varepsilon_{i_k}; \varepsilon_{n+1}).$$

PROOF. The case $\varepsilon_0 = 1$ and $\varepsilon_{n+1} = 0$ was settled above. The other cases follow immediately from (3.268). $\square$

EXAMPLE 3.271. If $n = 1$,

$$\begin{aligned}\Delta^\Gamma I(\varepsilon_0; \varepsilon_1; \varepsilon_2)&\\ &= I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes I(\varepsilon_0; \varepsilon_2) + I(\varepsilon_0; \varepsilon_1) I(\varepsilon_1; \varepsilon_2) \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_2)\\ &= I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes 1 + 1 \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_2),\end{aligned}$$

since $I(\varepsilon'; \varepsilon)$ is always equal to 1 regardless of the values of ε and ε'.

EXAMPLE 3.272. If $n = 2$, we get contributions from $k = 0, 1, 2$. As before, $k = 0$ corresponds to the choice of the empty subsequence and gives the value $I(\varepsilon_0; \varepsilon_1\varepsilon_2; \varepsilon_3) \otimes 1$, whereas $k = 2$ represents the choice of the whole sequence and contributes with $1 \otimes I(\varepsilon_0; \varepsilon_1\varepsilon_2; \varepsilon_3)$. For $k = 1$ we obtain two terms, corresponding to $i_1 = 1$ and $i_1 = 2$. In both cases, the product contains only one non-trivial factor ($p = 1$ if $i_1 = 1$ and $p = 0$ if $i_1 = 2$). Putting everything together,

$$\begin{aligned}\Delta^{\Gamma} I(\varepsilon_0; \varepsilon_1\varepsilon_2; \varepsilon_3) = I(\varepsilon_0; \varepsilon_1\varepsilon_2; \varepsilon_3) \otimes 1 \\ + I(\varepsilon_1; \varepsilon_2; \varepsilon_3) \otimes I(\varepsilon_0; \varepsilon_1; \varepsilon_3) \\ + I(\varepsilon_0; \varepsilon_1; \varepsilon_2) \otimes I(\varepsilon_0; \varepsilon_2; \varepsilon_3) \\ + 1 \otimes I(\varepsilon_0; \varepsilon_1\varepsilon_2; \varepsilon_3).\end{aligned} \tag{3.273}$$

Specializing formula (3.273) to the cases $(1; 1, 0; 0)$ and $(1; 0, 1; 0)$ we get

$$\Delta^{\Gamma}(x_0x_1) = x_0x_1 \otimes 1 + x_0 \otimes x_1 + x_1 \otimes x_0 + 1 \otimes x_0x_1,$$
$$\Delta^{\Gamma}(x_1x_0) = x_1x_0 \otimes 1 + 1 \otimes x_1x_0.$$

Just for fun, let us verify the compatibility with shuffle product. On the one hand,

$$\begin{aligned}&\Delta^{\Gamma}(x_0 \sqcup\!\sqcup x_1) \\ &\quad = \Delta^{\Gamma}(x_0x_1 + x_1x_0) \\ &\quad = (x_0x_1 + x_1x_0) \otimes 1 + 1 \otimes (x_0x_1 + x_1x_0) + x_0 \otimes x_1 + x_1 \otimes x_0.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\Delta^{\Gamma} x_1) \sqcup\!\sqcup (\Delta^{\Gamma} x_2) \\ &\quad = (1 \otimes x_0 + x_0 \otimes 1) \sqcup\!\sqcup (1 \otimes x_1 + x_1 \otimes 1) \\ &\quad = 1 \otimes (x_0 \sqcup\!\sqcup x_1) + x_0 \otimes x_1 + x_0 \otimes x_1 + (x_0 \sqcup\!\sqcup x_1) \otimes 1,\end{aligned}$$

and we see that the expressions are equal.

As the examples show, the formula for Goncharov's coproduct in Proposition 3.270 contains many trivial factors. Later in Chapter 5 we will give a linearization which is more suitable for computation.

⋆ ⋆ ⋆

Exercise 3.274. Prove formula 3.263.

Exercise 3.275. Calculate the number of terms appearing in Goncharov's coproduct.

4. Mixed Tate motives

The goal of this chapter is to give a precise meaning to the statement that the diagram D_U^H of Definition 3.258 has motivic origin. The theory of motives has been a very active area of research in the last decades. This is a rather abstract theory and it is remarkable that, up until today, the only proof that we have for the upper bound of the dimension of the space of multiple zeta values of a given weight uses the theory of motives. A proper treatment of the theory of motives falls outside the scope of this notes. We will use the theory of motives as a black-box and we will limit ourselves to give an idea of its origin and the properties that we will use. The interested reader is referred to the book [**And04**] and the references therein.

4.1. Tannakian formalism. The link between mixed Tate motives and multiple zeta values is made through the group of symmetries of mixed Tate motives. To make this idea precise we need the formalism of Tannakian categories that we summarize in this section.

The Galois group of a field extension is one of the basic tools in arithmetic and one of its more studied objects. In topology, the fundamental group of a topological space is the analogue of the absolute Galois group of a field and is one of the basic invariants of a topological space. Fueled by the utility of the Galois and fundamental groups it is natural to seek for analogues in other situations. The Tannakian formalism is the basic tool to define analogues of the Galois group in many algebro-geometric situations. The origin of this formalism is Pontryagin duality, according to which a locally compact abelian group is characterized by its character group, and the Tannaka-Krein duality that states that we can recover a compact Lie group from the category of its continuous finite-dimensional real representations. Grothendieck extended the Tannaka-Krein duality to affine algebraic groups. Saavedra-Rivano [**SR72**] encoded the properties of the category of linear representations of an algebraic group in the concept of a Tannakian category. Conversely, every Tannakian category is isomorphic to the category of linear representations of an algebraic group.

Note that the formalism of Tannakian categories is tailored to the study of affine group schemes. Thus we will not recover the "true" fundamental group of a topological space nor the Galois group of a field extension with this formalism, but only its so called pro-algebraic envelope.

We will follow the exposition in [**DM82**] to which the reader is referred for further details. Another nice reference is Chapter 6 of [**Sza09**]. Through this section we fix a field k (of any characteristic), that will play the role of field of coefficients.

4.1.1. *Tensor categories.* The definition of Tannakian category gathers together the properties of finite-dimensional k-linear representations of affine group schemes.

First of all, since morphisms between k-linear representations form a vector space, we need the concept of a k-linear category.

Definition 4.1. A *k-linear category* $\mathcal{C}$ is an additive category such that, for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the group $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a k-vector space and the composition maps are bilinear.

The tensor product of two representations is again a representation. Therefore, a Tannakian category should have a tensor product, which is a bilinear functor with some additional properties.

Definition 4.2. Let $\mathcal{C}$ be a k-linear category, together with a bilinear functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(a) An *associativity constraint* for $(\mathcal{C}, \otimes)$ is a natural transformation

$$\phi = \phi_{\cdot,\cdot,\cdot}\colon \ \cdot \otimes (\cdot \otimes \cdot) \longrightarrow (\cdot \otimes \cdot) \otimes \cdot$$

such that the following two conditions hold:

(1) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the map $\phi_{X,Y,Z}$ is an isomorphism.

(2) (*Pentagon axiom*) For all $X, Y, Z, T \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes:

$$\begin{tikzcd}
& X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, "\mathrm{Id} \otimes \phi_{Y,Z,T}"'] \arrow[dr, "\phi_{X,Y,Z\otimes T}"] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d, "\phi_{X,Y\otimes Z,T}"'] & & (X \otimes Y) \otimes (Z \otimes T). \arrow[d, "\phi_{X\otimes Y,Z,T}"] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z} \otimes \mathrm{Id}"'] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd}$$

(b) A *commutativity constraint* is a natural transformation

$$\psi = \psi_{\cdot,*}\colon \ \cdot \otimes * \longrightarrow * \otimes \cdot$$

such that, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the map $\psi_{X,Y}$ is an isomorphism, and the following composition is the identity:

$$\psi_{Y,X} \circ \psi_{X,Y}\colon X \otimes Y \longrightarrow X \otimes Y.$$

(c) An associativity and a commutativity constrain are said to be *compatible* if, for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes (*hexagon axiom*):

$$
\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\phi_{X,Y,Z}"] \arrow[dl, "\mathrm{Id} \otimes \psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X \otimes Y,Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & & & Z \otimes (X \otimes Y). \arrow[dl, "\phi_{Z,X,Y}"] \\
& (X \otimes Z) \otimes Y \arrow[r, "\psi_{X,Z} \otimes \mathrm{Id}"'] & (Z \otimes X) \otimes Y &
\end{tikzcd}
$$

(d) Finally, we say that a pair (U, u) consisting of an object U of $\mathcal{C}$ and an isomorphism $u\colon U \to U \otimes U$ is an *identity object* if the functor $X \mapsto U \otimes X$ is an equivalence of categories.

We now have all the ingredients to define one of the underlying structures of Tannakian categories.

DEFINITION 4.3. A *k-linear tensor category* is a tuple $(\mathcal{C}, \otimes, \phi, \psi)$ consisting of a k-linear category $\mathcal{C}$, a bilinear functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and compatible associativity and commutativity constraints ϕ and ψ, such that $\mathcal{C}$ contains an identity object.

The constraints ϕ and ψ are usually omitted from the notation and one simply denotes a k-linear tensor category by $(\mathcal{C}, \otimes)$.

REMARK 4.4. Two identity objects are canonically isomorphic. From now on, we will fix one and denote it by $(\mathbf{1}, e)$.

DEFINITION 4.5. An object L in $\mathcal{C}$ is called *invertible* if the functor $X \mapsto L \otimes X$ is an equivalence of categories.

One easily shows that an object L is invertible if and only if there exists an object L' such that $L \otimes L' \cong \mathbf{1}$. Then L' is also invertible.

4.1.2. *Rigid categories.* The set of k-linear maps between two representations is again a representation and, in particular, a representation on a vector space induces a representation on the dual vector space. Thus a Tannakian category should contain internal Hom's and duals.

Let $(\mathcal{C}, \otimes)$ be a tensor category and let $X, Y \in \mathrm{Ob}(\mathcal{C})$. We say that the functor $T \mapsto \mathrm{Hom}(T \otimes X, Y)$ is *representable* if there exist an object $Z \in \mathrm{Ob}(\mathcal{C})$ such that there are functorial isomorphisms

$$\mathrm{Hom}(T, Z) \longrightarrow \mathrm{Hom}(T \otimes X, Y) \tag{4.6}$$

for all $T \in \mathrm{Ob}(\mathcal{C})$. If this is the case, we denote Z by $\underline{\mathrm{Hom}}(X, Y)$ and we call it *internal Hom* between the objects X and Y. Note that any two such objects Z are related by a unique compatible isomorphism.

Taking $T = \underline{\mathrm{Hom}}(X, Y)$ in (4.6), the image of the identity $\mathrm{Id}_{\underline{\mathrm{Hom}}(X,Y)}$ is a morphism which will be denoted by

$$\mathrm{ev}_{X,Y}\colon \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y.$$

The *dual* of an object X is defined as $X^\vee = \underline{\mathrm{Hom}}(X, \mathbf{1})$. If $X^\vee$ and $(X^\vee)^\vee$ exist, there is a natural morphism $X \mapsto (X^\vee)^\vee$. We say that X is *reflexive* if this morphism is an isomorphism.

Example 4.7. In the category of groups, $\mathbb{Z}/2$ is not reflexive since its dual is 0. In the category of vector spaces, finite dimensional vector spaces are reflexive, whereas infinite dimensional ones are not.

Definition 4.8. A k-linear tensor category is said to be *rigid* if

(1) $\underline{\mathrm{Hom}}(X, Y)$ exists for all $X, Y \in \mathrm{Ob}(\mathcal{C})$;
(2) for all $X_1, X_2, Y_1, Y_2 \in \mathrm{Ob}(\mathcal{C})$, the natural morphism
$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \to \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$
is an isomorphism;
(3) all objects of $\mathcal{C}$ are reflexive.

4.1.3. *Neutral Tannakian categories.* The category of finite-dimensional k-linear representations $\mathbf{Rep}_k(G)$ of an algebraic group G over k has other relevant properties. First, it is an abelian category. Second, the one-dimensional representation given by the vector space k with trivial G-action is an identity object $\mathbf{1}$ that satisfies $\mathrm{End}(\mathbf{1}) = k$. Finally, the forgetful functor from $\mathbf{Rep}_k(G)$ to the category of finite-dimensional vector spaces $\mathbf{Vec}_k$ that consists in forgetting the action of G is exact, faithful and compatible with the tensor structure on both categories. These will turn out to be all the necessary ingredients to identify the categories of finite-dimensional representations of algebraic groups.

Definition 4.9. A *neutral Tannakian category* over k is a rigid k-linear abelian tensor category $\mathcal{C}$ such that $\mathrm{End}(\mathbf{1}) = k$ and that there exists an exact faithful k-linear tensor functor $\omega\colon \mathcal{C} \to \mathbf{Vec}_k$. Any such functor is called a *fibre functor*.

Since we shall never consider non-neutral Tannakian categories in the sequel, we will just refer to them as "Tannakian categories".

Examples 4.10.

(1) The category $\mathbf{Vec}_k$ of finite-dimensional vector spaces over k, together with the identity functor, is a Tannakian category.
(2) Let $\mathbf{GrVec}_k$ be the category of finite-dimensional graded vector spaces over k. The objects are finite-dimensional k-vector spaces V together with a direct sum decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$, and the

morphisms are homogeneous k-linear maps. The tensor structure comes from the tensor product of vector spaces, graded by

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.$$

The forgetful functor $\omega\colon \mathbf{GrVec}_k \to \mathbf{Vec}_k$ sending $(V, (V_n)_{n\in\mathbb{Z}})$ to V makes $\mathbf{GrVec}_k$ into a Tannakian category.

(3) Let G be any abstract group and $\mathbf{Rep}_k(G)$ the category of finite-dimensional k-linear representations of G. Let

$$\omega\colon\ \mathbf{Rep}_k(G) \to \mathbf{Vec}_k$$

be the functor that forgets the action of G. Then $\mathbf{Rep}_k(G)$ is a Tannakian category over k and ω is a fibre functor.

(4) Let $\mathbf{MHS}(\mathbb{Q})$ be the category of mixed Hodge structures over $\mathbb{Q}$ and let ω_{B} and ω_{dR} the forgetful functors of Definition 2.101. Then $\mathbf{MHS}(\mathbb{Q})$ is a Tannakian category over $\mathbb{Q}$ and both of the functors, ω_{B} and ω_{dR} are fibre functors.

(5) Let X be a path connected, locally path connected and locally simply connected topological space. The category $\mathbf{Loc}_k(X)$ of local systems of finite-dimensional k-vector spaces is a Tannakian category. For each point $x \in X$, the functor

$$\begin{array}{rccc} \omega_x\colon & \mathbf{Loc}_k(X) & \longrightarrow & \mathbf{Vec}_k \\ & V & \longmapsto & V_x \end{array}$$

that sends a local system V to its fibre at x is a fibre functor.

4.1.4. *The fundamental group of a Tannakian category.* Fix a Tannakian category $\mathcal{C}$ over k and a fibre functor ω.

Definition 4.11. For every k-algebra R, let $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ denote the set of families $(\lambda_X)_{X\in\mathrm{Ob}(\mathcal{C})}$ of R-linear automorphisms

$$\lambda_X : \omega(X) \otimes R \longrightarrow \omega(X) \otimes R$$

such that the following diagrams are commutative:

(1)

$$\begin{tikzcd}
\omega(X_1 \otimes X_2) \otimes R \arrow[r, "\lambda_{X_1\otimes X_2}"] \arrow[d] & \omega(X_1 \otimes X_2) \otimes R \arrow[d] \\
\omega(X_1) \otimes \omega(X_2) \otimes R \arrow[d] & \omega(X_1) \otimes \omega(X_2) \otimes R \arrow[d] \\
(\omega(X_1) \otimes R) \otimes_R (\omega(X_2) \otimes R) \arrow[r, "\lambda_{X_1}\otimes_R \lambda_{X_2}"] & (\omega(X_1) \otimes R) \otimes_R (\omega(X_2) \otimes R),
\end{tikzcd}$$

(2)

$$\begin{tikzcd} \omega(\mathbf{1}) \otimes R \arrow[r, "\lambda_{\mathbf{1}}"] \arrow[d] & \omega(\mathbf{1}) \otimes R \arrow[d] \\ R \arrow[r, "\mathrm{Id}"] & R, \end{tikzcd}$$

(3) for every morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

$$\begin{tikzcd} \omega(X) \otimes R \arrow[r, "\lambda_X"] \arrow[d, "\omega(\alpha)\otimes\mathrm{Id}"'] & \omega(X) \otimes R \arrow[d, "\omega(\alpha)\otimes\mathrm{Id}"] \\ \omega(Y) \otimes R \arrow[r, "\lambda_Y"] & \omega(Y) \otimes R. \end{tikzcd}$$

In the above diagrams, all unlabeled tensor products of vector spaces are over k and the unlabeled arrows are the obvious isomorphisms.

In particular, we define $\mathrm{Aut}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)(k)$. This is the group of k-linear automorphisms of the functor ω.

The main theorem of the theory of Tannakian categories is

Theorem 4.12. [**DM82**, Theorem 2.11] *Let $\mathcal{C}$ be a Tannakian category over k, together with a fibre functor ω. Then*

(1) the functor $R \mapsto \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is representable by an affine group scheme over k that we denote $\underline{\mathrm{Aut}}^{\otimes}(\omega)$;

(2) for every $X \in \mathrm{Ob}(\mathcal{C})$, the group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ acts naturally on $\omega(X)$ and the functor $\mathcal{C} \to \mathbf{Rep}_k(\underline{\mathrm{Aut}}^{\otimes}(\omega))$ sending X to the vector space $\omega(X)$ with this action of $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an equivalence of categories.

Definition 4.13. The affine group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is called the *Tannaka group* of $(\mathcal{C}, \omega)$. Whenever we want to stress the category we are considering, we will write $\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$.

Given a second fibre functor ω', the functor from k-algebras to sets

$$R \longmapsto \underline{\mathrm{Isom}}^{\otimes}(\omega, \omega')$$

is representable by an affine scheme which is a right torsor under $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ and a left torsor under $\underline{\mathrm{Aut}}^{\otimes}(\omega')$, see [**DM82**, Theorem 3.2].

4.1.5. *Matrix coefficients.* Instead of proving Theorem 4.12, we will content ourselves with a description of the Hopf algebra of the Tannaka group using the notion of matrix coefficients from[8] [**Del90**, §4.7].

Definition 4.14. Let $(\mathcal{C}, \omega)$ be a neutral Tannakian category over k, together with a fibre functor. A *matrix coefficient* in $(\mathcal{C}, \omega)$ is the data

$$(X, v, f)$$

of an object X of $\mathcal{C}$, and elements $v \in \omega(X)$ and $f \in \omega(X)^{\vee} = \mathrm{Hom}(\omega(X), k)$.

[8]See also [**Bro17**] and compare with the notion of framed objects from [**BGSV90**].

Let H be the k-vector space generated by all matrix coefficients, and $V \subseteq H$ the subspace spanned by

(1) (*bilinearity relations*) for every pair of matrix coefficients (X, v_1, f) and (X, v_2, f), and elements $\lambda, \mu \in k$, the relation

$$(X, \lambda v_1 + \mu v_2, f) - \lambda(X, v_1, f) - \mu(X, v_2, f) \in V.$$

Similarly, for every pair of matrix coefficients of the form (X, v, f_1) and (X, v, f_2), and elements $\lambda, \mu \in k$, the relation

$$(X, v, \lambda f_1 + \mu f_2) - \lambda(X, v, f_1) - \mu(X, v, f_2) \in V;$$

(2) (*compatibility relations*) for every pair of objects X, X', every morphism $\phi \in \operatorname{Hom}_{\mathcal{C}}(X, X')$, and $v \in \omega(X)$ and elements $f' \in \omega(X')^\vee$, the relation

$$(X, v, \omega(\phi)^\vee f') - (X', \omega(\phi)v, f') \in V.$$

We set $A = H/V$ and write $[X, v, f]$ for the class in A of a matrix coefficient (X, v, f). The vector space A comes with the following structures:

(1) *Product:* The tensor structure of $\mathcal{C}$ induces the product

$$[X, v, f] \cdot [X', v', f'] = [X \otimes X', v \otimes v', f \otimes f'].$$

The associativity and commutativity constraints together with the compatibility relation imply that this product is associative and commutative.

(2) *Unit:* Let $\mathbf{1}$ be an identity object. Then $\omega(\mathbf{1}) \simeq k$. Choose any $v \in \omega(\mathbf{1}) \setminus \{0\}$ and let $f \in \omega(\mathbf{1})^\vee$ be its dual, so that $f(v) = 1$. Then $[\mathbf{1}, v, f]$ is a unit for the product. By the bilinearity relations, this class does not depend on the choice of v.

(3) *Counit:* The counit is the map $A \to k$ given by $[X, v, f] \mapsto f(v)$.

(4) *Coproduct:* The coproduct is modeled on the Hopf algebra of GL_n (see Example 3.52). Given an object $X \in \mathcal{C}$, we choose a basis $(e_1, \ldots, e_n)$ of $\omega(X)$. If $(e_1^*, \ldots, e_n^*)$ is the dual basis, then

$$\Delta[X, v, f] = \sum_{j=1}^{n} [X, v, e_j^*] \otimes [X, e_j, f]. \tag{4.15}$$

One checks that (4.15) does not depend on the choice of the basis.

(5) *Antipode:* Finally, the rigidity of $\mathcal{C}$ allows us to define an antipode. If we identify $\omega(X^\vee)$ with $\omega(X)^\vee$, then

$$S([X, v, f]) = [X^\vee, f, v].$$

It is an easy verification to prove the following:

PROPOSITION 4.16. *Together with the above structures, A is a commutative Hopf k-algebra.*

Taking Theorem 4.12 for granted, we can show that A is the Hopf algebra of the Tannaka group $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. More precisely,

PROPOSITION 4.17. *The map $\varphi\colon A \to \mathcal{O}(G)$ given by*

$$\varphi([X, v, f])(\lambda) = f(\lambda_X(v))$$

is an isomorphism of Hopf algebras.

PROOF. We leave it to the reader to check that φ is a morphism of Hopf algebras. By Theorem 4.12, $\mathcal{C}$ is equivalent to the category $\mathbf{Rep}_k(G)$ of finite-dimensional k-representations of G, and we can identify ω with the forgetful functor $\mathbf{Rep}_k(G) \to \mathbf{Vec}_k$.

We first prove that φ is surjective. Note that there is a left group action of G on $\mathcal{O}(G)$ given by

$$(\lambda h)(\mu) = h(\mu\lambda).$$

By Lemma 3.87, $\mathcal{O}(G)$ is the union of its finite-dimensional subrepresentations. In other words, given $h \in \mathcal{O}(G)$, there exists a finite-dimensional subrepresentation (V, ρ) of $\mathcal{O}(G)$ containing h. It determines an object X of $\mathcal{C}$ such that h belongs to $\omega(X) = V$. Let $f \in V^\vee$ be the element given by $f(u) = u(e)$, where e is the unit of G and $u \in V \subseteq \mathcal{O}(G)$. Then, for each element $\lambda \in G$, we have

$$[X, h, f](\lambda) = f(\lambda h) = (\lambda h)(e) = h(e\lambda) = h(\lambda).$$

Therefore, $\varphi([X, h, f]) = h$ and φ is surjective.

We next prove the injectivity. Assume that $\varphi([X, v, f]) = 0$. We identify X with a finite-dimensional representation (V, ρ) of G such that $v \in V$. Let V' be the simple subrepresentation of V containing v. Then V' is generated by elements of the form λv for $\lambda \in G$. Since $\varphi([X, v, f]) = 0$, we deduce that $f|_{V'} = 0$. Let X' be the object of $\mathcal{C}$ corresponding to (V', ρ). By the compatibility relation

$$[X, v, f] = [X', v, f|_{V'}] = [X', v, 0] = 0.$$

This concludes the proof. $\square$

EXAMPLE 4.18. Let $(\mathbf{GrVec}_k, \omega)$ be the Tannakian category of finite-dimensional graded vector spaces from Example 4.10. It is equivalent to the semisimple category generated by objects k_n, $n \in \mathbb{Z}$ with

$$\mathrm{Hom}(k_n, k_m) = \begin{cases} k, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases} \qquad k_n \otimes k_m = k_{n+m}, \qquad \omega(k_n) \simeq k.$$

For each n choose a non-zero element $u_n \in \omega(k_n)$ and let $u_n^\vee \in \omega(k_n)^\vee$ be the element defined by $u_n^\vee(u_n) = 1$. Then every matrix coefficient in $\mathcal{T}$ can be written as a linear combination of the elements

$$[k_n, u_n, u_n^\vee], \quad n \in \mathbb{Z}.$$

Moreover,

$$[k_n, u_n, u_n^\vee] \cdot [k_m, u_m, u_m^\vee] = [k_{n+m}, u_{n+m}, u_{n+m}^\vee].$$

Thus, if we write $t = [k_1, u_1, u_1^\vee]$, there is an isomorphism of algebras

$$\mathcal{O}(\underline{\mathrm{Aut}}^{\otimes}(\omega)) = k[t, t^{-1}].$$

Moreover, the coproduct, the counit and the antipode are given by

$$\Delta t = t \otimes t, \quad \epsilon(t) = \epsilon(t^{-1}) = 1, \quad S(t) = t^{-1}.$$

From part (2) of Example 3.52, we deduce that $\underline{\mathrm{Aut}}^{\otimes}(\omega) = \mathbb{G}_m$, the multiplicative group. It is a general fact that the presence of a grading is related to an action of $\mathbb{G}_m$.

EXAMPLE 4.19. Consider the subgroup of $\mathrm{GL}_2(\mathbb{R})$ given by:

$$\{\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R}) \mid x^2 + y^2 > 0\}.$$

These are the real points of an affine algebraic group $\mathbb{S}$ over $\mathbb{R}$ called the *Deligne torus*. Alternatively, one can define it as

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m),$$

where $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ is the Weil restriction functor. This means that, if A is an $\mathbb{R}$-algebra, then $\mathbb{S}(A) = \mathbb{G}_m(A \otimes_{\mathbb{R}} \mathbb{C}) = (A \otimes_{\mathbb{R}} \mathbb{C})^\times$. The category of representations of $\mathbb{S}$ is equivalent to the category of split $\mathbb{R}$-mixed Hodge structures.

4.1.6. *Tannakian subcategories.* Let Y be an objects of a neutral Tannakian category $\mathcal{C}$, we denote by $\langle Y \rangle$ the full subcategory of $\mathcal{C}$ that contains Y and is stable by sums, tensor products, dual and subquotients. Then $\langle Y \rangle$, together with the restriction of any fibre functor ω on $\mathcal{C}$ is again a neutral Tannakian category. The action of $G = \underline{\mathrm{Aut}}_{\mathcal{C}}^{\otimes}(\omega)$ on the vector space $\omega(Y)$ induces a map $G \to \mathrm{GL}(\omega(Y))$. The following is shown in the proof of [**DM82**, Proposition 2.8]

LEMMA 4.20. *The image $G^Y \subset \mathrm{GL}(\omega(Y))$ of G by the above map is a closed subgroup of $\mathrm{GL}(\omega(Y))$ which agrees with the Tannaka group $\underline{\mathrm{Aut}}_{\langle Y \rangle}^{\otimes}(\omega)$ of the subcategory $\langle Y \rangle$.*

We can order the subcategories of the form $\langle Y \rangle$ for Y an object of $\mathcal{C}$ by inclusion. With this order they form a directed system. The following lemma exhibits the pro-algebraic nature of G.

LEMMA 4.21. *Let $(\mathcal{C}, \omega)$ be a neutral Tannakian category. Then:*

$$\underline{\mathrm{Aut}}_{\mathcal{C}}^{\otimes}(\omega) = \varprojlim_{\langle Y \rangle} \underline{\mathrm{Aut}}_{\langle Y \rangle}^{\otimes}(\omega) = \varprojlim_{\langle Y \rangle} G^Y.$$

PROOF. By Lemma 4.20, there is a surjection $G \to G^Y$ for every object Y of $\mathcal{C}$. These surjections are compatible with the maps $G^Z \to G^Y$ induced by an inclusion $\langle Y \rangle \subset \langle Z \rangle$. Therefore, there is a surjection

$$G \longrightarrow \varprojlim_{\langle Y \rangle} G^Y.$$

This map is also injective, because if an element of G is sent to the unit, then it acts trivially on $\omega(Y)$ for every object Y and is thus the unit of G. $\square$

4.1.7. *Tannakian categories and the fundamental group.* We next explore what can be recovered from the classical fundamental group of a topological space using the Tannakian formalism. This includes the pro-unipotent completion.

Let X be a path connected, locally path connected and locally simply connected topological space. Let x_0 be a point of X and $\pi_1(X, x_0)$ the fundamental group of X with base point x_0. By part (5) of Example 4.10, the category $\mathbf{Loc}_k(X)$ of local systems of finite dimensional k-vector spaces over X is a Tannakian category with fibre functor ω_{x_0}. Given a local system V, the fibre at x_0 is a k-vector space with an action of $\pi_1(X, x_0)$. This yields the so-called *monodromy representation*

$$\rho_V \colon \pi_1(X, x_0) \to \mathrm{GL}(\omega_{x_0}(V)).$$

It follows that $\mathbf{Loc}_k(X)$ is equivalent to the category of finite-dimensional k-linear representations of $\pi_1(X, x_0)$. However, since the fundamental group is not an affine group scheme, it cannot be the Tannaka group of the category $\mathbf{Loc}_k(X)$. In fact, as we will see, the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}(\omega_{x_0})$ is the *pro-algebraic completion* of $\pi_1(X, x_0)$.

Following Lemma 4.21, we can give the following description of the pro-algebraic completion of Γ. Let $Y = (V, \rho)$ be a k-linear finite-dimensional representation of Γ. The group G^Y from Lemma 4.21 is the Zariski closure $\overline{\rho(\Gamma)}^{\mathrm{Zar}}$ of the image of $\rho \colon \Gamma \to \mathrm{GL}(V)$. Let $Y' = (V', \rho')$ be another representation with $\langle Y' \rangle \subset \langle Y \rangle$. By Lemma 4.20, there is a restriction map $\overline{\rho(\Gamma)}^{\mathrm{Zar}} \to \overline{\rho'(\Gamma)}^{\mathrm{Zar}}$. The pro-algebraic completion is the projective limit

$$\Gamma^{\mathrm{alg}} = \varprojlim_{\langle (V,\rho) \rangle} \overline{\rho(\Gamma)}^{\mathrm{Zar}},$$

where the limit is taken with respect to the subcategories $\langle (V, \rho) \rangle$ ordered by inclusion.

Similarly, we can recover the pro-unipotent completion of Γ using the Tannakian formalism. A local system is called *unipotent* if its monodromy representation is unipotent (Definition 3.92). The category of unipotent local systems $\mathbf{ULoc}_k(X)$ on X is again a Tannakian category and ω_{x_0} is again a fibre functor. In this case, the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}(\omega_{x_0})$ is the pro-unipotent completion of Γ. It admits a similar description as the pro-algebraic completion but restricting to finite-dimensional unipotent representations:

$$\Gamma^{\mathrm{un}} = \varprojlim_{\langle (V,\rho)\rangle \text{ unip.}} \overline{\rho(\Gamma)}^{\mathrm{Zar}},$$

where the limit again is taken with respect to the subcategories $\langle (V, \rho)\rangle$ ordered by inclusion.

⋆ ⋆ ⋆

Exercise 4.22. Prove that

$$[X \oplus Y, u \oplus v, f \oplus g] = [X, u, f] + [Y, v, g].$$

Exercise 4.23. Consider the unit circle S^1 as a topological space. Its fundamental group is $\pi_1(S^1, 1) \simeq \mathbb{Z}$. Prove that the pro-algebraic completion $\mathbb{Z}^{\mathrm{alg}}$ is infinite-dimensional, while

$$\mathbb{Z}^{\mathrm{un}} \simeq \mathbb{G}_a,$$

the additive group. For the second part use that to give a unipotent representation of $\mathbb{Z}$ is equivalent to give a finite-dimensional vector space V together with a unipotent endomorphism of V and the fibre functor is just the forgetful functor. Then use the explicit description of the Hopf algebra of the Tannaka group.

Exercise 4.24. Consider the Tannakian category $\mathbf{Vec}_k$ with the identity as the fibre functor ω. Prove that $\underline{\mathrm{Aut}}^{\otimes}(\omega) = \mathrm{Spec}(k)$, the trivial group.

Exercise 4.25 (The pro-algebraic completion of a group). Let k be a field and Γ an abstract group. In this exercise, we present three equivalent constructions of the *pro-algebraic completion* of Γ, which is an affine group scheme $G = \Gamma^{\mathrm{alg}}$ over k together with a group morphism $\Gamma \to G(k)$.

(a) Let $\mathcal{C}$ be the category of finite-dimensional k-linear representations of Γ. Equipped with the forgetful functor, it is a Tannakian category, and one defines G as its fundamental group. A k-point of G is thus a collection $(\lambda_V)_{V\in\mathrm{Ob}(\mathcal{C})}$ of automorphisms $\lambda_V \colon V \to V$ satisfying the constraints of Definition 4.11. To each element $\gamma \in \Gamma$ one associates the collection of automorphisms $\lambda^\gamma = (\lambda_V^\gamma)_V$ defined as $\lambda_V^\gamma(v) = \lambda \cdot \gamma$. This yields the map $\Gamma \to G(k)$.

(b) Consider the collection of pairs (H, φ_H) consisting of an affine group scheme H over k and a group morphism $\varphi_H \colon \Gamma \to H(k)$ with Zariski dense image. We define a partial order by setting $(H, \varphi_H) \leq (H', \varphi_{H'})$ whenever there exists a morphism $f \colon H \to H'$ such that the induced map of k-points commutes with φ_H and $\varphi_{H'}$ and we define the pro-algebraic completion G as the limit:

$$G = \varprojlim H.$$

(c) The pro-algebraic completion G is an affine group scheme over k with a group morphism $\varphi\colon \Gamma \to G(k)$ such that, for any affine group scheme H over k and any group morphism $\varphi_H\colon \Gamma \to H(k)$, there exists a unique morphism $f\colon G \to H$ such that $f \circ \varphi = \varphi_H$.

Prove that the three constructions give the same pro-algebraic group.

4.2. Triangulated categories and t-structures.

4.2.1. *Triangulated categories.*

Definition 4.26 (Verdier). A *triangulated category* $\mathcal{T}$ is an additive category, together with the following extra data:

a) a self-equivalence of categories

$$\begin{aligned}[1]\colon \mathcal{T} &\longrightarrow \mathcal{T}\\ X &\longmapsto X[1].\end{aligned}$$

Once the self equivalence [1] is given, we shall call *triangles* all sequences of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r, "w"] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]. \end{tikzcd}$$

We will use the convention that an arrow decorated with [1] like $A \xrightarrow{[1]} B$ means a map $A \to B[1]$.

b) A class of triangles called *distinguished triangles*.

These data are required to satisfy the following axioms:

(T1) a) For any $X \in \mathrm{Ob}(\mathcal{T})$, the triangle

$$X \xrightarrow{\mathrm{Id}} X \to 0 \to X[1]$$

is distinguished.

b) Any triangle isomorphic to a distinguished one is distinguished.

c) Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

(T2) The triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.

(T3) Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1], \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1],$$

and morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ such that $g \circ u = u' \circ f$, there exists $h\colon Z \to Z'$ (not necessarily unique) such that

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1].
\end{tikzcd}$$

is a morphism of triangles.

(T4) Given a diagram of solid arrows

$$\begin{tikzcd}
 & & Y' \arrow[dddll, "n", "{[1]}"'] \arrow[drr, dashed, "g"] & & \\
Z' \arrow[urr, dashed, "f"] \arrow[d, "k", "{[1]}"'] & & & & X' \arrow[llll, "j{[1]}\circ i"', "{[1]}"] \arrow[ddll, "i", "{[1]}"'] \\
X \arrow[rrrr, "v\circ u"] \arrow[drr, "u"'] & & & & Z \arrow[u, "\ell"'] \arrow[uuull, "m"] \\
 & & Y, \arrow[urr, "v"'] \arrow[uull, "j"] & &
\end{tikzcd}$$

if the three triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} X[1]$$
$$Y \xrightarrow{v} Z \xrightarrow{\ell} X' \xrightarrow{i} Y[1]$$
$$X \xrightarrow{v\circ u} Z \xrightarrow{m} Y \xrightarrow{n} X[1]$$

are distinguished, then there exist dashed arrows f and g as in the diagram such that the triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{j[1]\circ i} Z'[1]$$

is distinguished and the following commutation relations hold:

$$k = n \circ f, \qquad \ell = g \circ m,$$
$$m \circ v = f \circ j, \qquad u[1] \circ n = i \circ g.$$

EXAMPLE 4.27. The main example of a triangulated category is the (bounded) derived category of an abelian category. We quickly recall the main structures associated to the definition of derived categories. Let $\mathcal{A}$ be an abelian category. The category of bounded cochain complexes $\mathcal{C}^b(\mathcal{A})$ consists of sequences of maps

$$\cdots \to A^k \xrightarrow{d^k} A^{k+1} \xrightarrow{d^{k+1}} A^{k+2} \xrightarrow{d^{k+2}} A^{k+3} \to \cdots$$

with $d^{k+1} \circ d^k = 0$ and such that $A^n = 0$ for large enough $|n|$. Morphisms of complexes are commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & A^k \arrow[r, "d^k"] \arrow[d, "f^k"] & A^{k+1} \arrow[r, "d^{k+1}"] \arrow[d, "f^{k+1}"] & A^{k+2} \arrow[r] \arrow[d, "f^{k+2}"] & \cdots \\
\cdots \arrow[r] & B^k \arrow[r, "d^k"] & B^{k+1} \arrow[r, "d^{k+1}"] & B^{k+2} \arrow[r] & \cdots
\end{tikzcd}$$

Complexes are denoted by A^* or, if we want to emphasize the differential, by (A^*, d). Given a complex A^*, its *cohomology groups* are

$$H^n(A^*) = \frac{\operatorname{Ker}(d^n \colon A^n \to A^{n+1})}{\operatorname{Im}(d^{n-1} \colon A^{n-1} \to A^n)}.$$

Given a complex (A^*, d), its *shift* $(A[1]^*, d[1])$ is defined as

$$A[1]^n = A^{n+1}, \text{ with } d[1] = -d.$$

A morphism of complexes $f \colon A^* \to B^*$ induces a morphism of cohomology groups

$$H(f) \colon H^*(A^*) \to H^*(B^*).$$

A morphism f is called a *quasi-isomorphism* whenever $H(f)$ is an isomorphism.

Another important construction in the category $\mathcal{C}^b(\mathcal{A})$ is the *cone* of a morphism of complexes. Let f be a morphism of complexes as before, then the cone of f is the complex

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n, \text{ with } d(a,b) = (-da, db - f(a)).$$

The cone is provided with two morphisms of complexes

$$\begin{aligned} B &\to \operatorname{cone}(f), & b &\mapsto (0, -b) \\ \operatorname{cone}(f) &\to A[1], & (a,b) &\mapsto a, \end{aligned}$$

that induce a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n(A^*) \xrightarrow{H(f)} H^n(B^*) \to H^n(\operatorname{cone}(f)) \to H^{n+1}(A^*) \longrightarrow \cdots$$

Given two morphisms of complexes $f, g \colon A^* \to B^*$, a *homotopy* between them is a collection of maps $s^n \colon A^n \to B^{n-1}$ such that

$$f^n - g^n = d^{n-1} \circ s^n + s^{n+1} \circ d^n.$$

If such a homotopy exists, we say that f and g are homotopicaly equivalent. Two homotopicaly equivalent morphisms induce the same morphism in cohomology groups.

The construction of the derived category is done in two steps. One first defines the homotopy category $\mathcal{K}^b(\mathcal{A})$ whose objects are the same as those of $\mathcal{C}^b(\mathcal{A})$ but whose morphisms are equivalence classes with respect to the homotopy equivalence of morphisms of $\mathcal{C}^b(\mathcal{A})$. In the second step, one constructs $\mathcal{D}^b(\mathcal{A})$ by inverting the quasi-isomorphisms. That is, the objects of $\mathcal{D}^b(\mathcal{A})$ are the same as the objects of $\mathcal{K}^b(\mathcal{A})$ (which are the same as the ones of $\mathcal{C}^b(\mathcal{A})$), while the morphism on $\mathcal{D}^b(\mathcal{A})$ between two objects A^* and B^* are equivalence classes of diagrams of the form

$$A^* \xleftarrow{\simeq} C^* \to B^*$$

where the arrow to the left is a quasi-isomorphism. The diagrams

$$A^* \xleftarrow{\simeq} C_1^* \to B^* \text{ and } A^* \xleftarrow{\simeq} C_2^* \to B^*$$

are equivalent if there is a third diagram of the same type such that

$$\begin{tikzcd} & C_1^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & C_3^* \arrow[l, "\simeq"'] \arrow[r] \arrow[u] \arrow[d] & B^* \\ & C_2^* \arrow[ul, "\simeq"] \arrow[ur] & \end{tikzcd} \tag{4.28}$$

commutes in $\mathcal{K}^b(\mathcal{A})$. This means that all the triangles in the diagram (4.28) are commutative up to homotopy but they are not necessarily commutative. To have such a simple description of the morphisms is the main reason to define the derived category in two steps. One can invert directly the quasi-isomorphisms in $\mathcal{C}^b(\mathcal{A})$, but then morphisms will be chains of the form

$$A^* \xleftarrow{\simeq} C_1^* \to C_2^* \dots C_{k-1}^* \xleftarrow{\simeq} C_k^* \to B^*$$

where all the arrows in the left direction are quasi-isomorphisms.

The category $\mathcal{D}^b(\mathcal{A})$ is a triangulated category, where the self equivalence [1] is defined by the shift, while the class of distinguished triangles are those triangles that are isomorphic (in $\mathcal{D}^b(\mathcal{A})$) to one of the form

$$A^* \xrightarrow{f} B^* \to \operatorname{cone}(f) \to A[1]^*.$$

4.2.2. *t-structures.* There are many natural situations where one is able to construct a triangulated category but would like to obtain an abelian category instead. In their work on perverse sheaves [**BBD82**], Beilinson, Bernstein and Deligne introduced the notion of *t-structure* as a way of extracting an abelian category from a triangulated category. This is how mixed Tate motives over a number field will be constructed in the sections to follow.

DEFINITION 4.29 (Beilinson-Bernstein-Deligne). Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is a pair of strictly full[9] subcategories

$$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$$

such that, defining for each integer n

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n], \quad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n],$$

the following three conditions are satisfied:

(1) One has $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$.
(2) (*Orthogonality*) If $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$, then $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$.
(3) Each object X of $\mathcal{T}$ fits into a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1] \tag{4.30}$$

with $Y \in \mathcal{T}^{\leq 0}$ and $Z \in \mathcal{T}^{\geq 1}$.

We say that the t-structure is *non-degenerate* if, moreover, the intersections $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\leq n}$ and $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\geq n}$ are reduced to zero.

DEFINITION 4.31. The *heart* of a t-structure on $\mathcal{T}$ is the full subcategory

$$\mathcal{T}^0 = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}.$$

A functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ between triangulated categories equipped with t-structures is said to be *t-exact* whenever $F(\mathcal{T}_1^{\leq 0}) \subseteq \mathcal{T}_2^{\leq 0}$ and $F(\mathcal{T}_1^{\geq 0}) \subseteq \mathcal{T}_2^{\geq 0}$. It restricts thus to a functor between the hearts.

Note that the objects Y and Z in the triangle (4.30) are not a priori required to be unique. However, this follows from the other axioms:

LEMMA 4.32.
(1) The inclusion of $\mathcal{T}^{\leq n}$ into $\mathcal{T}$ admits a right adjoint $t_{\leq n}\colon \mathcal{T} \to \mathcal{T}^{\leq n}$ and the inclusion $\mathcal{T}^{\geq n}$ into $\mathcal{T}$ admits a left adjoint $t_{\geq n}\colon \mathcal{T} \to \mathcal{T}^{\geq n}$.
(2) For each object X in $\mathcal{T}$, there exists a unique morphism

$$w \in \mathrm{Hom}_{\mathcal{T}}(t_{\geq 1}X, t_{\leq 0}X[1])$$

such that the following is a distinguished triangle:

$$t_{\leq 0}X \longrightarrow X \longrightarrow t_{\geq 1}X \xrightarrow{w} t_{\leq 0}X[1].$$

Up to unique isomorphism, this triangle is the only one satisfying the condition (3) in Definition 4.29.

Moreover, if $a \leq b$, there is a unique isomorphism

$$t_{\geq a}t_{\leq b}X \xrightarrow{\sim} t_{\leq b}t_{\geq a}X. \tag{4.33}$$

The standard example of t-structure is the following:

---

[9]By this we wean full and closed under isomorphism.

Example 4.34. Let $\mathcal{A}$ be an abelian category. Recall from Example 4.27 that the bounded derived category $D^b(\mathcal{A})$ is a triangulated category. It comes together with a canonical t-structure which measures how far a complex is from having cohomology concentrated in degree zero. Precisely, for each integer n, one considers the full subcategories

$$\mathcal{T}^{\leq n} = \{C^\bullet \in D^b(\mathcal{A}) \mid H^m(C^\bullet) = 0 \text{ for all } m > n\},$$
$$\mathcal{T}^{\geq n} = \{C^\bullet \in D^b(\mathcal{A}) \mid H^m(C^\bullet) = 0 \text{ for all } m < n\}.$$

It is easy to check that the pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\leq 0})$ satisfies the axioms (1)-(3) from Definition 4.29. Moreover, this t-structure is non-degenerate.

The functors $t_{\leq n}$ and $t_{\geq n}$ are given by the *canonical truncations*

$$(t_{\leq n}C)^m = \begin{cases} C^m, & \text{if } m < n, \\ \operatorname{Ker} d, & \text{if } m = n \\ 0, & \text{if } m > n. \end{cases} \quad (t_{\geq n}C)^m = \begin{cases} C^m, & \text{if } m > n, \\ C^n/dC^{n-1}, & \text{if } m = n \\ 0, & \text{if } m < n. \end{cases}$$

It follows that

$$t_{\leq 0}t_{\geq 0}C[n] = H^n(C^\bullet).$$

Viewing an object of $\mathcal{A}$ as a complex concentrated in degree zero, one gets an equivalence between $\mathcal{A}$ and the heart of $D^b(\mathcal{A})$.

For more general triangulated categories, the following theorem makes it possible to extract an abelian category [**BBD82**, Thm 1.3.6]. Recall that an abelian subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is said to be *admissible* whenever short exact sequences in $\mathcal{A}$ are exactly those sequences

$$0 \longrightarrow B \xrightarrow{u} C \xrightarrow{v} A \to 0$$

such that there exists a distinguished triangle

$$B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B[1]. \tag{4.35}$$

Remark 4.36. The extension to a distinguished triangle is not unique, unless $\mathcal{A}$ is a full subcategory, that is, $\operatorname{Hom}_{\mathcal{A}}(X,Y) = \operatorname{Hom}_{\mathcal{T}}(X,Y)$ for all objects $X, Y \in \mathcal{A}$. Indeed, it follows from axiom (T3) in the definition of triangulated categories that, given two extensions as in (4.35), the identity maps $B \to B$ and $C \to C$ can be completed to a morphism of triangles

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, equal] & C \arrow[r, "v"] \arrow[d, equal] & A \arrow[r, "w"] \arrow[d, "h"'] & B[1] \arrow[d, equal] \\ B \arrow[r, "u"] & C \arrow[r, "v"] & A \arrow[r, "w'"] & B[1] \end{tikzcd}$$

in $\mathcal{T}$. In particular, $w = w' \circ h$ and uniqueness amounts to proving that h is the identity. Since $\mathcal{A}$ is a full subcategory, $h\colon A \to A$ is a morphism in $\mathcal{A}$ such that $v \circ h = h$, and the surjectivity of v implies $h = \operatorname{Id}_A$.

The following theorem is proved in [**BBD82**, Thm. 1.3.6]:

THEOREM 4.37 (Beilinson-Bernstein-Deligne). *The heart of a t-structure on a triangulated category is a full admissible abelian subcategory.*

REMARK 4.38. It is not true, however, that $\mathcal{T}$ is equivalent, as triangulated category, to the derived category of the heart of a t-structure. Usually, one does not even have a functor $D^b(\mathcal{T}^0) \to \mathcal{T}$ (see Exercice 4.44).

DEFINITION 4.39. Let n be an integer. The n-th cohomology of $X \in \mathcal{T}$ with respect to the t-structure is the following object of the heart:

$$h^n(X) = t_{\leq 0} t_{\geq 0} X[n] \in \mathcal{T}^0. \tag{4.40}$$

This yields a *cohomological* functor $h^n \colon \mathcal{T} \to \mathcal{T}^0$, in the sense that it maps distinguished triangles $X \to Y \to Z \to X[1]$ to long exact sequences

$$\cdots \to h^n(X) \longrightarrow h^n(Y) \longrightarrow h^n(Z) \longrightarrow h^{n+1}(X) \to \dots$$

4.2.3. *Extensions.* We now explain the relation between Hom groups in a triangulated category and extensions in its abelian subcategories.

DEFINITION 4.41. Let $\mathcal{A}$ be an abelian category. Given two objects A and B, a degree n extension of A by B is an exact sequence

$$E\colon \quad 0 \longrightarrow B \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow A \longrightarrow 0.$$

Two extensions of the same degree E and E' are said to be equivalent if there exists a commutative diagram

$$\begin{tikzcd}
E\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
E'\colon & 0 \arrow[r] & B \arrow[r] & C'_{n-1} \arrow[r] & \cdots \arrow[r] & C'_0 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

We consider the equivalence relation generated by such relations. The set of equivalence classes of degree n extensions of A by B forms a group $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ with respect to the *Baer sum*.

Consider a full admissible abelian subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$. Let $0 \to B \to C \to A \to 0$ be an extension in $\mathcal{A}$. By Remark 4.36, it extends to a *unique* distinguished triangle $B \to C \to A \to B[1]$, yielding a map $w \colon A \to B[1]$. Moreover, the same argument shows that two equivalent extensions give rise to the same w. We thus obtain a homomorphism

$$\varphi_1 \colon \mathrm{Ext}^1_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A, B[1]).$$

More generally, breaking a degree n extension

$$0 \to B \to C_{n-1} \to \cdots \to C_0 \to A \to 0$$

into several short exact sequences gives a morphism $A \to B[n]$ which only depends on the equivalence class of the extension. For instance, if $n = 2$, one associates to $0 \to B \to C_1 \xrightarrow{a} C_0 \to A \to 0$ the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & C_1 \arrow[r] & \mathrm{Im}(a) \arrow[r] \arrow[d, equal] & 0 & & \\ & & 0 \arrow[r] & \mathrm{Ker}(a) \arrow[r] & C_0 \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

Setting $D = \mathrm{Im}(a) = \mathrm{Ker}(a)$ and applying φ_1 to the rows of the above diagram, we get maps $\alpha\colon D \to B[1]$ and $\beta\colon A \to D[1]$. Then we form

$$\alpha[1] \circ \beta\colon A \to B[2].$$

Proposition 4.42. *Let $\mathcal{A}$ be a full admissible abelian subcategory of a triangulated category $\mathcal{T}$. Assume that $\mathcal{A}$ is stable under extension. Then*

$$\varphi_n\colon \mathrm{Ext}^n_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A, B[n]).$$

is an isomorphism for $n = 1$ and an injection for $n = 2$.

Proof. See [**Lev93**, Prop. 1.6]. □

⋆⋆⋆

Exercise 4.43. Show that the distinguished triangle (4.30) in the definition of t-structure is uniquely determined by X up to a unique isomorphism. Thus, it makes sense to write $Y = X^{\leq 0}$ and $Z = X^{\geq 1}$. Moreover, the assignments $X \mapsto X^{\leq 0}$ and $X \mapsto X^{\geq 1}$ determine functors $t_{\leq 0}$ and $t \geq 0$.

Exercise 4.44 (A t-structure such that the derived category of the heart is not equivalent to the original triangulated category). Let X be a connected finite CW-complex and let $\mathrm{Sh}(X)$ be the abelian category of sheaves of $\mathbb{Q}$-vector spaces on X. Consider the full subcategory

$$\mathcal{T} \subseteq D^b(\mathrm{Sh}(X))$$

consisting of complexes of sheaves C such that all the cohomology sheaves $\mathcal{H}^i(C)$ are constant. Then $\mathcal{T}$ inherits a structure of triangulated category. We define

$$\mathcal{T}^{\leq 0} = \{C \mid \mathcal{H}^i(C) = 0 \text{ for } i > 0\},$$
$$\mathcal{T}^{\geq 0} = \{C \mid \mathcal{H}^i(C) = 0 \text{ for } i < 0\}.$$

(1) Show that the pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ forms a t-structure on $\mathcal{T}$, whose heart is equivalent to the category $\mathbf{Vec}_{\mathbb{Q}}$ of finite-dimensional $\mathbb{Q}$-vector spaces.
(2) Let $\mathbb{Q}_X$ be the constant sheaf on X. Show that

$$\mathrm{Hom}_{\mathcal{T}}(\mathbb{Q}_X, \mathbb{Q}_X[2]) = H^2(X, \mathbb{Q}).$$

However, using the fact that $D^b(\mathcal{T}^0)$ is equivalent to the category $D^b(\mathbf{Vec}_{\mathbb{Q}})$, we have

$$\mathrm{Hom}_{D^b(\mathcal{T}^0)}(\mathbb{Q}_X, \mathbb{Q}_X[2]) = 0.$$

Deduce that, as long as $H^2(X, \mathbb{Q}) \neq 0$, the triangulated category $\mathcal{T}$ is not equivalent to the derived category of the heart.

4.3. Voevodsky's category of motives.

4.3.1. *A universal cohomology.* Different cohomology theories have been proved useful in the study of algebraic varieties. For instance, as we saw in Chapter 2, to any variety X over a subfield k of $\mathbb{C}$, it is attached the Betti cohomology

$$H^*_{\mathrm{B}}(X) = H^*(X(\mathbb{C}), \mathbb{Q}),$$

which is a finite-dimensional graded $\mathbb{Q}$-vector space. If, in addition, X is smooth, one has also at disposal the de Rham cohomology

$$H^*_{\mathrm{dR}}(X) = \mathbb{H}^*(X, \Omega^*_X),$$

which is now a finite-dimensional graded k-vector space. Recall from Theorem 2.60 that both cohomologies are related, after complexification, by the period isomorphism

$$H^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.45}$$

Another important example is ℓ-adic cohomology defined, for a variety X over a field k of arbitrary characteristic p, a choice of a separable closure $\bar{k}$ of k, and a prime number ℓ different from p, by

$$H^*_\ell(X) = \varprojlim H^*_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

When $\bar{k}$ is embeddable into $\mathbb{C}$, Artin proved that there exists a canonical isomorphism

$$H_\ell(X) \simeq H_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \tag{4.46}$$

All the cohomology theories we have mentioned satisfy similar properties, such as homotopy invariance, Poincaré duality, Künneth formulas, Mayer-Vietoris exact sequences etc. A fundamental feature is that the corresponding vector spaces usually come together with extra structures. We have already seen that Betti cohomology can be provided with a mixed Hodge structure, and ℓ-adic cohomology carries a continuous $\mathbb{Q}_\ell$-linear action of the Galois group $\mathrm{Gal}(\bar{k}/k)$.

The similarities between different cohomology theories, as well as the existence of comparison isomorphisms such as (4.45) or (4.46), led Grothendieck to postulate the existence of a universal cohomology theory which factors all the others: this should be the *motive* of the variety. Since its introduction by Grothendieck, the theory of motives has inspired a wealth of research but, although we have advanced a lot in our understanding, many fundamental questions remain still unanswered.

Restricting to the case of smooth projective varieties, Grothendieck constructed a category of *pure* motives over a field k with some of the desired properties. However, in order to prove that it has all of them, he stated a set of conjectures, the *standard conjectures*, that have proved to be very difficult and seem to be still out of reach. Nevertheless some of the sought properties of the category of pure motives, like the fact that the category of motives modulo *numerical equivalence* is semi-simple [**Jan92**], have been proved without the use of the standard conjectures.

The terminology "pure" comes from the fact that for any smooth projective variety, its n-th cohomology group always has certain properties that are encoded in the statement "$H^n(X)$ is of pure weight n". For instance, if X is a smooth projective complex variety, the group $H^n_{\mathrm{B}}(X, \mathbb{C})$ has a Hodge decomposition

$$H^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{p+q=n} H^{p,q}(X).$$

The fact that only factors with $p+q=n$ appear means that its Hodge structure is pure of weight n. For varieties over a finite field, the corresponding purity is reflected by the fact that the eigenvalues of the action of Frobenius on étale cohomology have absolute value $q^{\frac{n}{2}}$.

Using resolution of singularities, we can express the cohomology of a singular quasi-projective variety in terms of the cohomology of smooth projective varieties, but in this expression cohomologies of different degrees get mixed. As we have seen in Section 2.5.2 this gives rise to a mixed Hodge structure in the cohomology of X. Thus, the motive of a smooth projective variety should be *pure* while the motive of a singular or quasi-projective variety should be *mixed*. Since Grothendieck, there has been a great effort to develop a theory of mixed motives.

Abstractly we can think of a cohomology theory in the following way. Fix a field k, denote by $\mathbf{Var}_k$ the category of varieties over k, and let $\mathcal{A}$ be an abelian category (or more precisely a Tannakian category). Denote by $\mathcal{DA}$ the derived category of $\mathcal{A}$. Then $\mathcal{DA}$ is a triangulated category provided with a t-structure (see Section 4.2.2 for a definition) that allows us to recover $\mathcal{A}$ from $\mathcal{DA}$. A cohomology theory (with values in $\mathcal{A}$) is a contravariant functor

$$H\colon \mathbf{Var}_k \longrightarrow \mathcal{DA}$$

satisfying certain properties. We can recover the "cohomology groups" of X from $H(X)$ using the t-structure:

$$H^n(X) = t_{\leq n} t_{\geq n} H(X) \in \mathcal{A}.$$

Voevodsky was able to define a triangulated category $\mathbf{DM}_{gm}(k)$, which is a candidate for the derived category of mixed motives over k. The main missing piece is a suitable "motivic" t-structure. Recently, Beilinson [**Bei12**] showed that, when k has characteristic zero, the existence of such motivic t-structure implies the standard conjectures. Conversely, Hanamura proved in [Han99] that, over any field k, the conjunction of the standard conjectures and conjectures by Murre and Beilinson-Soulé implies the existence of the motivic t-structure. Thus we are back to Grothendieck insight that to have a full theory of motives we need to prove the standard conjectures.

4.3.2. *The derived category of mixed motives.* Let k be a field. In what follows, we give a sketch of Voevodsky's construction of a derived category of mixed motives over k with rational coefficients, which will be denoted by

$$\mathbf{DM}(k) = \mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}.$$

For more details we refer the reader to the original paper [Voe00], the lecture notes [MVW06] or part II of the introductory book [And04].

We start with the category $\mathbf{Sm}(k)$ of smooth varieties over k. This category is not additive, for it does not make sense to "sum" two morphisms of schemes. The first step of the construction will be to enlarge the set of morphisms through the notion of finite correspondence.

4.3.3. *First step: the category of finite correspondences.*

Definition 4.47. Let X and Y be objects of $\mathbf{Sm}(k)$. A *finite correspondence* from X to Y is a $\mathbb{Z}$-linear combination of integral closed subschemes $W \subseteq X \times Y$ such that the projection $W \to X$ is finite and surjective over a connected component of X.

Finite correspondences form an abelian subgroup of the group of algebraic cycles $\mathcal{Z}^{\dim Y}(X \times Y)$, which will be denoted by $c(X, Y)$.

Example 4.48. Given any morphism of schemes $f \colon X \to Y$, the graph $\Gamma_f \subseteq X \times Y$ is a finite correspondence. In general, we can think of finite correspondences as multivalued maps on a connected component of X.

Given $X, Y, Z \in \mathbf{Sm}(k)$, we will denote by p_{XY}, p_{XZ} and p_{YZ} the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively.

Lemma 4.49. *Let X, Y, Z be objects in $\mathbf{Sm}(k)$. Consider finite correspondences $W \in c(X, Y)$ and $W' \in c(Y, Z)$. Then the cycles $p_{XY}^*(W)$ and $p_{YZ}^*(W')$ intersect properly on $X \times Y \times Z$. Moreover, the projection of the cycle $p_{XZ}(p_{XY}^*\alpha \cdot p_{YZ}^*\beta)$ is finite over X and surjective over a connected component.*

Thanks to the above lemma, we can define the composition

$$\circ : c(X, Y) \ \times \ c(Y, Z) \to c(X, Z)$$

by

$$\alpha \circ \beta = p_{XZ}(p_{XY}^*\alpha \cdot p_{YZ}^*\beta). \tag{4.50}$$

The category $\mathbf{SmCor}(k)$ has the same objects as $\mathbf{Sm}(k)$, but the morphisms are given by finite correspondences with $\mathbb{Q}$-coefficients:

$$\mathrm{Hom}_{\mathbf{SmCor}(k)}(X, Y) = c(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

There is a functor $\mathbf{Sm}(k) \to \mathbf{SmCor}(k)$ that is the identity on objects and sends a map $f\colon X \to Y$ to its graph Γ_f. By Exercise 4.71, the composition of maps is compatible with the composition (4.50) of finite correspondences. We denote by $[X]$ the image in $\mathbf{SmCor}(k)$ of a smooth variety X.

The direct sum in $\mathbf{SmCor}(k)$ is given by the disjoint union of varieties. This category is also equipped with the tensor product

$$[X] \otimes [Y] = [X \times_k Y].$$

4.3.4. *Second step: A triangulated category with homotopy invariance and Mayer Vietoris.* The second step is similar to the construction of the derived category of an abelian category. We start with the category

$$C^b(\mathbf{SmCor}(k))$$

of bounded chain complexes in $\mathbf{SmCor}(k)$. The objects are diagrams

$$\cdots \to [X_n] \xrightarrow{\partial_n} [X_{n-1}] \to \cdots ,$$

where X_i is in $\mathbf{Sm}(k)$ and $\partial_n \in c(X_n, X_{n-1}) \otimes \mathbb{Q}$ are finite correspondences such that $\partial_{n-1} \circ \partial_n = 0$. Then we define the homotopy category $K^b(\mathbf{SmCor}(k))$ as the one having the same objects as $C^b(\mathbf{SmCor}(k))$, and morphisms given by homotopy classes of morphisms of complexes.

Two examples of objects of $K^b(\mathbf{SmCor}(k))$ are:

(1) (homotopy complex) for any X in $\mathbf{Sm}(k)$, the complex

$$[X \times \mathbb{A}^1] \xrightarrow{\mathrm{pr}} [X]$$

placed in degrees 1 and 0.

(2) (Mayer-Vietoris complex) for any X in $\mathbf{Sm}(k)$ and any open cover $X = U \cup V$, the complex

$$[U \cap V] \xrightarrow{i_{U\cap V,U} + i_{U\cap V,V}} [U] \oplus [V] \xrightarrow{i_{U,X} - i_{V,X}} [X],$$

where $[U \cap V]$ sits in degree 2, and the arrows $i_{U,X}, i_{V,X}, i_{U\cap V,U}$ and $i_{U\cap V,V}$ are the obvious inclusions.

We want to force the homotopy invariance and the Mayer-Vietoris property, which mean that the above two complexes are acyclic. To this end, we take the quotient of $K^b(\mathbf{SmCor}(k))$ by the thick triangulated subcategory generated by all homotopy and Mayer-Vietoris complexes. It has the structure of a triangulated category.

4.3.5. *Third step: The pseudo-abelian envelope.* The next step is to take the pseudo-abelian envelope of the quotient obtained in the previous step. The resulting category is denoted by $DM^{\text{eff}}_{gm}(k)$.

Recall the construction of the pseudo-abelian envelope

DEFINITION 4.51. Let $\mathcal{C}$ be an additive category. The *pseudo-abelian envelope* of $\mathcal{C}$ is the category with

- *objects*: (X, p) where X is an object of $\mathcal{C}$ and $p \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ is an idempotent, that is, $p^2 = p$.
- *morphisms*: $\mathrm{Hom}((X, p), (Y, q)) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the subgroup of those f such that $f = q \circ f \circ p$.

There is a fully faithful functor $\mathcal{C} \to \mathcal{C}_{pa}$ sending X to (X, id). Passing to the pseudo-abelian envelope allows us to consider the kernel of each idempotent $p \colon X \to X$ as a subobject of X. This will be crucial when we want to talk about "pieces of the cohomology".

REMARK 4.52. By a result of Balmer and Schlichting [**BS01**], the pseudo-abelian envelope of a triangulated category remains triangulated. Thus, $DM^{\text{eff}}_{gm}(k)$ is still a triangulated category.

We have a functor $M \colon \mathbf{Sm}(k) \to DM^{\text{eff}}_{gm}(k)$ sending X to $[X]$, regarded as a complex concentrated in degree zero. The category $DM^{\text{eff}}_{gm}(k)$ is also equipped with a tensor product that is characterized by the property

$$M(X) \otimes M(Y) = M(X \times Y).$$

The unit object is the motive of the base field, which will be denoted by

$$\mathbb{Q}(0) = M(\mathrm{Spec}(k)).$$

Note also that there is a functor

$$C^b(\mathbf{SmCor}(k)_{\mathrm{pa}}) \longrightarrow DM^{\text{eff}}_{gm}(k) \tag{4.53}$$

from the category of bounded complexes in the pseudo-abelian envelope of $\mathbf{SmCor}(k)$ to the category of effective motives $DM^{\text{eff}}_{gm}(k)$.

4.3.6. *Fourth step: inversion of the Tate motive.* Given X in $\mathbf{Sm}(k)$, let $X \to \mathrm{Spec}(k)$ denote the structural morphism. We can think of it as a complex sitting in degrees 0 and -1:

$$[X] \longrightarrow [\mathrm{Spec}\, k]. \tag{4.54}$$

DEFINITION 4.55. The *reduced motive* of X is the object $\widetilde{M}(X)$ of $DM^{\text{eff}}_{gm}(k)$ determined by the complex (4.54).

When X has a k-rational point, there is a direct sum decomposition (see exercise 4.72)

$$M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X).$$

Definition 4.56. The *Tate motive* $\mathbb{Q}(1)$ is $\widetilde{M}(\mathbb{P}^1_k)[-2]$. For $n \geq 0$, one defines $\mathbb{Q}(n)$ as $\mathbb{Q}(1)^{\otimes n}$.

The last step of the construction of $\mathbf{DM}(k)$, necessary to obtain a rigid tensor category, is to formally invert the motive $\mathbb{Q}(1)$. By this we mean the following: an object of the new category $\mathbf{DM}(k)$ is a pair (M, m), where M is an object of $DM^{\text{eff}}_{gm}(k)$ and $m \in \mathbb{Z}$. Morphisms are given by

$$\begin{aligned}&\operatorname{Hom}_{\mathbf{DM}(k)}((M,m),(N,n))\\ &\qquad = \varinjlim_{r\geq -m,-n} \operatorname{Hom}_{DM^{\text{eff}}_{gm}(k)}(M\otimes\mathbb{Q}(m+r), N\otimes\mathbb{Q}(n+r)).\end{aligned}$$

The resulting category has the following property:

Theorem 4.57 (Voevodsky). *The category $\mathbf{DM}(k)$ is a rigid tensor $\mathbb{Q}$-linear triangulated category.*

Proof. See [**MVW06**, Theorem 20.17]. □

4.3.7. *Properties of $\mathbf{DM}(k)$.* All the usual machinery to compute the homology of algebraic varieties is still available in the derived category of motives:

(1) (Künneth): $M(X \times Y) = M(X) \otimes M(Y)$.
(2) ($\mathbb{A}^1$-homotopy invariance): $M(X \times \mathbb{A}^1) = M(X)$.
(3) (Mayer-Vietoris): For $X = U \cup V$ as before, there is a distinguished triangle
$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1].$$
(4) (Gysin) If $Z \subset X$ is a smooth closed subscheme of codimension c of a smooth scheme X, then there is a distinguished triangle
$$M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1].$$
(5) (Blow-ups) Let $Z \subseteq X$ be a smooth closed subscheme of a smooth scheme, $\mathrm{Bl}_Z X$ the blow-up of X along Z, and E the exceptional divisor. Then there is a distinguished triangle
$$M(E) \to M(\mathrm{Bl}_Z X) \oplus M(Z) \to M(X) \to M(E)[1].$$
Moreover, if Z has codimension c in Z, the triangle yields a canonical isomorphism
$$M(\mathrm{Bl}_Z X) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$
(6) (Duality) There is a duality $A \mapsto A^\vee$ that, for X smooth and projective of dimension d, satisfies
$$M(X)^\vee = M(X)(-d)[-2d].$$

(7) (Adjunction) The duality and tensor product are related by the adjunction formulas

$$\mathrm{Hom}(A \otimes B^\vee, C) = \mathrm{Hom}(A, C \otimes B),$$
$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(B, A^\vee \otimes C).$$

REMARK 4.58. We observe that the functor from $\mathbf{Sm}(k)$ to $\mathbf{DM}(k)$ is covariant, thus is a "homological" functor in contrast to the contravariant functor chosen by Grothendieck for pure motives that was cohomological.

EXAMPLE 4.59. Let us use some of these properties to show that

$$M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \oplus \cdots \oplus \mathbb{Q}(n)[2n].$$

This should be compared with Example 2.92, where the cohomology of $\mathbb{P}^n$ is computed, but noting that $M(\mathbb{P}^n)$ is to be seen as the homology of $\mathbb{P}^n$.

We proceed by induction on n, the case $n = 1$ being reduced to the definition of $\mathbb{Q}(1)$. For $n \geq 2$, the standard closed immersion $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ satisfies $\mathbb{P}^n \setminus \mathbb{P}^{n-1} = \mathbb{A}^n$. By the Gysin property, we have the distinguished triangle

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\mathbb{P}^{n-1})(1)[2] \to M(\mathbb{A}^n)[1]. \tag{4.60}$$

Note that $M(\mathbb{A}^n) = \mathbb{Q}(0)$, as one can prove by repeatedly applying the $\mathbb{A}^1$-homotopy property. Moreover, the composition

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\mathrm{Spec}(k))$$

is the identity $\mathbb{Q}(0) \to \mathbb{Q}(0)$. Thus, the triangle (4.60) is split and $M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus M(\mathbb{P}^{n-1})(1)[2]$. The result follows by induction hypothesis.

REMARK 4.61. To understand the different roles of the twist and the shift, it is instructive to compare the reduced motives of $\mathbb{P}^1$ and $\mathbb{G}_m$. In the first case, we have $\widetilde{M}(\mathbb{P}^1) = \mathbb{Q}(1)[2]$. For the second case, one can use the Mayer-Vietoris triangle for the open covering $\mathbb{P}^1 = U \cup V$, with $U = \mathbb{P}^1 \setminus \{0\}$ and $V = \mathbb{P}^1 \setminus \{\infty\}$. One gets an exact triangle

$$M(\mathbb{G}_m) \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) \longrightarrow \mathbb{Q}(1)[2] \longrightarrow M(\mathbb{G}_m)[1],$$

from which it follows that $M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$, thus $\widetilde{M}(\mathbb{G}_m) = \mathbb{Q}(1)[1]$. This can be compared with the fact that, for any cohomology theory, $H^1(\mathbb{G}_m)$ and $H^2(\mathbb{P}^1)$ are isomorphic, but they lie in different degree. In particular, the Hodge structure $H^2(\mathbb{P}^1)$ is pure of weight 2 and Hodge type $(1,1)$. The same is true for $H^1(\mathbb{G}_m)$, but, since this last group lies in degree one, we consider it as a mixed Hodge structure.

4.3.8. *Motivic cohomology.* Voevodsky also computed some morphism groups in the category $\mathbf{DM}(k)$. In particular, he defined:

DEFINITION 4.62. The *motivic cohomology* of X is

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) = \mathrm{Hom}_{\mathbf{DM}(k)}(M(X), \mathbb{Q}(p)[n]).$$

Using Bloch's formula relating higher Chow groups and K-theory he proves ([**Voe02**], [**Blo86**], [**Lev94**])

THEOREM 4.63. *Given a smooth variety X, there is an isomorphism*

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) = (K_{2p-n}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(p)},$$

where $K_\bullet(X)$ denotes Quillen's K-theory of X and the index (p) means the eigenspace for the Adams operations.

4.3.9. *The normalization of a cosimplicial scheme.* To every variety X, not necessarily smooth, it is attached a motive $M(X)$ in Voevodsky's category. Using tools from homological algebra, one can construct more general motives, for instance the motive of a cosimplicial variety.

Recall that in Section 3.5.4 we defined the normalized complex associated to a cosimplicial object in an abelian category. It turns out that it is enough to work in a pseudo-abelian category.

LEMMA 4.64. *Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. Given integers $m > n \geq 0$, the following endomorphism in $\mathbf{SmCor}(k)$ is idempotent:*

$$p_n = (1 - \delta^0\sigma^0)(1 - \delta^1\sigma^1)\cdots(1 - \delta^n\sigma^n) : [X^m] \to [X^m].$$

PROOF. We argue by induction on n. For $n = 0$, note that the relation $\sigma^0\delta^0 = \mathrm{Id}$ implies that $\delta^0\sigma^0$ is an idempotent, hence the same holds for $1 - \delta^0\sigma^0$. Let us now assume that p_{n-1} is idempotent. We next observe that for $i = 0, \dots, n-1$, the face σ^n commutes with $\delta^i\sigma^i$. Indeed, by relations (c) and (b) in (3.185),

$$\sigma^n(\delta^i\sigma^i) = \delta^i\sigma^{n-1}\sigma^i = (\delta^i\sigma^i)\sigma^n.$$

Moreover, relation (d) in (3.185) implies $\sigma^n(1 - \delta^n\sigma^n) = 0$. These two equations together imply

$$\sigma^n(1 - \delta^0\sigma^0)\cdots(1 - \delta^n\sigma^n) = 0. \tag{4.65}$$

We now compute, using equation (4.65), and the induction hypothesis,

$$\begin{aligned} p_n^2 = {} & \underbrace{(1 - \delta^0\sigma^0)\cdots(1 - \delta^{n-1}\sigma^{n-1})}_{p_{n-1}}(1 - \delta^n\sigma^n) \\ & \qquad \underbrace{(1 - \delta^0\sigma^0)\cdots(1 - \delta^{n-1}\sigma^{n-1})}_{p_{n-1}}(1 - \delta^n\sigma^n) \\ & \qquad\qquad = p_{n-1}^2(1 - \delta^n\sigma^n) = p_{n-1}(1 - \delta^n\sigma^n) = p_n, \end{aligned}$$

as we wanted to show. ∎

Since p_n is idempotent, $\mathrm{Im}(p_n)$ is an object of the pseudo-abelian enveloppe of $\mathbf{SmCor}(k)$. By convention, we write $p_{-1} = \mathrm{Id}$.

DEFINITION 4.66. Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. The *normalization* of $X^\bullet$ is the complex in $\mathbf{SmCor}(k)_{\mathrm{pa}}$ given by

$$\mathcal{N}(X^\bullet)^n = \mathrm{Im}(p_{n-1}\colon [X^n] \to [X^n]),$$

together with the differential

$$d = \sum_{i=0}^{n+1} (-1)^i \delta^i \colon \mathcal{N}(X^\bullet)^n \to \mathcal{N}(X^\bullet)^{n+1}.$$

In general, the complex $\mathcal{N}(X^\bullet)$ is not bounded. To obtain a bounded complex, we consider the *bête* truncation $\sigma_{\leq N}\mathcal{N}(X^\bullet)$, that is,

$$\sigma_{\leq N}\mathcal{N}(X^\bullet)^n = \begin{cases} \mathcal{N}(X^\bullet)^n & n \leq N, \\ 0 & n > N. \end{cases}$$

This is now an element of $C^b(\mathbf{SmCor}(k)_{\mathrm{pa}})$. For each $N \geq 0$, applying the functor (4.53), we obtain a motive

$$[\sigma_{\leq N}\mathcal{N}(X^\bullet)].$$

Clearly, given integers $M \geq N \geq 0$, there is a morphism of complexes

$$\sigma_{\leq M}\mathcal{N}(X^\bullet) \to \sigma_{\leq N}\mathcal{N}(X^\bullet).$$

The system $([\sigma_{\leq N}\mathcal{N}(X^\bullet)])_{N \geq 0}$ is a pro-object in $\mathbf{DM}(k)$.

REMARK 4.67. The advantage of using Lemma 4.64 is that it provides us with an explicit idempotent cutting out the normalized complex from the cochain complex. However, we could have also constructed it directly by abstract means, as we now explain[10]. Recall that a category is said to be *preadditive* if the morphism sets are abelian groups and the composition of maps is bilinear. Given a preadditive category $\mathcal{A}$, let $\mathbf{Ab}(\mathcal{A})$ denote the category of presheaves of abelian groups on $\mathcal{A}$, by which we simply mean additive contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$. Then $\mathbf{Ab}(\mathcal{A})$ is an abelian category, and the Yoneda lemma ensures that the natural functor

$$h\colon \mathcal{A} \longrightarrow \mathbf{Ab}(\mathcal{A})$$

which sends X to $\mathrm{Hom}(-, X)$ is fully faithful. Assume now that $\mathcal{A}$ is pseudo-abelian. If Y' is a direct factor of an object of the form $h(X)$, then projecting to the complement one gets an idempotent p of $h(X)$ such that $Y' = \mathrm{Ker}(p)$. By fully-faithfulness, we can see p as an idempotent of X, and the object $Y = \mathrm{Ker}(p)$ in $\mathcal{A}$, determined up to unique isomorphism, satisfies $h(Y) = Y'$. If $X^\bullet$ is a cosimplicial object in $\mathcal{A}$, the associated cochain complex CX^* is a complex in $\mathcal{A}$ whose formation commutes with the functor h, in the sense that $h(CX^*) = C^*(h(X^\bullet))$. Since $\mathbf{Ab}(\mathcal{A})$ is abelian, the normalized complex $\mathcal{N}^*(h(X^\bullet))$, as introduced in Section 3.5.4, is a direct factor of $C^*(h(X^\bullet))$. Proceeding as above, one gets a complex (up to unique isomorphism) $\mathcal{N}X^*$ such that $h(\mathcal{N}X^*) = \mathcal{N}^*(h(X^\bullet))$.

[10] We thank J. Ayoub for pointing this argument to us.

4.3.10. *Hodge realization.* From now on, we assume that k has characteristic zero and comes with an embedding $k \hookrightarrow \mathbb{C}$. We end this section recalling the existence of the Hodge realization functor.

THEOREM 4.68. *There is a covariant functor of $\mathbb{Q}$-linear rigid tensor triangulated categories*

$$R^{\mathrm{H}}\colon \mathbf{DM}(k) \longrightarrow D^b(\mathbf{MHS}(k)).$$

The proof of this theorem is sketched in [DG05, §1.5]. The main difficulty is the covariance of the de Rham complex for finite correspondences. A more detailed version of the argument is exposed in [Bou09].

We now give a sketch of the construction of the Hodge realization functor in the case of the motive $[\sigma_{\leq N}\mathcal{N}(X^\bullet)]$ from the previous section. Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. Assume that there is an embedding of cosimplicial smooth varieties over k,

$$j_\bullet\colon X^\bullet \to \overline{X}^\bullet,$$

such that all the $\overline{X}^n$ are smooth projective varieties and $D^n = \overline{X}^n \setminus X^n$ is a simple normal crossing divisor. The Hodge realization of $[\sigma_{\leq N}\mathcal{N}(X^\bullet)]$ is constructed as follows.

(1) *Betti part R^{B}.* For each n, let $\mathcal{C}^*(X^n(\mathbb{C}), \underline{\mathbb{Q}})$ be the Godement canonical flasque resolution of the locally constant sheaf $\mathbb{Q}$ of the complex manifold $X^n(\mathbb{C})$ and let $j_{n,*}\mathcal{C}^*(X^n(\mathbb{C}), \underline{\mathbb{Q}})$ be the complex of sheaves on $\overline{X}^n$ obtained by direct image by the inclusion $j_n\colon X_n \to \overline{X}_n$. On this complex of sheaves we put the canonical increasing filtration

$$W_m j_{n,*}\mathcal{C}^k(X^n(\mathbb{C}), \underline{\mathbb{Q}}) = \begin{cases} j_{n,*}\mathcal{C}^k(X^n(\mathbb{C}), \underline{\mathbb{Q}}), & \text{if } k < m, \\ \operatorname{Ker} d, & \text{if } k = m, \\ 0, & \text{if } k > m. \end{cases}$$

We construct filtered acyclic resolutions $(K^*_{\mathrm{B},n}, W)$ of the complex $(j_{n,*}\mathcal{C}(X^n(\mathbb{C}), \underline{\mathbb{Q}}), W)$ in a functorial way. For instance using again the Godement canonical flasque resolution, this time on $\overline{X}^n$. Taking now global sections we obtain a filtered simplicial complex $(\Gamma(\overline{X}^\bullet, K^*_{\mathrm{B},n}), W)$. Finally, taking the normalization, the truncation and the total complex of the resulting double complex we obtain a filtered complex

$$(\operatorname{Tot} \sigma_{\leq N}\mathcal{N}\Gamma(\overline{X}^\bullet, K^*_{\mathrm{B},n}), W).$$

Finally, since we want the realization functor to be covariant, so we write

$$(R^{\mathrm{B}}(\sigma_{\leq N}\mathcal{N}X^\bullet), W) = (\operatorname{Tot} \sigma_{\leq N}\mathcal{N}\Gamma(\overline{X}^\bullet, K^*_{\mathrm{B},n}), W)^\vee. \tag{4.69}$$

Here it is important to note that the normalization of simplicial and cosimplicial objects are dual of each other.

(2) *de Rham part* R^{dR}. For each n let $\Omega^*_{\overline{X}^n}(\log D^n)$ be the de Rham complex of algebraic forms on $\overline{X}^n$ with logarithmic poles along D^n. This complex has a decreasing Hodge filtration F that counts the number of differentials and an increasing weight filtration W that counts the number of poles of a differential form. We construct an acyclic bifiltered resolution (K_{dR}, F, W) again in a functorial way. We now repeat the process done in the Betti case: we take global sections, the normalization and truncation on the simplicial direction, the total complex and the dual to obtain a bifiltered complex

$$(R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet), F, W) = ((\mathrm{Tot}\,\sigma_{\leq N}\mathcal{N}\Gamma(X^\bullet, K_{\mathrm{dR}})^\vee, F, W). \tag{4.70}$$

(3) *The comparison isomorphism.* Going to the cosimplicial complex manifold $\overline{X}^\bullet(\mathbb{C})$, we can construct a bifiltered complex

$$(R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet(\mathbb{C})), F, W),$$

that is the analogue of the complex $(R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet), F, W)$ but using holomorphic forms. Then the maps

$$\begin{tikzcd} (R^{\mathrm{B}}(\sigma_{\leq N}\mathcal{N}X^\bullet), W) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r] & (R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet(\mathbb{C})), W) \\ & (R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet), W) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[u] \end{tikzcd}$$

are filtered quasi-isomorphisms giving the comparison isomorphism.

⋆ ⋆ ⋆

Exercise 4.71. Prove that the composition of the finite correspondences given by the graphs of two morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$, as defined in (4.50), is the graph of $g \circ f\colon X \to Z$.

Exercise 4.72. Let X be a smooth variety over k, together with a rational point $x\colon \mathrm{Spec}(k) \to X$. Consider the composition

$$p\colon X \longrightarrow \mathrm{Spec}(k) \xrightarrow{x} X.$$

(1) Show that p is a projector and that the class of $(X, 1-p)$ agrees with the reduced motive $\widetilde{M}(X)$ from Definition 4.55. Thus there is a decomposition

$$M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X).$$

(2) Show that the reduced motive $\widetilde{M}(X)$ is independent of the choice of the rational point x.

4.4. Mixed Tate motives. As was mentioned in the previous section, it is not known how to construct a motivic t-structure yielding the desired abelian category of mixed motives. However, when k is number field, one can extract from $\mathbf{DM}(k)$ an abelian category of mixed Tate motives with similar properties to mixed Hodge Tate structures. The keystone is Borel's computation of the K-theory of number fields.

4.4.1. *The derived category of mixed Tate motives.* The motives $\mathbb{Q}(n)$ are the simplest non-trivial objects of the category $\mathbf{DM}(k)$. It is thus reasonable to figure out what can be built starting from them.

DEFINITION 4.73. The *derived category of mixed Tate motives over* k is the smallest triangulated subcategory $\mathbf{DMT}(k)$ of $\mathbf{DM}(k)$ containing the objects $\mathbb{Q}(n)$, for all $n \in \mathbb{Z}$, and stable under extensions.

Recall that the latter condition means that if $A \to B \to C \to A[1]$ is a distinguished triangle in $\mathbf{DM}(k)$ and two objects among A, B, C belong to $\mathbf{DMT}(k)$, then so does the third.

Thanks to the comparison between motivic cohomology and K-theory (Theorem 4.63), the extension groups of simple objects in the category $\mathbf{DMT}(k)$ are given by

$$\begin{aligned}\operatorname{Ext}^i(\mathbb{Q}(l), \mathbb{Q}(m)) &= \operatorname{Ext}^i(\mathbb{Q}(0), \mathbb{Q}(m-l)) \\ &= \operatorname{Hom}_{\mathbf{DM}(k)}(M(\operatorname{Spec}(k)), \mathbb{Q}(m-l)[i]) \\ &= (K_{2(m-l)-i}(k) \otimes \mathbb{Q})^{(m-l)}.\end{aligned}$$

The K-theory groups of general fields are still largely unknown, but, when k is a number field, Borel computed their ranks:

THEOREM 4.74 (Borel, [Bor74]). *Let k be a number field with r_1 (resp. $2r_2$) real (resp. complex) embeddings. Then:*

$$(K_{2(m-l)-i}(k) \otimes \mathbb{Q})^{(m-l)} = \begin{cases} \mathbb{Q}, & \text{if } i = 0,\ m-l = 0, \\ k^\times \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } i = 1,\ m-l = 1, \\ \mathbb{Q}^{r_1 + r_2}, & \text{if } i = 1,\ m-l \geq 3 \text{ odd}, \\ \mathbb{Q}^{r_2}, & \text{if } i = 1,\ m-l \geq 2 \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

The important information we should get from this is

(1) the only non-zero groups Ext^i occur for $i = 0, 1$;
(2) $\operatorname{Ext}^0(\mathbb{Q}(l), \mathbb{Q}(m)) = \operatorname{Hom}(\mathbb{Q}(l), \mathbb{Q}(m)) = 0$ unless $m = l$, for which it is equal to $\mathbb{Q}$;
(3) if $\operatorname{Ext}^1(\mathbb{Q}(l), \mathbb{Q}(m)) \neq 0$, then $m > l$;
(4) the only infinite-dimensional group is $\operatorname{Ext}^1(\mathbb{Q}(l), \mathbb{Q}(l+1))$.

In particular, when $k = \mathbb{Q}$, we have $r_1 = 1$ and $r_2 = 0$, so

$$\mathrm{Ext}^1_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } n = 1, \\ \mathbb{Q} & \text{if } n \geq 3 \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

This and the fact that $\mathrm{Ext}^i_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ for $i \geq 2$ will determine the structure of the category of mixed Tate motives over $\mathbb{Q}$.

Example 4.75 (Kummer motives). Since

$$\mathrm{Ext}^1_{\mathbf{DMT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) = k^\times \otimes_{\mathbb{Z}} \mathbb{Q},$$

there are plenty of non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. They are all rational linear combinations of Kummer motives. For each $t \in k^\times \setminus \{1\}$, consider the complex K_t in $\mathbf{SmCor}(k)$ given by

$$\mathrm{Spec}(k) \oplus \mathrm{Spec}(k) \xrightarrow{f_t} \mathbb{P}^1_k \setminus \{0, \infty\},$$

where $\mathrm{Spec}(k) \oplus \mathrm{Spec}(k) = \{*_1, *_2\}$ sits in degree 0 and the finite correspondence f_t is defined by the cycle $[(*_1, t)] - [(*_2, 1)]$.

The class of K_t in $\mathbf{DM}(k)$ belongs to $\mathbf{DMT}(k)$ and the degrees are chosen so that it belongs to $\mathbf{MT}(k)$. The Kummer motive K_t^{Mot} is the class of K_t in $\mathbf{MT}(k)$. For $t = 1$ we write K_1^{Mot} for the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. The Hodge realization of the Kummer motive is the Kummer mixed Hodge structure of Example 2.158.

Another well understood case is the K-theory of finite fields, which was completely computed by Quillen in [**Qui72**, Thm. 8], shortly after he introduced the definition of higher algebraic K-theory:

Theorem 4.76 (Quillen, [**Qui72**]). *Let $\mathbb{F}_q$ be the finite field with q elements. Then:*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/(q^n - 1) & i = 2n - 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Conjecture 4.77 (Beilinson-Soulé). *If k is a field, then $K_n(k)^{(r)}_{\mathbb{Q}}$ vanishes for all $n > 2r$.*

An immediate corollary of Borel and Quillen's theorems is:

Corollary 4.78. *The Beilinson-Soulé conjecture holds when k is either a number field or a finite field.*

4.4.2. *A t-structure on* $\mathbf{DMT}(k)$ *after Levine* [**Lev93**]. For each pair of integers a and b, let us denote by $\mathcal{T}_{[a,b]}$ the strictly full triangulated subcategory of $\mathbf{DMT}(k)$ generated by the objects $\mathbb{Q}(n)$ for $a \leq -2n \leq b$. We denote $\mathcal{T}_{[a,a]}$ simply by $\mathcal{T}_a$, and we extend the definition to cover the cases $a = -\infty$ or $b = \infty$ as well. In particular, $\mathcal{T}_{(-\infty,\infty)} = \mathbf{DMT}(k)$.

Lemma 4.79. *Let $a \leq b \leq c$ be integers (the cases $a = -\infty$ and $c = \infty$ are also allowed). Then $(\mathcal{T}_{[a,b-1]}, \mathcal{T}_{[b,c]})$ is a t-structure on $\mathcal{T}_{[a,c]}$.*

In particular, for each integer b, the pair $(\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)})$ provides a t-structure on $\mathbf{DMT}(k)$. Let us emphasize that this is not the t-structure we are looking for, since its heart is reduced to zero. However, it will allow us to define a *weight structure*.

The truncation functors for the t-structure $(\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)})$ on $\mathbf{DMT}(k)$ will be denoted by

$$W_{\leq b}\colon \mathbf{DMT}(k) \longrightarrow \mathcal{T}_{(-\infty,b]}$$
$$W^{>b}\colon \mathbf{DMT}(k) \longrightarrow \mathcal{T}_{[b+1,\infty)}.$$

The reason for the subindex or superindex is that one will give an increasing filtration whereas the other will give a decreasing filtration.

Let $W^{\geq b}$ denote $W^{>b-1}$ and define

$$\mathrm{Gr}_b^W(M) = W^{\geq b} W_{\leq b}(M).$$

For each even integer a, let $\mathcal{T}_a^{\leq 0}$ (resp. $\mathcal{T}_a^{\geq 0}$) be the full subcategory of $\mathcal{T}_a$ generated by $\mathbb{Q}(-a/2)[n]$ for $n \leq 0$ (resp. $n \geq 0$). Finally, let $\mathcal{T}_{[a,b]}^{\leq 0}$ (resp. $\mathcal{T}_{[a,b]}^{\geq 0}$) be the full subcategory of $\mathcal{T}_{[a,b]}$ generated by the objects M such that $\mathrm{Gr}_c^W(M)$ belongs to $T_c^{\leq 0}$ (resp. $\mathcal{T}_c^{\geq 0}$) for all $a \leq c \leq b$.

Theorem 4.80 (Levine). *Assume that the field k satisfies the Beilinson-Soulé conjecture. Then the pair of strictly full subcategories*

$$(\mathcal{T}_{(-\infty,\infty)}^{\leq 0}, \mathcal{T}_{(-\infty,\infty)}^{\geq 0})$$

forms a non-degenerate t-structure on $\mathbf{DMT}(k)$.

Definition 4.81. The category $\mathbf{MT}(k)$ of *mixed Tate motives over k* is the heart of the above t-structure.

The category $\mathbf{MT}(k)$ has the following properties:

(1) It is a neutral Tannakian category generated under extensions by the objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.
(2) Each object M of $\mathbf{MT}(k)$ has an increasing *weight filtration* $W_\bullet M$ such that

$$\mathrm{Gr}_{2n}^W M \simeq \mathbb{Q}(n)^{\oplus k_n}, \qquad \mathrm{Gr}_{2n+1}^W = 0$$

for some natural numbers k_n.

(3) A fibre functor is given by

$$\omega(M) = \bigoplus_n \operatorname{Hom}(\mathbb{Q}(n), \operatorname{Gr}^W_{2n} M). \tag{4.82}$$

Moreover, Wildeshaus [**Wil09**, Théorème 1.3] proved that there exists a canonical equivalence of categories

$$F\colon D^b(\mathbf{MT}(k)) \longrightarrow \mathbf{DMT}(k). \tag{4.83}$$

The functor F is t-exact, induces the identity on the heart $\mathbf{MT}(k)$, and has the property that the composition with the cohomology functor H^0 associated to the t-structure as in (4.40) coincides with the canonical cohomology functor $D^b(\mathbf{MT}(k)) \to \mathbf{MT}(k)$. In view of Remark 4.38, the main difficulty does not lie in proving that the two categories are equivalent but in constructing a functor between them.

4.4.3. *Examples.* If the motive of a variety X is of mixed Tate type, *i.e.* belongs to $\mathbf{DMT}(k)$, then decomposing $M(X)$ (or rather its dual) by means of Levine's t-stucture we obtain the cohomology motives

$$h^i(X) = t_{\leq 0}t_{\geq 0}(M(X)^\vee[i]) \in \mathbf{MT}(k).$$

Thus we can isolate the different cohomological degrees, something we do not know how to do for general motives.

Example 4.84. By Example 4.59, the motive of the projective space $M(\mathbb{P}^n_k)$ is of mixed Tate type and one has

$$h^i(\mathbb{P}^n_k) = \begin{cases} \mathbb{Q}(-m) & i = 2m,\ 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Using properties of $\mathbf{DMT}(k)$ such as the homotopy invariance or the long exact sequence of a closed immersion, we can show that certain motives are mixed Tate. For instance, if a variety X possesses a stratification such that the motive of each locally closed stratum is mixed Tate, then the whole $M(X)$ is a mixed Tate motive.

Example 4.85. Let $n \geq 3$ be an integer and consider the moduli space $M_{0,n}$ of distinct n-points in $\mathbb{P}^1$. It is a smooth variety of dimension $n-3$ which is defined over $\mathbb{Q}$. Since any three points can be sent to $0, 1, \infty$ by a projective transformation, one has $M_{0,3} = \operatorname{Spec}(\mathbb{Q})$ and $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In general, $M_{0,n} = (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus$ diagonals. We will write elements of $M_{0,n}$ as tuples $(0, 1, \infty, x_4, \ldots, x_n)$.

Let us show by induction that $M(M_{0,n})$ belongs to $\mathbf{DMT}(\mathbb{Q})$. The result is clear for $n = 3$ and 4. For $n \geq 5$, we can decompose $M_{0,n}$ as follows:

$$M_{0,n} \simeq (M_{0,4} \times M_{0,n-1}) \setminus \bigsqcup_{i=5}^{n} \{x_i = x_4\}.$$

By the Künneth formula and the induction hypothesis, the motive of $X = M_{0,4} \times M_{0,n-1}$ belongs to $\mathbf{DMT}(\mathbb{Q})$. The same is true for the motive of $Z = \bigsqcup_{i=5}^{n} \{x_i = x_4\}$. Now the Gysin triangle reads

$$M(M_{0,n}) \to M(X) \to M(Z)(1)[2] \to M(M_{0,n})[1]$$

and since $M(X)$ and $M(Z)(1)[2]$ belong to $\mathbf{DMT}(\mathbb{Q})$, so does $M(M_{0,n})$.

Example 4.86. Let $L = L_0 \cup \dots \cup L_n$ and $M = M_0 \cup \dots \cup M_n$ be hyperplanes in the projective space $\mathbb{P}^n$. Assume that they are in general position, meaning that the divisor $L \cup M$ has normal crossings. Then the following motive belongs to $\mathbf{MT}(k)$:

$$H^2(\mathbb{P}^n \setminus L, M \setminus (M \cap L)).$$

4.4.4. *Realizations.* Recall that in Definition 2.102 we introduced a category $\mathbf{MHTS}(\mathbb{Q})$ of mixed Hodge Tate structures over $\mathbb{Q}$. Then the functor R^{H} of Theorem 4.68 restricts to a functor

$$\mathbf{DMT}(\mathbb{Q}) \to D^b(\mathbf{MHTS}(\mathbb{Q})).$$

As explained in Example 4.34, the category appearing on the right-hand side has a canonical t-structure. We have also defined a t-structure on $\mathbf{DMT}(\mathbb{Q})$. Since it is motivic, any realization functor is t-exact in the sense of Definition 4.31, hence restricts to a functor on the hearts. Specializing to R^{H}, we obtain a functor from $\mathbf{MT}(\mathbb{Q})$ to $\mathbf{MHS}(\mathbb{Q})$. Taking into account that the Hodge realization of a mixed Tate motive is a mixed Hodge Tate structure, we actually get a functor

$$R^{\mathrm{H}} \colon \mathbf{MT}(\mathbb{Q}) \longrightarrow \mathbf{MHTS}(\mathbb{Q}) \tag{4.87}$$

which respects the weight filtrations.

It is important to note that the category $\mathbf{MHTS}(\mathbb{Q})$ is much bigger than $\mathbf{MT}(\mathbb{Q})$. For instance compare the set of extensions of $\mathbb{Q}(m)$ and $\mathbb{Q}(n)$ in the category $\mathbf{MHTS}(\mathbb{Q})$ given by Theorem 2.154, that is uncountable, with the set of extensions in $\mathbf{MT}(\mathbb{Q})$ given by Theorem 4.74, that is countable. Thus it is important to know which mixed Hodge structures come from geometry. This leads to the precise meaning to the word “motivic” when speaking about a mixed Hodge Tate structure:

Definition 4.88. We say that a mixed Hodge Tate structure over $\mathbb{Q}$ is *motivic* if it lies in the essential image of the functor R^{H}. The same definition applies to pro-mixed Hodge Tate structures. More generally, we say that a diagram of pro-mixed Hodge Tate structures is motivic if it is isomorphic to the image by the functor R^{H} of a diagram of pro-mixed Tate motives.

Even if $\mathbf{MHTS}(\mathbb{Q})$ is much bigger than $\mathbf{MT}(\mathbb{Q})$, the realization functor between them is fully faithful and stable by subobjects. This is a very useful result to prove that many mixed Hodge structures have motivic origin. We should mention that to determine whether the Hodge realization functor from the hypotetical category of mixed motives is fully faithful (*i.e.* bijective on Hom sets) would be a extremely difficult problem. For instance, if one restricts to the category of pure motives it amounts to the Hodge conjecture. That we can do it for $\mathbf{MT}(\mathbb{Q})$ relies again on Borel's results about the K-theory of number fields.

PROPOSITION 4.89 (Deligne-Goncharov). *The realization functor* (4.87) *is fully faithful and its essential image is stable under subobjects.*

PROOF. The key point of the argument is that the realization functor R^{H} determines injections

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \tag{4.90}$$

into the extension groups which were computed in Theorem 2.154. For $n = 1$, this follows from the injectivity of

$$\log|\cdot|\colon \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{C}/2\pi i\mathbb{Q}.$$

For $n > 1$, the injectivity follows by interpreting $\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$ as a part of the motivic cohomology of $\mathrm{Spec}(\mathbb{Q})$, which can be computed using K-theory:

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}(n)) = K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q},$$

then interpreting $\mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n))$ as Deligne cohomology groups:

$$\mathrm{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{Q}), \mathbb{Q}(n)).$$

Under this interpretation, the realization map (4.90) should correspond to the Borel regulator map mentioned in Digression 1.14, which is known to be injective by the work of Borel.

Consider now the fibre functors ω_{dR} on $\mathbf{MHS}(\mathbb{Q})$ (Definition 2.101) and ω on $\mathbf{MT}(\mathbb{Q})$ (4.82). These fibre functors are compatible and induce maps at the level of Tannaka groups

$$G^{\mathrm{H}}_{\omega_{\mathrm{dR}}} = \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MHTS}(\mathbb{Q})}(\omega_{\mathrm{dR}}) \to \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}(\mathbb{Q})}(\omega) = G_\omega. \tag{4.91}$$

By the Tannakian dictionary, the functor R^H is fully faithful if and only if the morphism (4.91) is surjective.

To show this we argue as follows: both $G^{\mathrm{H}}_{\omega_{\mathrm{dR}}}$ and G_ω can be written as the semidirect product of $\mathbb{G}_m$ and a pro-unipotent group.

$$G^{\mathrm{H}}_{\omega_{\mathrm{dR}}} = U^{\mathrm{H}}_{\omega_{\mathrm{dR}}} \rtimes \mathbb{G}_m, \quad G_\omega = U_\omega \rtimes \mathbb{G}_m.$$

Then the injectivity of (4.90) implies the surjectivity of (4.91) (see the proof of Theorem 4.123 for the precise relationship between the Ext groups and the Lie algebra of U_ω). □

EXAMPLE 4.92. Let $n > 0$ be an even integer and H a mixed Hodge structure over $\mathbb{Q}$ that is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. If this extension is non-trivial then it is not motivic over $\mathbb{Q}$, in the sense that it can not be the

Hodge realization of a motive over $\mathbb{Q}$. Indeed, assume that there is a mixed Tate motive over $\mathbb{Q}$ whose Hodge realization is H. Since the realization functor is fully faithful, from the exact sequence

$$0 \to \mathbb{Q}(n) \to H \to \mathbb{Q}(0) \to 0$$

corresponds an exact sequence of mixed Tate motives

$$0 \to \mathbb{Q}(n) \to M \to \mathbb{Q}(0) \to 0.$$

Since $\mathrm{Ext}^1_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ this extension is split. Hence the sequence of mixed Hodge structures is also split.

Of course, there exist motivic non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ defined over non-totally real number fields.

⋆⋆⋆

Exercise 4.93. Prove that the pair of subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of Example 4.34 forms indeed a t-structure.

Exercise 4.94. Let $\mathrm{Gr}(d, n)$ be the Grassmanian scheme of d-planes in k^n. Show that the motive of $\mathrm{Gr}(d, n)$ belongs to $\mathbf{DMT}(k)$.

4.5. Mixed Tate motives over $\mathbb{Z}$. From now on, we specialize further to the case $k = \mathbb{Q}$. The category $\mathbf{MT}(\mathbb{Q})$ is still too big for our purposes since the extension group

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \simeq \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \bigoplus_{p \text{ prime}} \mathbb{Q}$$

is infinite-dimensional. To remedy this, Goncharov [**Gon01**, §3] introduced a subcategory of "mixed Tate motives over $\mathbb{Z}$".

4.5.1. *Definition and basic properties.*

Definition 4.95. A motive M in $\mathbf{MT}(\mathbb{Q})$ is said to be *everywhere unramified* if, given any integer n, there is no subquotient E of M which fits into a non-split extension $0 \to \mathbb{Q}(n+1) \to E \to \mathbb{Q}(n) \to 0$. The full subcategory $\mathbf{MT}(\mathbb{Z})$ of $\mathbf{MT}(\mathbb{Q})$ consisting of everywhere unramified motives is called the category of *mixed Tate motives over* $\mathbb{Z}$.

To a motive M over $\mathbb{Q}$ and a prime number ℓ, we can associate the ℓ-adic realization of M. For instance, to the motive corresponding to a smooth variety X over $\mathbb{Q}$ we associate the dual of the ℓ-adic cohomology $H^*_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. The ℓ-adic realization is a $\mathbb{Q}_\ell$-vector space, together with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let p be a prime number distinct from ℓ. The choice of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a field embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ allows one to see the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}_p)$ as a subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By restriction, we obtain a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Recall that the Galois group of the maximal unramified extension $\mathbb{Q}_p \subset \mathbb{Q}_p^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ is isomorphic to $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The *inertia subgroup* I_p is defined by

$$1 \to I_p \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1.$$

DEFINITION 4.96. Let $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V)$ be an ℓ-adic representation, and p a prime number distinct from ℓ. We say that ρ is *unramified* at p if its restriction to the inertia subgroup $I_p \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is trivial.

We have at our disposal the following criterion to decide whether a mixed Tate motive over $\mathbb{Q}$ belongs to $\mathbf{MT}(\mathbb{Z})$.

PROPOSITION 4.97 (Deligne-Goncharov). *A mixed Tate motive M over $\mathbb{Q}$ belongs to $\mathbf{MT}(\mathbb{Z})$ if and only if, for each prime number p, there exists a prime $\ell \neq p$ such that the ℓ-adic realization $\omega_\ell(M)$ is unramified at p.*

PROOF. See [**DG05**, Prop. 1.8]. □

EXAMPLE 4.98. Let K_t^{Mot} be the Kummer motive associated to an element $t \in \mathbb{Q}^\times$ as in Example 4.75. For each prime ℓ, the ℓ-adic realization of K_t^{Mot} is the extension

$$0 \to \mathbb{Q}_\ell(1) \to K_t^\ell \xrightarrow{f} \mathbb{Q}(0) \to 0$$

corresponding to the $\mathbb{Q}_\ell(1)$-torsor given by the projective limit of ℓ^n-th roots of unity of t. This is unramified everywhere if and only if $t \in \mathbb{Z}^\times$. Thus, taking into account that $\mathbb{Z}^\times \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, the only Kummer motive that belongs to $\mathbf{MT}(\mathbb{Z})$ is the trivial one K_1^{Mot}. This solves the problem of the extension groups being infinite-dimensional.

The main properties of the category $\mathbf{MT}(\mathbb{Z})$ are summarized in the following theorem

THEOREM 4.99.

(1) $\mathbf{MT}(\mathbb{Z})$ *is a Tannakian category generated by the objects $\mathbb{Q}(n)$ for all integers $n \in \mathbb{Z}$.*

(2) Each object M of $\mathbf{MT}(\mathbb{Z})$ has a canonical increasing weight filtration W indexed by even integers, and such that

$$\mathrm{Gr}_{2n}^W M \cong \mathbb{Q}(-n)^{\oplus k_n}$$

for some integers $k_n \geq 0$.

(3) The extension groups in the category $\mathbf{MT}(\mathbb{Z})$ are given by

$$\mathrm{Ext}^i_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(l), \mathbb{Q}(m)) = \begin{cases} \mathbb{Q}, & \text{if } i = 0,\ m - l = 0, \\ \mathbb{Q}, & \text{if } i = 1,\ m - l \geq 3 \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence all of them are finite-dimensional.

Since $\mathbf{MT}(\mathbb{Z}) \subset \mathbf{MT}(\mathbb{Q})$ is stable under subobjects, we immediately deduce from Proposition 4.89:

Corollary 4.100. *The realization functor*

$$R\colon \mathbf{MT}(\mathbb{Z}) \to \mathbf{MHTS}(\mathbb{Q})$$

is fully faithful with essential image stable under subobjects.

4.5.2. *Fibre functors.* In this section, we introduce various fibre functors on the category $\mathbf{MT}(\mathbb{Z})$ and compute the corresponding Tannaka groups. The first one is defined using the weight structure on $\mathbf{MT}(\mathbb{Z})$ given by part (2) of Theorem 4.99. For each motive M in $\mathbf{MT}(\mathbb{Z})$ and each integer $n \in \mathbb{Z}$, we write

$$\omega_n(M) = \mathrm{Hom}_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathrm{Gr}^W_{-2n}(M))$$

and define a fibre functor $\omega\colon \mathbf{MT}(\mathbb{Z}) \to \mathrm{Vec}_{\mathbb{Q}}$ by

$$\omega(M) = \bigoplus_n \omega_n(M). \tag{4.101}$$

Observe that ω factors through the category of graded $\mathbb{Q}$-vector spaces.

From the Hodge realization of a motive we obtain two fibre functors. The de Rham fibre functor, denoted by ω_{dR}, is the de Rham part of the Hodge structure. For a motive $M \in \mathbf{MT}(\mathbb{Z})$, the vector space $\omega_{\mathrm{dR}}(M)$ comes equipped with two filtrations, the decreasing Hodge filtration F, and the increasing weight filtration W. Since $(\omega_{\mathrm{dR}}(M), F, W)$ is part of a mixed Tate Hodge structure, these filtrations are opposed in the sense that, if we write

$$\omega_{\mathrm{dR}}(M)^n = F^{-n}\omega_{\mathrm{dR}}(M) \cap W_{-2n}\omega_{\mathrm{dR}}(M),$$

then

$$\omega_{\mathrm{dR}}(M) = \bigoplus_n \omega_{\mathrm{dR}}(M)^n,$$

$$F^{-p}\omega_{\mathrm{dR}}(M) = \bigoplus_{m \le p} \omega_{\mathrm{dR}}(M)^m,$$

$$W_{-2n}\omega_{\mathrm{dR}}(M) = \bigoplus_{m \ge n} \omega_{\mathrm{dR}}(M)^m.$$

Thus the de Rham fibre functor ω_{dR} also factors through the category of graded vector spaces.

Lemma 4.102. *The de Rham fibre functor ω_{dR} is canonically isomorphic to the fibre functor ω.*

There is also a Betti fibre functor ω_{B} given by the Betti part of the Hodge realization. The rational vector space ω_{B} is provided with a weight filtration W, but not a Hodge filtration. Note that ω_{B} does not factor canonically through the category of graded vector spaces.

Finally there is a comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}}\colon \omega_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \omega_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.103}$$

EXAMPLE 4.104. In this example we compute explicitly the de Rham and Betti realizations of $\mathbb{Q}(1)$ and the comparison isomorphism. First we need a variety whose motive contains $\mathbb{Q}(1)$. Let

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\} = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \mathbb{G}_{m,\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[x, x^{-1}]).$$

Recall from Remark 4.61 that $M(X) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$, hence

$$t_0(M(X)[-i]) = \begin{cases} \mathbb{Q}(i), & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

We already have a nice compactification $X \subset \mathbb{P}^1_{\mathbb{Q}}$. We can write down explicitly the complex of differential forms on $\mathbb{P}^1_{\mathbb{Q}}$ with logarithmic poles along $\{0, \infty\}$. The sheaf $\Omega^0_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})$ is $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$, the sheaf of rational functions on $\mathbb{P}^1_{\mathbb{Q}}$. The sheaf $\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})$ is the $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$-module generated by the differential form $\frac{dx}{x} = -\frac{dx^{-1}}{x^{-1}}$. Thus, as a sheaf, is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$. Since

$$H^i(\mathbb{P}^1_{\mathbb{Q}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}) = 0, \quad \text{for } i > 0,$$

there is no need to search for a resolution of the complex $\Omega^*_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})$ and we can use directly the complex of global sections to compute de Rham cohomology. We have

$$\Gamma(\mathbb{P}^1_{\mathbb{Q}}, \Omega^0_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})) = \mathbb{Q}[x, x^{-1}],$$
$$\Gamma(\mathbb{P}^1_{\mathbb{Q}}, \Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})) = \mathbb{Q}[x, x^{-1}]\frac{dx}{x}.$$

The differential map is given by $dx^n = nx^{n-1}$. Hence

$$H^0_{\mathrm{dR}}(X) = \mathbb{Q}, \qquad H^1_{\mathrm{dR}}(X) = \mathbb{Q}\frac{dx}{x}.$$

Therefore

$$\omega_{\mathrm{dR}}(\mathbb{Q}(1)) = \left(\mathbb{Q}\frac{dx}{x}\right)^{\vee}.$$

Thus $\omega_{\mathrm{dR}}(\mathbb{Q}(1))$ is a one dimensional vector space and we have identified a canonical generator $(dx/x)^{\vee}$.

The Betti realization is given by the singular homology of the space of complex points. Thus

$$\omega_{\mathrm{B}}(\mathbb{Q}(1)) = H_1(\mathbb{C} \setminus \{0\}, \mathbb{Q})$$

This is again a rational vector space of dimension 1. A generator of it is given by the unit circle traveled in the counterclockwise direction, that we denote γ.

The comparison isomorphism is obtained from the integration of differential forms along singular chains. Since

$$\int_\gamma \frac{dx}{x} = 2\pi i$$

we deduce that $\mathrm{comp}_{\mathrm{dR,B}}(\gamma) = (dx/x)^\vee \otimes (2\pi i)$.

4.5.3. *Tannaka groups of* $\mathbf{MT}(\mathbb{Z})$. We now turn to the description of the affine group schemes associated to the various fibre functors on the category of mixed Tate motives over $\mathbb{Z}$.

NOTATION 4.105. The following notation will be used throughout:

$$G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{dR}}), \tag{4.106}$$

$$G_{\mathrm{B}} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{B}}), \tag{4.107}$$

$$P_{\mathrm{B,dR}} = \underline{\mathrm{Isom}}^{\otimes}(\omega_{dR}, \omega_{\mathrm{B}}), \tag{4.108}$$

$$P_{\mathrm{dR,B}} = \underline{\mathrm{Isom}}^{\otimes}(\omega_{\mathrm{B}}, \omega_{dR}). \tag{4.109}$$

Observe that both $P_{\mathrm{B,dR}}$ and $P_{\mathrm{dR,B}}$ are G_{dR}-torsors and $\mathrm{comp}_{\mathrm{B,dR}}$ (resp. $\mathrm{comp}_{\mathrm{dR,B}}$) is a complex point of $P_{\mathrm{B,dR}}$ (resp. $P_{\mathrm{dR,B}}$).

In what follows, we will use the subscript dR/B for properties which are common to G_{dR} and G_{B}.

LEMMA 4.110. *The groups $G_{\mathrm{dR/B}}$ fit into an exact sequence*

$$1 \longrightarrow U_{\mathrm{dR/B}} \longrightarrow G_{\mathrm{dR/B}} \longrightarrow \mathbb{G}_m \longrightarrow 1, \tag{4.111}$$

where $U_{\mathrm{dR/B}}$ is a pro-unipotent group.

PROOF. Recall that the category $\mathbf{MT}(\mathbb{Z})$ contains the object $\mathbb{Q}(1)$. Since $\omega_{\mathrm{dR/B}}(\mathbb{Q}(1))$ is a one-dimensional $\mathbb{Q}$-vector space, we obtain a morphism

$$t_{\mathrm{dR/B}} \colon G_{\mathrm{dR/B}} \to \mathrm{GL}(\omega_{\mathrm{dR/B}}(\mathbb{Q}(1))) = \mathbb{G}_m. \tag{4.112}$$

We define $U_{\mathrm{dR/B}}$ as the kernel of this morphism.

Since the action of $G_{\mathrm{dR/B}}$ is compatible with the tensor product, an element $g \in G_{\mathrm{dR/B}}$ acts on $\omega_{\mathrm{dR/B}}(\mathbb{Q}(n))$ as $t_{\mathrm{dR/B}}(g)^n$. Since the weight filtration is a filtration in the category of motives, $G_{\mathrm{dR/B}}$ respects the weight filtration. This means that, if $g \in G_{\mathrm{dR/B}}$ and $X \in \mathrm{Ob}(\mathbf{MT}(\mathbb{Z}))$, the action of g in $\omega_{\mathrm{dR/B}}(X)$ sends $W_n\omega_{\mathrm{dR/B}}(X) = \omega_{\mathrm{dR/B}}(W_nX)$ to $W_n\omega_{\mathrm{dR/B}}(X)$. Therefore, it acts on $\mathrm{Gr}_n^W \omega_{\mathrm{dR/B}}(X)$. Since $\mathrm{Gr}_n^W \omega_{\mathrm{dR/B}}(X)$ is a sum of copies of $\omega_{\mathrm{dR/B}}(\mathbb{Q}(n))$, g acts on $\mathrm{Gr}_n^W \omega_{\mathrm{dR/B}}(X)$ as $t_{\mathrm{dR/B}}(g)^n$ and the action of an element $u \in U_{\mathrm{dR/B}}$ on the same space is trivial. This implies that $U_{\mathrm{dR/B}}$ is a pro-unipotent group, that is, an inverse limit of unipotent affine algebraic groups. □

At this level, an advantage of using the de Rham fibre functor $\omega = \omega_{\mathrm{dR}}$ instead of the Betti one ω_{B} is that the exact sequence (4.111) admits a canonical splitting $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$. Indeed:

LEMMA 4.113. *One has*

$$G_{\mathrm{dR}} = U_{\mathrm{dR}} \rtimes \mathbb{G}_m.$$

PROOF. We use the fact that $\omega = \omega_{\mathrm{dR}}$ factors through the category of graded vector spaces. Given $t \in \mathbb{G}_m$, let $\tau(t) \in G_{\mathrm{dR}}$ denote the element that acts as multiplication by t^n on ω_n. This defines a section $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$ of t_{dR}. Hence G_{dR} is a semidirect product. □

COROLLARY 4.114. *Any G_{dR}-torsor is trivial.*

PROOF. We assume that the reader is familiar with the vanishing of the Galois cohomology groups

$$H^1(\mathbb{Q}, \mathbb{G}_m) = H^1(\mathbb{Q}, \mathbb{G}_a) = 0$$

(see for instance [**Wat79**, 18.2] or [**Ser94**, Chap. II, §1.2, Prop. 1]). It follows that, for any unipotent group U or any group G that is an extension of $\mathbb{G}_m$ by U, the Galois cohomology groups are also trivial

$$H^1(\mathbb{Q}, U) = H^1(\mathbb{Q}, G) = 0.$$

Now, the group G_{dR} can be written as

$$G_{\mathrm{dR}} = \varprojlim_N G^N_{\mathrm{dR}},$$

where each G^N_{dR} is an extension of $\mathbb{G}_m$ by a unipotent group and all the transition maps are surjective. By Mittag-Leffler we deduce that

$$H^1(\mathbb{Q}, G_{\mathrm{dR}}) = \varprojlim_N H^1(\mathbb{Q}, G^N_{\mathrm{dR}}) = 0,$$

which implies that any G_{dR}-torsor defined over $\mathbb{Q}$ is trivial. □

The corollary has the important following consequence, which will be exploited in the next chapter.

PROPOSITION 4.115. *There exists an element $a \in G_{\mathrm{dR}}(\mathbb{C})$ such that, for all motives M of $\mathbf{MT}(\mathbb{Z})$, one has*

$$\omega_B(M) = (\mathrm{comp}_{B,\mathrm{dR}} \circ a)(\omega_{dR}(M)). \tag{4.116}$$

Moreover, a can be chosen of the form $a = u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$.

PROOF. We follow [**Del89**, §8.10]. Recall from (4.108) that

$$P_{\mathrm{B,dR}} = \underline{\mathrm{Isom}}^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})$$

is a G_{dR}-torsor with a complex point $\mathrm{comp}_{\mathrm{B,dR}} \in P_{\mathrm{B,dR}}(\mathbb{C})$. In particular, $P_{\mathrm{B,dR}}$ is non-empty. This implies that $P_{\mathrm{B,dR}}$ has a $\overline{\mathbb{Q}}$-rational point, hence it is a trivial torsor over $\overline{\mathbb{Q}}$. By Corollary 4.114, the torsor has to be trivial already over $\mathbb{Q}$, which implies the existence of a rational point, that is an isomorphism of fibre functors $\alpha\colon \omega_{\mathrm{dR}} \xrightarrow{\sim} \omega_{\mathrm{B}}$. Define

$$a = \mathrm{comp}_{\mathrm{dR,B}} \circ \alpha. \tag{4.117}$$

By construction, a is an element of $G_{\mathrm{dR}}(\mathbb{C})$ and $\mathrm{comp}_{\mathrm{B,dR}} \circ a = \alpha$, from which (4.116) follows. Note also that any other element of $G_{\mathrm{dR}}(\mathbb{C})$ satisfying this propery is of the form $a\gamma$ with $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$.

Let us now turn to the second assertion, that a can be chosen of the form $u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$. This uses in a crucial way the compatibility between the comparison isomorphism and complex conjugation explained in Proposition 2.73. Interpreted in our context, it says that the following diagram of fibre functors is commutative:

$$\begin{tikzcd}
\omega_{\mathrm{dR}} \arrow[r, "\alpha"] \arrow[rrr, bend left, "a"] & \omega_{\mathrm{B}} \arrow[r, hook] & \omega_{\mathrm{B}} \otimes \mathbb{C} \arrow[r, "\mathrm{comp}"] & \omega_{\mathrm{dR}} \otimes \mathbb{C} \\
\omega_{\mathrm{dR}} \arrow[r, "\alpha"'] \arrow[rrr, bend right, "a"'] & \omega_{\mathrm{B}} \arrow[r, hook] \arrow[u, "\rho"] & \omega_{\mathrm{B}} \otimes \mathbb{C} \arrow[r, "\mathrm{comp}"] \arrow[u, "\rho\otimes c"] & \omega_{\mathrm{dR}} \otimes \mathbb{C}, \arrow[u, "\mathrm{Id}\otimes c"']
\end{tikzcd}$$

where ρ is the map induced from complex conjugation on the topological space and c is complex conjugation on the coefficients. Note that ρ is a rational point of G_{B}. The complex conjugate of a is $\overline{a} = \mathrm{Id} \otimes \sigma \circ a$. Define $x = a^{-1}\overline{a}$. By the commutativity of the diagram, $x = \alpha^{-1}\rho\alpha$. Thus $x \in G_{\mathrm{dR}}(\mathbb{Q})$ and has order two.

Let us apply (4.117) to the motive $\mathbb{Q}(1)$. Since

$$\mathrm{comp}_{\mathrm{dR,B}}\colon \omega_{\mathrm{B}}(\mathbb{Q}(1)) \longrightarrow \omega_{\mathrm{dR}}(\mathbb{Q}(1))$$

is multiplication by $2\pi i$ by Example 4.104 and $\alpha(\mathbb{Q}(1))$ is an invertible map of one-dimensional $\mathbb{Q}$-vector spaces, it follows that $t_{\mathrm{dR}}(a) \in \mathbb{G}_m$ lies in $2\pi i \mathbb{Q}^\times$. Thus, up to replacing a by $a\gamma$ with $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$, we can assume that

$$a^{-1}\overline{a} = \tau(-1). \tag{4.118}$$

Any other element satisfying both (4.116) and (4.118) is of the form $a\gamma$ for some $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$ such that $\gamma^{-1}\tau(-1)\gamma = \tau(-1)$. In particular, any $\gamma \in \tau(\mathbb{Q}^\times)$ works. Therefore, replacing a by $a\gamma$ with $\gamma \in \tau(\mathbb{Q}^\times)$, one can choose a such that $t_{\mathrm{dR}}(a) = 2\pi i$. This amounts to saying that $a = u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{C})$.

It remains to show that $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$. By (4.118),

$$\tau(2\pi i)^{-1} u_0^{-1} \overline{u_0} \tau(-2\pi i) = \tau(-1)$$

and writing $\tau(-1) = \tau(2\pi i)^{-1}\tau(-2\pi i)$ one gets $u_0 = \overline{u_0}$. $\square$

4.5.4. *The period map and the period conjecture.* Recall from the previous sections that $P_{\mathrm{dR,B}}$ denotes the scheme of tensor isomorphisms between ω_{B} and ω_{dR}, which has the structure of a pro-algebraic variety over $\mathbb{Q}$. The ring of regular functions $\mathcal{O}(P_{\mathrm{dR,B}})$ forms an ind-object in the category of $\mathbb{Q}$-algebras of finite type.

Definition 4.119. The *period map* is the ring morphism

$$\mathrm{per}\colon \mathcal{O}(P_{\mathrm{dR,B}}) \to \mathbb{C} \tag{4.120}$$

given by evaluation at the point $\mathrm{comp}_{\mathrm{dR,B}}$:

$$\mathrm{per}(f) = f(\mathrm{comp}_{\mathrm{dR,B}}).$$

Similarly, evaluation at the point $\mathrm{comp}_{\mathrm{B,dR}}$ yields a period map

$$\mathcal{O}(P_{\mathrm{B,dR}}) \to \mathbb{C}.$$

The following is a variant of Grothendieck's period conjecture for the category of mixed Tate motives over $\mathbb{Z}$ (cf. also [**And04**, 25.2]).

Conjecture 4.121 (Grothendieck). *The point* $\mathrm{comp}_{\mathrm{dR,B}}$ *is generic.*

To give a meaning to the word "generic", observe that, as in Lemma 4.21, $P_{\mathrm{B,dR}}$ can be written as the projective system of torsors $P^Y_{\mathrm{B,dR}}$ for mixed Tate motives Y. Then, by "generic" we mean that, for every quotient $P_{\mathrm{B,dR}} \to P^Y_{\mathrm{B,dR}}$ the image $\mathrm{comp}^Y_{\mathrm{B,dR}}$ of the point $\mathrm{comp}_{\mathrm{B,dR}}$ in $P^Y_{\mathrm{B,dR}}$ is not contained in any proper subvariety defined over $\mathbb{Q}$. Therefore $\mathrm{comp}_{\mathrm{B,dR}}$ is generic if and only if, for every mixed Tate motive, the period map

$$\mathrm{per} = \mathrm{ev}_{\mathrm{comp}^Y_{\mathrm{B,dR}}}\colon \mathcal{O}(P^Y_{\mathrm{B,dR}}) \longrightarrow \mathbb{C}$$

is injective. Moreover, if $\mathrm{comp}_{\mathrm{B,dR}}$ is generic, then the transcendence degree of the residue field of $\mathrm{comp}^Y_{\mathrm{B,dR}}$ is equal to the dimension of $P^Y_{\mathrm{B,dR}}$.

From the previous discussion, we see that Grothendieck's period conjecture for mixed Tate motives is equivalent to the following:

Conjecture 4.122. *The period map* (4.120) *is injective.*

4.5.5. *Lie algebras.* Let $\mathfrak{u}_{\mathrm{dR}}$ be the Lie algebra of U_{dR}. The decomposition $G_{\mathrm{dR}} = U_{\mathrm{dR}} \rtimes \mathbb{G}_m$ from Lemma 4.113 yields an action of $\mathbb{G}_m$ on $\mathfrak{u}_{\mathrm{dR}}$ which is compatible with the Lie algebra structure in that $t\cdot[a,b] = [t\cdot a, t\cdot b]$ for all $t \in \mathbb{G}_m$ and all $a, b \in \mathfrak{u}_{\mathrm{dR}}$. Let $\mathfrak{u}^n_{\mathrm{dR}} \subseteq \mathfrak{u}_{\mathrm{dR}}$ be the subspace where $t \in \mathbb{G}_m$ acts as multiplication by t^n. Then

$$[\mathfrak{u}^n_{\mathrm{dR}}, \mathfrak{u}^m_{\mathrm{dR}}] \subseteq \mathfrak{u}^{n+m}_{\mathrm{dR}}$$

and therefore we get a graded Lie algebra

$$\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{u}^n_{\mathrm{dR}}.$$

The fibre functor ω_{dR} induces an equivalence of categories between finite-dimensional graded vector spaces together with an action of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ compatible with the gradings and the category $\mathbf{MT}(\mathbb{Z})$.

The main result of this section is the following:

THEOREM 4.123. *The graded Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is free with one generator in each positive odd degree $n \geq 3$.*

The theorem will be a consequence of Lemma 4.126 below. Since we have not found a suitable reference, we include a proof of it.

Recall that a finite-dimensional Lie algebra $\mathfrak{L}$ is said to be *nilpotent* if there exists an integer n such that $[a, [a, \overset{n}{\ldots}, [a, b] \ldots]$ for all $a, b \in \mathfrak{L}$. This definition admits several generalizations to infinite-dimensional Lie algebras. The one that will be useful for us is the following:

DEFINITION 4.124. A Lie algebra $\mathfrak{L}$ is called *quasi-nilpotent* if

$$\bigcap_n [\mathfrak{L}, [\mathfrak{L}, \overset{n}{\ldots}, [\mathfrak{L}, \mathfrak{L}] \ldots] = 0.$$

EXAMPLES 4.125. Any nilpotent Lie algebra is quasi-nilpotent. A pro-nilpotent Lie algebra is quasi-nilpotent. The graded Lie algebra associated to a pro-nilpotent graded Lie algebra is also quasi-nilpotent. Any subalgebra of a quasi-nilpotent Lie algebra is quasi-nilpotent.

LEMMA 4.126. *Let $\mathfrak{L} = \bigoplus_n \mathfrak{L}_n$ be a quasi-nilpotent graded Lie algebra over $\mathbb{Q}$ with $H_1(\mathfrak{L}, \mathbb{Q})$ concentrated in positive degrees and $H_2(\mathfrak{L}, \mathbb{Q}) = 0$. Then $\mathfrak{L}$ is isomorphic to the free algebra generated by $H_1(\mathfrak{L}, \mathbb{Q})$.*

PROOF. We use the Koszul complex of $\mathfrak{L}$ to compute its homology

$$\ldots \longrightarrow \mathfrak{L} \wedge \mathfrak{L} \wedge \mathfrak{L} \longrightarrow \mathfrak{L} \wedge \mathfrak{L} \xrightarrow{[\ ,\ ]} \mathfrak{L} \xrightarrow{0} \mathbb{Q},$$

where the last map in the complex is the zero map and the previous to the last is given by the Lie bracket. From this complex we derive the well known identity

$$H_1(\mathfrak{L}, \mathbb{Q}) = \mathfrak{L}/[\mathfrak{L}, \mathfrak{L}].$$

The map $\mathfrak{L} \to H_1(\mathfrak{L}, \mathbb{Q})$ is homogeneous and surjective, thus we can choose a homogeneous lifting $H_1(\mathfrak{L}, \mathbb{Q}) \to \mathfrak{L}$. In general, this lifting is non-canonical. Let $\mathfrak{F}$ be the free Lie algebra generated by $H_1(\mathfrak{L}, \mathbb{Q})$. It is a graded algebra. By the universal property of free Lie algebras, the chosen lifting defines a graded map $\mathfrak{F} \to \mathfrak{L}$. We want to show that this map is an isomorphism.

Let F_n denote the increasing filtration of $\mathfrak{L}$ and $\mathfrak{F}$ given by the degree:

$$F_n \mathfrak{L} = \bigoplus_{n' \leq n} \mathfrak{L}_{n'}, \quad F_n \mathfrak{F} = \bigoplus_{n' \leq n} \mathfrak{F}_{n'}.$$

We prove by induction on $n \geq 0$ that the map $F_n \mathfrak{F} \to F_n \mathfrak{L}$ is surjective. By construction, $F_0 \mathfrak{F} = 0$. Since $\mathfrak{L}$ is graded, we deduce that $F_0 \mathfrak{L}$ is a Lie subalgebra. Since $\mathfrak{L}$ is quasi-nilpotent, the same is true for $F_0 \mathfrak{L}$. Since

$H_1(\mathfrak{L}, \mathbb{Q})$ is concentrated in positive degrees, $F_0\mathfrak{L}$ is also perfect: $F_0\mathfrak{L} = [F_0\mathfrak{L}, F_0\mathfrak{L}]$. This implies that $F_0\mathfrak{L} = \{0\}$ so we get the case $n = 0$ in the induction process.

We assume now that $F_{n'}\mathfrak{F} \to F_{n'}\mathfrak{L}$ is surjective for all $n' < n$. Since we can write

$$F_n\mathfrak{L}/F_{n-1}\mathfrak{L} = H_1(\mathfrak{L}, \mathbb{Q})_n + [\mathfrak{L}, \mathfrak{L}]_n$$

the definition of $\mathfrak{F}$, the fact that $F_0\mathfrak{L} = 0$ and the induction hypothesis imply that the map $F_n\mathfrak{F} \to F_n\mathfrak{L}$ is surjective. Since $\mathfrak{L}$ is graded,

$$\mathfrak{L} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{L}_n = \bigcup_{n \geq 0} F_n\mathfrak{L},$$

and we conclude the surjectivity of $\mathfrak{F} \to \mathfrak{L}$.

Let now $\mathfrak{k} \subset \mathfrak{F}$ denote the kernel of the map $\mathfrak{F} \to \mathfrak{L}$. We have a commutative diagram

$$\begin{tikzcd}
& \mathfrak{F} \wedge \mathfrak{k} \arrow[r] \arrow[d, hook] & \mathfrak{k} \arrow[d, hook] & \\
\mathfrak{F} \wedge \mathfrak{F} \wedge \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \wedge \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[r, two heads] \arrow[d] & H_1(\mathfrak{L}, \mathbb{Q}) \arrow[d, equal] \\
\mathfrak{L} \wedge \mathfrak{L} \wedge \mathfrak{L} \arrow[r] & \mathfrak{L} \wedge \mathfrak{L} \arrow[r] & \mathfrak{L} \arrow[r, two heads] & H_1(\mathfrak{L}, \mathbb{Q}),
\end{tikzcd}$$

where $\mathfrak{F} \wedge \mathfrak{k}$ is the image of $\mathfrak{F} \otimes \mathfrak{k}$ in $\mathfrak{F} \wedge \mathfrak{F}$. The long vertical sequences and the upper long horizontal sequence are exact by definition. The lower long sequence is exact because $H_2(\mathfrak{L}, \mathbb{Q}) = 0$. From this we deduce

$$\mathfrak{k} \subset [\mathfrak{k}, \mathfrak{F}].$$

Since $\mathfrak{F}$ is also quasi-nilpotent we conclude that $\mathfrak{k} = 0$, thus showing the injectivity of the map $\mathfrak{F} \to \mathfrak{L}$. $\square$

Proof of Theorem 4.123. We start by computing the Lie algebra cohomology of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$. To this end, let $\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})$ (respectively $\mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}})$) denote the category of continuous $\mathbb{Q}$-linear representations of U_{dR} (respectively G_{dR}), not necessarily of finite dimension. We have a fully faithful functor

$$\mathbf{MT}(\mathbb{Z}) = \mathbf{Rep}_{\mathbb{Q}}(G_{\mathrm{dR}}) \longrightarrow \mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}}).$$

In particular, there are representations $\mathbb{Q}(n)$ of G_{dR} on which G_{dR} acts through its quotient $\mathbb{G}_m$. Then

$$H^i(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) = \mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathbb{Q}),$$

where $\mathbb{Q}$ is viewed as the trivial representation of U_{dR}.

In order to compute the groups $\mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathbb{Q})$ we will use the theory of induction and restriction of representations. From the inclusion $U_{\mathrm{dR}} \to G_{\mathrm{dR}}$ we have a functor from the category of representations of G_{dR}

to the category of representations of U_{dR} that consist simply in restricting the group that act. This functor is denoted $\mathrm{Res}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}$. This functor admits a left adjoint denoted $\mathrm{Ind}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}$.

The properties we need are the computations

$$\mathrm{Res}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}) = \mathbb{Q}, \quad \text{and} \quad \mathrm{Ind}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}) = \prod_{n\in\mathbb{Z}} \mathbb{Q}(n),$$

and the adjoint property. Then

$$\begin{aligned}
\mathrm{Ext}^i_{\mathbf{Rep}^\infty_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q},\mathbb{Q}) &= \mathrm{Ext}^i_{\mathbf{Rep}^\infty_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathrm{Res}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}))\\
&= \mathrm{Ext}^i_{\mathbf{Rep}^\infty_{\mathbb{Q}}(G_{\mathrm{dR}})}(\mathrm{Ind}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}), \mathbb{Q})\\
&= \mathrm{Ext}^i_{\mathbf{Rep}^\infty_{\mathbb{Q}}(G_{\mathrm{dR}})}(\prod_{n\in\mathbb{Z}} \mathbb{Q}(n), \mathbb{Q})\\
&= \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_{\mathbf{Rep}^\infty_{\mathbb{Q}}(G_{\mathrm{dR}})}(\mathbb{Q}(n), \mathbb{Q}).
\end{aligned}$$

It follows from part (3) of Theorem 4.99 that

$$\begin{aligned}
H^1(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) &= \bigoplus_{\substack{n\leq -3\\ n \text{ odd}}} \mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathbb{Q}(0)),\\
H^2(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) &= 0,
\end{aligned}$$

where each summand $\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathbb{Q}(0))$ is one-dimensional and sits in odd degree $n \leq -3$. Going to homology we deduce that

$$H_1(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) = \bigoplus_{\substack{n\geq 3\\ n \text{ odd}}} \mathbb{Q}, \tag{4.127}$$

$$H_2(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) = 0. \tag{4.128}$$

To prove the theorem we only need to show that $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ satisfies the hypothesis of Lemma 4.126. By definition, it is a graded Lie algebra. Since U_{dR} is pro-unipotent, we deduce that $\mathfrak{u}_{\mathrm{dR}}$ is pro-nilpotent, hence $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is quasi-nilpotent. The other assumptions of the lemma are nothing but conditions (4.127) and (4.128) above. □

Remarks 4.129.

(1) Following [**DG05**] and [**Del13**], the grading on $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ that we consider is the one coming from the action of $\mathbb{G}_m$, where t acts as t on $\mathbb{Q}(1)$. This is why we obtain a positively graded Lie algebra in contrast with [**And04**] or [**Bro12**] that have a negatively graded Lie algebra.

(2) Consider the abelianization

$$(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}} = \mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}/[\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}],$$

which is a graded vector space. The proof of Theorem 4.123 yields a canonical identification

$$(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}_n = (\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)))^\vee.$$

Moreover, $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is isomorphic to the free Lie algebra generated by $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$. Nevertheless, there is no canonical lifting from $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$ to $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$, hence no canonical isomorphism between $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ and the free Lie algebra generated by $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$.

(3) Note also that $\mathfrak{u}_{\mathrm{dR}}$ and $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ are not isomorphic. In fact, $\mathfrak{u}_{\mathrm{dR}}$ is the completion of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ with respect to the grading, which implies that $\mathfrak{u}_{\mathrm{dR}}$ is not a free Lie algebra.

4.5.6. *The Hilbert-Poincaré series.* From Theorem 4.123, we deduce that the universal enveloping algebra $\mathcal{U}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is the free associative graded algebra with one generator in each odd degree $n \geq 3$. The algebra of regular functions $\mathcal{O}(U_{\mathrm{dR}})$ is also graded and is the dual of the completed universal enveloping algebra $\hat{\mathcal{U}}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ in the graded sense.

For simplicity we will consider the grading by the codegree in $\mathcal{O}(U_{\mathrm{dR}})$ that is the opposite of the one induced by the grading of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$. Thus it is also positively graded. We can compute its Hilbert-Poincaré series

$$\begin{aligned} H_{\mathcal{O}(U_{\mathrm{dR}})}(t) &= \frac{1}{1-t^3-t^5-t^7-\dots} \\ &= \frac{1-t^2}{1-t^2-t^3} \end{aligned} \tag{4.130}$$

from the dimension of the graded pieces of $\mathcal{U}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$.

Let us now, somehow artificially, introduce the algebra

$$\mathcal{H}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}}) \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{4.131}$$

where f_2 is in degree 2. From (4.130) we immediately deduce:

LEMMA 4.132. *The Hilbert-Poincaré series of $\mathcal{H}^{\mathcal{MT}}$ is given by*

$$H_{\mathcal{H}^{\mathcal{MT}}}(t) = \frac{1}{1-t^2-t^3} = \sum_{k\geq 0} d_k t^k,$$

where the integers d_k are the same as in Zagier's Conjecture 1.71.

Following Deligne, Goncharov and Terasoma, in order to prove the upper bound $\dim \mathcal{Z}_k \leq d_k$ of Theorem 1.95, we will construct in Chapter 5 a $\mathbb{Q}$-algebra $\mathcal{H}$, which injects into $\mathcal{H}^{\mathcal{MT}}$, and comes together with a surjective graded map $\mathcal{H} \to \bigoplus \mathcal{Z}_k$. This will imply immediately the bound. The reason we have changed the grading of $\mathcal{O}(U_{\mathrm{dR}})$ is precisely to make this map compatible with the degree. We have already seen that multiple zeta values appear as periods of the pro-unipotent completion of the fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. The motivic interpretation will give the link between $\mathcal{H}$ and $\bigoplus \mathcal{Z}_k$.

⋆⋆⋆

EXERCISE 4.133. Find examples which show that all the hypothesis in Lemma 4.126 are needed.

4.6. The motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. We continue considering the algebraic variety

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$$

over $\mathbb{Q}$ and the complex manifold

$$M = X(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}.$$

As in Section 3.9, we set:

$\mathbf{0}$=the tangential base point $(0,1)$, *i.e.* the tangent vector 1 at 0,

$\mathbf{1}$=the tangential base point $(1,-1)$, *i.e.* the tangent vector -1 at 1.

Let $\boldsymbol{x}, \boldsymbol{y} \in X(\mathbb{Q}) \cup \{\mathbf{0}, \mathbf{1}\}$ be rational or tangential base points. The aim of this section is to explain that the pro-unipotent completion of the torsor of paths from $\boldsymbol{x}$ to $\boldsymbol{y}$, as well as the extra structures given by composition of paths and local monodromy, are motivic in the sense of Definition 4.88. In fact, we want to add to Summary 3.254 a motivic side whose Betti and de Rham realizations give the Betti and de Rham sides of that summary. To exhibit the motivic nature of the group schemes and torsors in that summary, it seems necessary to use the language of algebraic geometry over a Tannakian category [**Del89**, §6]. In order to avoid this language, we will only consider the motivic analogues of ${}_{\bullet}U^{?}_{\bullet}$ and ${}_{\bullet}\mathcal{L}^{?}_{\bullet}$.

4.6.1. *The pro-mixed Tate motive* ${}_yU^{\mathrm{Mot}}_x$. We start with the case of two rational base points $x, y \in X(\mathbb{Q}) \subseteq M$. Recall the cosimplicial manifold ${}_yM^{\bullet}_x$ from Construction 3.189. As we already used in Section 3.6.1, when endowing the fundamental group with a mixed Hodge structure over $\mathbb{Q}$, all the maps involved in ${}_yM^{\bullet}_x$ are algebraic and, the points x, y being rational, defined over $\mathbb{Q}$. We will denote by ${}_yX^{\bullet}_x$ the corresponding cosimplicial object in the category $\mathbf{Sm}(\mathbb{Q})$.

As explained in Section 4.3.9, to ${}_yX^{\bullet}_x$ one associates a family of motives

$$\{[\sigma_{\leq N}\mathcal{N}_y X^{\bullet}_x]\}_{N\geq 0}.$$

By construction, given integers $M \geq N \geq 0$, there is a morphism

$$\sigma_{\leq M}\mathcal{N}_y X^{\bullet}_x \to \sigma_{\leq N}\mathcal{N}_y X^{\bullet}_x$$

making $\{[\sigma_{\leq N}\mathcal{N}_y X^{\bullet}_x]\}_{N\geq 0}$ into a projective system of motives.

LEMMA 4.134. *The object* $[\sigma_{\leq N}\mathcal{N}_y X^{\bullet}_x]$ *belongs to* $\mathbf{DMT}(\mathbb{Q})$.

PROOF. Exercise 4.162. □

We can therefore consider its cohomology with respect to the t-structure of $\mathbf{DMT}(\mathbb{Q})$.

DEFINITION 4.135. For each $N \geq 0$, we define a mixed Tate motive

$$ {}_yU_x^{\mathrm{Mot},N} = H_0([\sigma_{\leq N}\mathcal{N}\,{}_yX_x^\bullet]) \in \mathbf{MT}(\mathbb{Q}). $$

As N varies, these motives fit into a pro-mixed Tate motive ${}_yU_x^{\mathrm{Mot}}$.

We also consider the constant cosimplicial variety $\mathrm{Spec}(\mathbb{Q})^\bullet$ given by $\mathrm{Spec}(\mathbb{Q})$ in all degrees, with coface and codegeneracy maps all equal to the identity. Applying the previous construction to $\mathrm{Spec}(\mathbb{Q})^\bullet$, one easily finds (Exercise 4.163) that, for all $N \geq 0$,

$$ H_0([\sigma_{\leq N}\mathcal{N}\,\mathrm{Spec}(\mathbb{Q})^\bullet]) = \mathbb{Q}(0). $$

4.6.2. *The structures of* ${}_yU_x^{\mathrm{Mot}}$. We next introduce some extra structures carried by ${}_yU_x^{\mathrm{Mot}}$: the unit and counit, the completed coproduct, the composition of paths and the antipode. The idea is to give a geometric analogue of the constructions in the reduced bar complex of a connected dg-algebra (see Definition 3.115), in such a way that they are compatible with the isomorphism from Lemma 3.193.

We start with the unit and counit. Each point $x \in X(\mathbb{Q})$ determines a morphism of cosimplicial varieties

$$ \eta_x^\vee \colon \mathrm{Spec}(\mathbb{Q})^\bullet \longrightarrow {}_xX_x^\bullet \tag{4.136} $$

which sends $\mathrm{Spec}(\mathbb{Q})^n = \mathrm{Spec}(\mathbb{Q})$ to the point $(x,\dots,x) \in {}_xX_x^n$. Besides, we have for each pair of points $x, y \in X(\mathbb{Q})$ a map of cosimplicial varieties

$$ \epsilon^\vee \colon {}_yX_x^\bullet \longrightarrow \mathrm{Spec}(\mathbb{Q})^\bullet \tag{4.137} $$

given by the structural map in all degrees. These induce morphisms

$$ \eta_x^\vee \colon \mathbb{Q}(0) \longrightarrow {}_xU_x^{\mathrm{Mot}}, $$
$$ \epsilon^\vee \colon {}_yU_x^{\mathrm{Mot}} \longrightarrow \mathbb{Q}(0), $$

which are called *unit* and *counit* respectively.

REMARK 4.138. To understand the notation we will use in the following constructions, recall from 4.3.3 that the direct sum in in the category $\mathbf{SmCor}(\mathbb{Q})$ corresponds to the disjoint union of varieties, whereas the tensor product is given by the cartesian product of varieties. Note also that the description we will give of morphisms should be understood as correspondences. For instance, the map for the antipode below is the cycle in $X^n \times X^n$ given by $(-1)^{\frac{n(n+1)}{2}}\Gamma$, where Γ is the graph of the map $(x_1,\dots,x_n) \mapsto (x_n,\dots,x_1)$.

For any two rational points $x, y \in X(\mathbb{Q})$, consider the unbounded complex $\mathcal{C}^*({}_yX_x^\bullet)$ in the category $\mathbf{SmCor}(\mathbb{Q})$ given by

$$\mathcal{C}^n({}_yX_x^\bullet) = {}_yX_x^n,$$

together with the differential

$$d = \sum_{i=0}^{n+1}(-1)^i\delta^i\colon\ \mathcal{C}^n({}_yX_x^\bullet) \longrightarrow \mathcal{C}^{n+1}({}_yX_x^\bullet).$$

We consider the morphism

$$[X]^{\otimes n} \longrightarrow \bigoplus_{p+q=n}[X]^{\otimes p}\otimes[X]^{\otimes q}$$

in $\mathbf{SmCor}(\mathbb{Q})$ that sends the point $(x_1, \dots, x_n)$ to

$$\sum_{p+q=n}\sum_{\sigma\in\sqcup\!\sqcup(p,q)}(-1)^\sigma(x_{\sigma(1)},\dots,x_{\sigma(p)})\otimes(x_{\sigma(p+1)},\dots,x_{\sigma(n)}), \tag{4.139}$$

where $(-1)^\sigma$ is the sign of the permutation σ.

Remark 4.140. Notice that what appears in this product is the permutation σ instead of σ^{-1} as in Proposition 1.151 or Definition 3.115. This is due to the contravariant nature of differential forms.

One can check that this map induces a morphism of complexes

$$\nabla^\vee\colon \mathcal{C}^*({}_yX_x^\bullet) \longrightarrow \mathcal{C}^*({}_yX_x^\bullet)\otimes\mathcal{C}^*({}_yX_x^\bullet).$$

Now, for points $x, y, z \in X(\mathbb{Q})$, and integers $p, q \geq 0$, we consider the map

$$[X]^{\otimes p}\otimes[X]^{\otimes q} \to [X]^{\otimes(p+q)}$$

given by

$$(x_1,\dots,x_p)\otimes(y_1,\dots,y_q) \longmapsto (x_1,\dots,x_p,y_1,\dots,y_q). \tag{4.141}$$

Varying p, q we obtain a morphism of complexes

$$\Delta^\vee\colon \mathcal{C}^*({}_zX_y^\bullet)\otimes\mathcal{C}^*({}_yX_x^\bullet) \longrightarrow \mathcal{C}^*({}_zX_x^\bullet).$$

Finally, the correspondence $[X]^{\otimes n} \to [X]^{\otimes n}$ given by

$$(x_1,\dots,x_n) \longmapsto (-1)^{\frac{n(n+1)}{2}}(x_n,\dots,x_1) \tag{4.142}$$

defines a morphism of complexes, called the dual antipode,

$$S^\vee\colon \mathcal{C}^*({}_yX_x^\bullet) \longrightarrow \mathcal{C}^*({}_xX_y^\bullet).$$

The next step is to induce morphisms at the level of the normalized complexes $\mathcal{N}({}_yX_x^\bullet)$. For this, one needs to check that the chain morphisms commute with the projector p_n of Lemma 4.64 and take care of the truncations. The precise statement is the following lemma whose proof is elementary.

Lemma 4.143. *Let $N, M \geq 0$ be integers.*

(1) If $N \geq 2M$, the map $\nabla^\vee$ induces a morphism of complexes

$$\nabla^\vee \colon \sigma_{\leq N}\mathcal{N}({}_yX_x^\bullet) \longrightarrow \sigma_{\leq M}\mathcal{N}({}_yX_x^\bullet) \otimes \sigma_{\leq M}\mathcal{N}({}_yX_x^\bullet).$$

(2) If $N \geq M$, the map $\Delta^\vee$ induces a morphism of complexes

$$\Delta^\vee \colon \sigma_{\leq N}\mathcal{N}({}_zX_y^\bullet) \otimes \sigma_{\leq N}\mathcal{N}({}_yX_x^\bullet) \longrightarrow \sigma_{\leq M}\mathcal{N}({}_zX_x^\bullet).$$

(3) If $N \geq M$, the map $S^\vee$ induces a morphism of complexes

$$S^\vee \colon \sigma_{\leq N}\mathcal{N}({}_yX_x^\bullet) \longrightarrow \sigma_{\leq M}\mathcal{N}({}_xX_y^\bullet).$$

Moreover, when N and M vary within the above constraints, the three morphisms yield maps of projective systems.

As a consequence of Lemma 4.143 we obtain the following result.

Proposition 4.144. *Given any three points $x, y, z \in X(\mathbb{Q})$, there are morphisms of pro-mixed Tate motives*

(1) a composition of paths

$$\Delta^\vee \colon {}_zU_y^{\mathrm{Mot}} \otimes {}_yU_x^{\mathrm{Mot}} \longrightarrow {}_zU_x^{\mathrm{Mot}};$$

(2) a unit

$$\eta_x^\vee \colon \mathbb{Q}(0) \longrightarrow {}_xU_x^{\mathrm{Mot}};$$

(3) a completed coproduct

$$\nabla^\vee \colon {}_yU_x^{\mathrm{Mot}} \longrightarrow {}_yU_x^{\mathrm{Mot}} \hat{\otimes} {}_yU_x^{\mathrm{Mot}};$$

(4) a counit

$$\epsilon^\vee \colon {}_yU_x^{\mathrm{Mot}} \longrightarrow \mathbb{Q}(0);$$

(5) a dual antipode

$$S^\vee \colon {}_yU_x^{\mathrm{Mot}} \longrightarrow {}_xU_y^{\mathrm{Mot}}.$$

4.6.3. *The motivic nature of the fundamental groupoid of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$.*

Theorem 4.145 (Deligne-Goncharov [**DG05**]). *For $x, y \in X(\mathbb{Q})$, the Hodge realization of ${}_yU_x^{\mathrm{Mot}}$ agrees with the pro-mixed Hodge structure ${}_yU_x^{\mathrm{H}}$ described in Summary 3.254:*

$$R^{\mathrm{H}}({}_yU_x^{\mathrm{Mot}}) = {}_yU_x^{\mathrm{H}}.$$

Moreover, R^{H} is compatible with the composition of paths, the unit, the completed coproduct, the counit and the dual antipode. In particular, the diagram ${}_\bullet U_*^{\mathrm{H}}$ for $\bullet, *$ varying in rational base points, is motivic.

Proof. Let A^* be the differential graded algebra given in Example 2.130. Recall that it is given by

$$A^0 = \mathbb{Q}, \quad A^1 = \mathbb{Q}\,\omega_0 \oplus \mathbb{Q}\,\omega_1$$

with zero differential. The product in this algebra satisfies $\omega_0 \wedge \omega_1 = 0$. The Hodge filtration is given by

$$F^0 = A^* \supset F^1 = A^1 \supset F^2 = 0$$

and the weight filtration by

$$W_{-1} = 0 \subset W_0 = A^0 \subset W_1 = A^*.$$

As we have seen in Proposition 2.132, the differential graded algebra A^* allows us to compute the de Rham cohomology of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ with its weight and Hodge filtration. We have seen also in Section 3.6.2 that it can be used to compute the de Rham side of ${}_\bullet U^{\mathrm{H}}_*$.

We will now use this algebra to compute the de Rham side of $R^{\mathrm{H}}({}_y U_x^{\mathrm{Mot}})$.

Consider the variety $(\mathbb{P}^1_{\mathbb{Q}})^n$ and the divisor D_n consisting of all points with one coordinate equal to $0, 1$ or ∞. This is a simple normal crossing divisor. Then, for every pair of rational points $x, y \in X(\mathbb{Q})$, the n-th component of the cosimplicial scheme ${}_y X_x^\bullet$ is given by

$${}_y X_x^n = (\mathbb{P}^1_{\mathbb{Q}})^n \setminus D_n.$$

Let $(E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n), F, W)$ be the de Rham algebra of complex valued smooth differential forms on $(\mathbb{P}^1(\mathbb{C}))^n$ with logarithmic poles along D_n with its Hodge and weight filtration (see Section 2.6.1). We now denote

$$A^*({}_y X_x^n) = A^* \otimes \overset{n}{\cdots} \otimes A^*.$$

The Hodge and weight filtrations of A^* induce Hodge and weight filtrations on $A^*({}_y X_x^n)$. For all rational points x, y and integer $n \geq 0$, there is an inclusion

$$A^*({}_y X_x^n) \hookrightarrow E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n)$$

given by

$$1 \otimes \cdots \otimes \omega_{\varepsilon_i} \otimes \cdots \otimes 1 \mapsto \omega_{\varepsilon_i}(t_i),$$

where $\varepsilon_i = 0, 1$, the 1-form ω_{ε_i} is in the position i and t_i is the i-th coordinate of $\mathbb{A}^n_{\mathbb{C}} \subset (\mathbb{P}^1_{\mathbb{C}})^n$. From the fact that

$$A^* \otimes \mathbb{C} \to E^*_{\mathbb{P}^1(\mathbb{C})}(\log D_n)$$

is a bifiltered quasi-isomorphism (see the end of Example 2.130) we deduce that the map

$$A^*({}_y X_x^n) \otimes \mathbb{C} \hookrightarrow E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n)$$

is also a bifiltered quasi-isomorphism. Thus $A^*({}_y X_x^n)$ determine the Hodge and weight filtration of the de Rham cohomology of ${}_y X_x^n$, even with its $\mathbb{Q}$-structure. The important point to note now, that is easy to check, is that the previous inclusions are functorial with respect to any morphism involved in the structures of ${}_y X_x^n$. More precisely

LEMMA 4.146. *The family of inclusions*

$$A^*({}_yX_x^n) \hookrightarrow E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n), \tag{4.147}$$

for $x, y \in X(\mathbb{Q})$ and $n \geq 0$ is functorial with respect to

(1) *the coface and codegeneracy maps of the cosimplicial schemes ${}_yX_x^\bullet$;*
(2) *the maps (4.136) and (4.137), where we identify* $\operatorname{Spec}(\mathbb{Q})$ *with ${}_yX_x^0$ through the structure map of $\mathbb{Q}$-schemes;*
(3) *the maps (4.139), (4.141) and (4.142) that will induce the product, the coproduct and the antipode.*

Moreover, each map in the family is a filtered quasi-isomorphism.

PROOF. The fact that each map in the family is a quasi-isomorphism has already been discussed. To be precise of the meaning of functoriality in this lemma we spell out the case of a coface, being all the other maps treated in a similar way. Consider the coface

$$\delta^0 \colon {}_yX_x^n \longrightarrow {}_yX_x^{n+1}$$

given by $\delta^0(x_1, \dots, x_n) = (y, x_1, \dots, x_n)$. Then there is a diagram

$$\begin{tikzcd}
A^*({}_yX_x^{n+1}) \arrow[r] & E^*_{\mathbb{P}^1(\mathbb{C})^{n+1}}(\log D_{n+1}) \arrow[d, "(\delta^0)^*"] \\
A^*({}_yX_x^n) \arrow[r] & E^*_{\mathbb{P}^1(\mathbb{C})^n}(\log D_n).
\end{tikzcd}$$

The statement of the lemma means that there is a unique morphism, also denoted by $(\delta^0)^*$,

$$A^*({}_yX_x^{n+1}) \longrightarrow A^*({}_yX_x^n)$$

completing the diagram to a commutative square. By the fact that the horizontal arrows are injective the unicity is clear and we have to show the existence. The needed map is obviously given by

$$(\delta^0)^*(a_1 \otimes \dots \otimes a_{n+1}) = \varepsilon(a_1)a_2 \otimes \dots \otimes a_{n+1},$$

where ε is the augmentation of A^* given by (3.197). All the remaining maps are defined in a similar way. The compatibility of all the morphims with the composition of maps is just a consequence of the injectivity of the morphisms (4.147). □

The main consequence of Lemma 4.146 is that to compute the de Rham realization functor as explained in section 4.3.10 we can use the algebras $A^*({}_yX_x^n)$ and we deduce that

$$R^{\mathrm{dR}}({}_yU_x^{\mathrm{Mot}})^\vee = \varinjlim_N H_0\big(\operatorname{Tot}\sigma_{\leq N}\mathcal{N}A^*({}_yM_x^\bullet)\big).$$

By Lemma 3.193, there is a canonical isomorphism

$$\operatorname{Tot}\mathcal{N}A^*_\bullet \xrightarrow{\sim} B^*(A^*).$$

Taking the truncation, the cohomological functor H^0 and the inductive limit we deduce that $R^{dR}({}_yU_x^{\mathrm{Mot}})^\vee = {}_yA_x^{\mathrm{dR}}$. By duality we get

$$R^{dR}({}_yU_x^{\mathrm{Mot}}) = {}_yU_x^{\mathrm{dR}}.$$

The next step is to check the compatibility with the structures on both sides. This is the content of next lemma.

LEMMA 4.148. *The morphism ψ of Lemma 3.193 is compatible with the shuffle product, the coproduct and the antipude.*

PROOF. Since the different structures do not depend on the rational points x, y we omit them from the notation. We begin by proving the compatibility with the shuffle product. For non-negative integers p, q, r, s, the map (4.139) induces a map

$$\nabla\colon A^r(X^p) \otimes A^s(X^q) \longrightarrow A^{r+s}(X^{p+q})$$

given by

$$\nabla\left((\omega_1(x_1) \wedge \dots \wedge \omega_p(x_p)) \otimes (\omega_{p+1}(x_{p+1}) \wedge \dots \wedge \omega_{p+q}(x_{p+q}))\right) = \sum_{\sigma \in ⧢(p,q)} (-1)^\sigma (-1)^{ps} \omega_1(x_{\sigma(1)}) \wedge \dots \wedge \omega_{p+q}(x_{\sigma(p+q)})$$

The sign $(-1)^\sigma$ comes from the definition of the map (4.139), while the sign $(-1)^{ps}$ comes from the fact that we have to swap the simplicial degree p with the differential degree s. We now compute

$$\nabla(\psi([\omega_1|\dots|\omega_p]) \otimes \psi([\omega_{p+1}|\dots|\omega_{p+q}])) = \sum_{\sigma \in ⧢(p,q)} (-1)^{\sum_{i=1}^{p+q} i \deg(\omega_i)} (-1)^\sigma \omega_1(x_{\sigma(1)}) \wedge \dots \wedge \omega_{p+q}(x_{\sigma(p+q)}).$$

In this equality we have used that

$$\sum_{i=1}^{p} i \deg(\omega_i) + \sum_{j=1}^{q} j \deg(\omega_{p+j}) + p \sum_{j=1}^{q} \deg(\omega_{p+j}) = \sum_{i=1}^{p+q} i \deg(\omega_i) \tag{4.149}$$

We also compute

$$\psi(\nabla([\omega_1|\dots|\omega_p] \otimes [\omega_{p+1}|\dots|\omega_{p+q}])) = \sum_{\sigma \in ⧢(p,q)} \eta(\sigma)(-1)^{\sum_{i=1}^{p+q} i \deg(\omega_{\sigma^{-1}(i)})} \omega_{\sigma^{-1}(1)}(x_1) \wedge \dots \wedge \omega_{\sigma^{-1}(p+q)}(x_{p+q}),$$

where $\eta(\sigma)$ is the sign determined by equation (3.120). In order to see that the signs in both expressions agree we introduce formal variables $a_1 \dots a_{p+q}$

of degree -1, and put $a = a_1 \wedge \cdots \wedge a_{p+q}$. Then, on the one hand,

$$\begin{aligned}\eta(\sigma)(-1)^{\sum_{i=1}^{p+q} i\deg(\omega_{\sigma^{-1}(i)})}&\omega_{\sigma^{-1}(1)}(x_1)\wedge\cdots\wedge\omega_{\sigma^{-1}(p+q)}(x_{p+q})\wedge a\\ &= \eta(\sigma)a_1\wedge\omega_{\sigma^{-1}(1)}(x_1)\wedge\cdots\wedge a_{p+q}\wedge\omega_{\sigma^{-1}(p+q)}(x_{p+q})\\ &= a_{\sigma(1)}\wedge\omega_1(x_{\sigma(1)})\wedge\cdots\wedge a_{\sigma(p+q)}\wedge\omega_{p+q}(x_{\sigma(p+q)}),\end{aligned}$$

while, on the other hand,

$$\begin{aligned}&(-1)^{\sum_{i=1}^{p+q} i\deg(\omega_i)}(-1)^{\sigma}\omega_1(x_{\sigma(1)})\wedge\cdots\wedge\omega_{p+q}(x_{\sigma(p+q)})\wedge a\\ &= (-1)^{\sum_{i=1}^{p+q} i\deg(\omega_i)}\omega_1(x_{\sigma(1)})\wedge\cdots\wedge\omega_{p+q}(x_{\sigma(p+q)})\wedge a_{\sigma(1)}\wedge\cdots\wedge a_{\sigma(p+q)}\\ &= a_{\sigma(1)}\wedge\omega_1(x_{\sigma(1)})\wedge\cdots\wedge a_{\sigma(p+q)}\wedge\omega_{p+q}(x_{\sigma(p+q)}),\end{aligned}$$

proving the compatibility with the shuffle product.

We next prove the compatibility with the coproduct. The maps (4.141) induce morphisms

$$\Delta\colon A^t(X^n)\longrightarrow\bigoplus_{r+s=t}\bigoplus_{p+q=n}A^r(X^p)\otimes A^s(X^q)$$

given by

$$\begin{aligned}&\Delta(\omega_1(x_1)\wedge\cdots\wedge\omega_n(x_n))\\ &=\sum_{p=0}^{n}(-1)^{p\sum_{i=p+1}^{n}\deg(\omega_i)}\omega_1(x_1)\wedge\cdots\wedge\omega_p(x_p)\otimes\omega_{p+1}(x_{p+1})\wedge\cdots\wedge\omega_n(x_n),\end{aligned}$$

where the sign comes again from the fact that we are swapping a simplicial degree with a differential degree. Then $\Delta\circ\psi=\psi\circ\Delta$ is easily checked using equation (4.149).

Finally, the map (4.142) induces a morphisms

$$S\colon A^*(X^n)\longrightarrow A^*(X^n)$$

given by

$$S(\omega_1(x_1)\wedge\cdots\wedge\omega_n(x_n))=(-1)^{\frac{n(n+1)}{2}}\omega_1(x_n)\wedge\cdots\wedge\omega_n(x_1).$$

The proof of the compatibility of the antipode S with the the map ψ follows the same method as the previous compatibilities. □

As a consequence of this lemma we know that the de Rham realization $R^{dR}({}_yU_x^{\mathrm{Mot}})$ agrees with $= {}_yU_x^{\mathrm{dR}}$ including all the structures.

To conclude, the fact that

$$R^{\mathrm{B}}({}_yU_x^{\mathrm{Mot}})={}_yU_x^{\mathrm{B}}$$

follows from Theorem 3.162, Lemma 3.192, Proposition 3.188 and the description of the Betti realization functor in Section 4.3.10. □

4.6.4. *The case of tangential base points.* We next have to consider the case of tangential base points and prove that the space of paths with tangential base points is also motivic.

We start with the particular case of $\mathbb{G}_m = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\}$ and the tangential base point $\mathbf{0} = (0, 1)$. Recall that in Variant 3.256 we have stated that the method used to study $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ can be used to study $\mathbb{G}_m$. In this case the dg-algebra we use is $A(\mathbb{G}_m) = \mathbb{Q} \oplus \mathbb{Q}\,\omega_0$ and we obtain that ${}_{\boldsymbol{y}}U(\mathbb{G}_m)^{\mathrm{dR}}_{\boldsymbol{x}} = \mathbb{Q}[\![e_0]\!]$

PROPOSITION 4.150. *There is an isomorphism*

$$ {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} \xrightarrow{\sim} {}_{1}U(\mathbb{G}_m)^{\mathrm{H}}_{1}. $$

Moreover, if $x \in \mathbb{G}_m(\mathbb{Q})$, then there is an isomorphism

$$ {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{x} \xrightarrow{\sim} {}_{1}U(\mathbb{G}_m)^{\mathrm{H}}_{x}. $$

PROOF. We only prove the second statement. The proof of the first one is similar. We define the de Rham component of the sought isomorphism as the identity. Clearly it is compatible with the Hodge and the weight filtrations. This is justified because as was the case of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, the de Rham side is independent of the base points.

We have introduced the straight path $\mathbf{dch}$ between 0 and 1, given by $\mathbf{dch}(t) = t$ for $t \in [0, 1]$. We define the Betti part of the isomorphism as the map induced by the composition of paths which sends a path $\gamma \in \pi_1(\mathbb{G}_m; \mathbf{0}, x)$ to the path $\mathbf{dch} \cdot \gamma \in \pi_1(\mathbb{G}_m; 1, x)$. We need to prove that both isomorphisms are compatible with the comparison isomorphism. The comparison map $\mathrm{comp} = \mathrm{comp}_{\mathrm{dR,B}}$ is given by the iterated integral map

$$ \mathrm{comp}(\gamma) = \sum_{n \geq 0} e_0^n \int_\gamma \omega_0 \overset{n}{\cdots} \omega_0 $$

and satisfies $\mathrm{comp}(\gamma \cdot \gamma') = \mathrm{comp}(\gamma)\,\mathrm{comp}(\gamma')$. Thus we only need to check that $\mathrm{comp}(\mathbf{dch}) = 1$. This last equality follows by taking the limit $z \to 1$ in Example 3.234.

That the Betti part of the isomorphism is compatible with the weight filtration is now a consequence of the fact that the de Rham side is. □

From the proposition we immediately deduce:

COROLLARY 4.151. *The pro-mixed Hodge structures ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{x}$ and ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}}$ are motivic (i.e. they are in the essential image of R^H).*

The next lemma describes the structure of ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}}$.

LEMMA 4.152. *The pro-mixed Hodge structure ${}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}}$ is split and agrees with*

$$ \prod_{n \geq 0} \mathbb{Q}(n). $$

In particular, ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} = \mathbb{Q}(1)$.

PROOF. Let f_n and b_n be generators of $\mathbb{Q}(n)_{dR}$ and $\mathbb{Q}(n)_B$ respectively; they satisfy $\mathrm{comp}(b_n) = (2\pi i)^n f_n$. Let γ_0 be the generator of $\pi_1(\mathbb{G}_m, \mathbf{0})$ introduced in Section 3.8.1. By Example 3.236, we know that $\mathrm{comp}_{\mathrm{dR,B}}(\gamma_0) = \exp(2\pi i e_0)$. Consider the power series

$$\log(\gamma_0) = \log(1 + (\gamma_0 - 1)) \in \mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge.$$

For each n, we define a map

$$\varphi_n \colon \mathbb{Q}(n) \to {}_{\mathbf{0}}U(\mathbb{G}_m)^{\mathrm{H}}_{\mathbf{0}} \tag{4.153}$$

which sends f_n to $e_0^n \in \mathbb{Q}\langle\langle e_0 \rangle\rangle$ and b_n to $\log(\gamma_0)^n \in \mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge$. This map is compatible with the comparison isomorphism:

$$\begin{aligned} \mathrm{comp}_{\mathrm{dR,B}}(\varphi_n(b_n)) &= \mathrm{comp}_{\mathrm{dR,B}}(\log(\gamma_0)^n) \\ &= (2\pi i)^n e_0^n \\ &= \varphi_n(\mathrm{comp}_{\mathrm{dR,B}}(b_n)). \end{aligned}$$

Moreover, taking into account that

$$\log(\gamma_0)^n \in J^n \mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge = W_{-2n}\mathbb{Q}[\pi_1(\mathbb{G}_m, \mathbf{0})]^\wedge$$

and $e_0^n \in F^{-n} \cap W_{-2n}\mathbb{Q}\langle\langle e_0 \rangle\rangle$, the map (4.153) is a morphism of mixed Hodge structures. The maps φ_n induce the sought isomorphism of pro-mixed Hodge structures. The second statement follows immediately from the first one. $\square$

We next reduce the question of showing that the mixed Hodge structure of the universal enveloping algebra is motivic to the question that the one of the Lie algebra is motivic.

LEMMA 4.154. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two base points of M (tangential or not). Then the pro-mixed Hodge structure ${}_{\boldsymbol{y}}U^{\mathrm{H}}_{\boldsymbol{x}}$ is motivic if and only if the structure ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}}$ is.*

PROOF. Since ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}}$ is a sub-mixed Hodge structure of ${}_{\boldsymbol{y}}U^{\mathrm{H}}_{\boldsymbol{x}}$, by Proposition 4.89, if ${}_{\boldsymbol{y}}U^{\mathrm{H}}_{\boldsymbol{x}}$ is motivic, then ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}}$ is also motivic.

Conversely, assume that ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}}$ is motivic. Recall that ${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}}$ is an inverse limit

$${}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}} = \varprojlim_N {}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}} / ({}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}})_{\geq N+1}.$$

By Proposition 4.89, each quotient in this limit is motivic. Since

$${}_{\boldsymbol{y}}A^{\mathrm{H}}_{\boldsymbol{x}} = \varinjlim_N \mathrm{Sym}^*({}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}} / ({}_{\boldsymbol{y}}\mathcal{L}^{\mathrm{H}}_{\boldsymbol{x}})_{\geq N+1})^\vee,$$

we deduce that ${}_{\boldsymbol{y}}A^{\mathrm{H}}_{\boldsymbol{x}}$ is also motivic. By duality, we conclude that ${}_{\boldsymbol{y}}U^{\mathrm{H}}_{\boldsymbol{x}}$ is also motivic. $\square$

Now let $x \in X(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{0,1,\infty\}$ be a rational point and $\mathbf{0}$ the tangential base point $(0,1)$. By Lemma 4.154, to show that ${}_{\mathbf{0}}U_x^{\mathrm{H}}$ is motivic, it is enough to show that ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}}$ is. To show that ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}}$ is motivic, we will embed it in a mixed Hodge structure that we know is motivic. Once this is proved, that ${}_{\mathbf{1}}U_x^{\mathrm{H}}$ is motivic follows from the symmetry of X that sends x to $1-x$.

Let $f\colon X \to \mathbb{G}_m$ be the natural inclusion. Then f induces a morphism of mixed Hodge structures

$$\varphi_1\colon {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_x^{\mathrm{H}}. \tag{4.155}$$

The map f also induces a local monodromy map

$$f^*\colon {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}U_{\mathbf{0}}^{\mathrm{H}}.$$

Consider the composition of morphisms of mixed Hodge structures

$$\begin{aligned} {}_{\mathbf{0}}U_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} &\xrightarrow{\nabla^\vee\otimes \mathrm{Id}} {}_{\mathbf{0}}U_x^{\mathrm{H}} \hat{\otimes} {}_{\mathbf{0}}U_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \\ &\xrightarrow{S^\vee\otimes \mathrm{Id}\otimes\mathrm{Id}} {}_xU_{\mathbf{0}}^{\mathrm{H}} \hat{\otimes}_{\mathbf{0}} U_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}U(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_xU_x^{\mathrm{H}}, \end{aligned}$$

where the last morphism is induced by the composition of paths

$$\gamma_1 \otimes \gamma_2 \otimes \gamma_3 \mapsto \gamma_1 \cdot f^*(\gamma_3)\cdot \gamma_2.$$

Restricting to Lie type elements we obtain a map

$${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \otimes {}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} \longrightarrow {}_x\mathcal{L}_x^{\mathrm{H}}. \tag{4.156}$$

Now the identification ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{\mathbf{0}}^{\mathrm{H}} = \mathbb{Q}(1)$ yields a morphism of pro-mixed Hodge structures

$$\varphi_2\colon {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \longrightarrow {}_x\mathcal{L}_x^{\mathrm{H}}(-1). \tag{4.157}$$

LEMMA 4.158. *The following morphism of pro-mixed Hodge structures is injective:*

$$\varphi_1+\varphi_2\colon {}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}} \longrightarrow {}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_{x'}^{\mathrm{H}} \oplus {}_x\mathcal{L}_x^{\mathrm{H}}(-1).$$

PROOF. It is enough to check the injectivity on the de Rham side. Let $\mathcal{L}$ be the free Lie algebra with generators e_0 and e_1 on degree -1. Let $\widehat{\mathcal{L}}$ be the completion of $\mathcal{L}$ with respect to this grading. Then we have that ${}_{\mathbf{0}}\mathcal{L}_x^{dR} = \widehat{\mathcal{L}}$ and and ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_x^{\mathrm{H}} = \mathbb{Q}\, e_0$. Clearly, the map φ_1 is the projection to the e_0 component. By construction, the map (4.156), is given by $a\otimes e_0 \mapsto [e_0, a]$. Therefore, the map $\varphi_2\colon \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}$ is given by $a \mapsto [e_0,a]$. Denote by $\varphi_2'\colon \mathcal{L}\to\mathcal{L}$ the map given by the same formula. By [**Reu93**, Theorem 2.10] the kernel of the map φ_2' is $\mathbb{Q}\, e_0$. It is an easy exercise on inverse limits to show that this implies that the kernel of φ_2 is also $\mathbb{Q}\, e_0$. Since φ_1 does not vanish on the kernel of φ_2 we deduce the lemma. □

Combining Proposition 4.150 and Theorem 4.145 we know that the pro-mixed Hodge structure ${}_{\mathbf{0}}\mathcal{L}(\mathbb{G}_m)_x^{\mathrm{H}} \oplus {}_x\mathcal{L}_x^{\mathrm{H}}(-1)$ is motivic. By Proposition 4.89, we deduce that ${}_{\mathbf{0}}\mathcal{L}_x^{\mathrm{H}}$ is motivic and by Lemma 4.154 that ${}_{\mathbf{0}}U_x^{\mathrm{H}}$ is motivic.

We now have to consider the case of two tangential base points. Let $\boldsymbol{x}, \boldsymbol{y} \in \{\boldsymbol{0}, \boldsymbol{1}\}$ two tangential base points of X. Let $z \in X(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ be a rational point. The composition of paths gives us a surjection

$$ {}_{\boldsymbol{y}}U_z^{\mathrm{H}} \otimes {}_zU_{\boldsymbol{x}}^{\mathrm{H}} \longrightarrow {}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}. $$

Since we already know that the structures on the left-hand side are motivic, we deduce that ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$ is also motivic. Once we know that, for all $\boldsymbol{x}, \boldsymbol{y} \in \{\boldsymbol{0}, \boldsymbol{1}\}$, the mixed Hodge structure ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$ is motivic, the realization functor R^{H} being fully faithful, any morphism among them is also motivic. Therefore, the composition of paths, the completed coproduct, the antipode, the unit and the counit, and the local monodromy maps are motivic.

4.6.5. *The main theorem and some consequences.* From the previous discussion we deduce

THEOREM 4.159 (Deligne-Goncharov [**DG05**]). *For each pair of tangential base points $\boldsymbol{x}, \boldsymbol{y} \in \{\boldsymbol{0}, \boldsymbol{1}\}$ of X, there is a pro-mixed Tate motive ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{Mot}}$ whose Hodge realization is isomorphic to ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{H}}$. By the fully faithfulness of the realization functor, ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{Mot}}$ is unique up to unique isomorphism. Moreover, the unit and the counit, the composition of paths, the completed coproduct, the antipode, and the local monodromy maps are motivic.*

In fact we can do even more

THEOREM 4.160 (Deligne-Goncharov [**DG05**]). *For each pair of tangential base points $\boldsymbol{x}, \boldsymbol{y} \in \{\boldsymbol{0}, \boldsymbol{1}\}$, the pro-mixed Tate motive ${}_{\boldsymbol{y}}U_{\boldsymbol{x}}^{\mathrm{Mot}}$ is a pro-object in the category $\mathbf{MT}(\mathbb{Z})$. The motive ${}_{\boldsymbol{0}}U(\mathbb{G}_m)_{\boldsymbol{0}}^{\mathrm{Mot}}$ belongs to $\mathbf{MT}(\mathbb{Z})$.*

PROOF. The proof of this theorem relies on showing that the ℓ-adic realizations of these motives are unramified (see [**DG05**, Proposition 4.17]) and using Proposition 4.97. ☐

COROLLARY 4.161. *The diagram D_U^{H} of definition 3.258 is motivic and defined over $\mathbb{Z}$.*

The importance of this result is that it connects a very abstract and non-explicit group $G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{dR}})$, but with known structure (see sections 4.5.3 and 4.5.5), with a very concrete combinatorial group $\mathrm{Aut}(D_U^{\mathrm{dR}})$ (see section 3.9.3). The group G_{dR} is the group of symmetries of the category $\mathbf{MT}(\mathbb{Z})$. Therefore it acts on every motive defined over $\mathbb{Z}$ or even in any diagram of motives defined over $\mathbb{Z}$. Since the diagram D_U^{dR} is motivic, the group G_{dR} acts on it and we obtain a group homomorphism

$$ G_{\mathrm{dR}} \longrightarrow \mathrm{Aut}(D_U^{\mathrm{dR}}). $$

The subgroup $U_{\mathrm{dR}} \subset G_{\mathrm{dR}}$ acts trivially on the motive $\mathbb{Q}(1)$, which implies that its image acts trivially on ${}_{\boldsymbol{0}}\mathcal{L}(\mathbb{G}_m)_{\boldsymbol{0}}^{\mathrm{dR}}$, hence on ${}_{\boldsymbol{0}}U(\mathbb{G}_m)_{\boldsymbol{0}}^{\mathrm{H}}$. Therefore the image of U_{dR} is contained in $\mathrm{Aut}^0(D^{\mathrm{dR}})$ and we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U_{\mathrm{dR}} \arrow[r] \arrow[d] & G_{\mathrm{dR}} \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Aut}^0(D^{\mathrm{dR}}) \arrow[r] & \mathrm{Aut}(D^{\mathrm{dR}}) \arrow[r] & \mathbb{G}_m \arrow[r] & 0
\end{tikzcd}$$

The next chapter will be mainly devoted to extract consequences of this diagram.

⋆ ⋆ ⋆

Exercise 4.162. Use that $[X]$ belongs to $\mathbf{DMT}(\mathbb{Q})$ and the fact that $\mathbf{DMT}(\mathbb{Q})$ is closed under products and extensions to prove by induction that $[\sigma_{\leq N}\mathcal{N}_y X_x^\bullet]$ belongs to $\mathbf{DMT}(\mathbb{Q})$.

Exercise 4.163. Show that the complex $\mathcal{N}\,\mathrm{Spec}(\mathbb{Q})^\bullet$ is isomorphic in $C(\mathbf{SmCor}(\mathbb{Q})_{\mathrm{pa}})$ to the complex $\mathrm{Spec}(\mathbb{Q})$ concentrated in degree zero. Conclude that $H_0(\sigma_{\leq N}\mathcal{N}\,\mathrm{Spec}(\mathbb{Q})^\bullet) = \mathbb{Q}(0)$ for all $N \geq 0$.

5. Motivic multiple zeta values (after Brown, Deligne and Goncharov)

In this final chapter, we pull together all the techniques developed so far to prove theorems A and B from the preface. An important part of the strategy will be to upgrade multiple zeta values to motivic multiple zeta values, which are elements of a Hopf algebra. At the end of the chapter, we will state some remarkable consequences of both theorems, among which are the fact that periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[\frac{1}{2\pi i}]$-linear combinations of multiple zeta values, and that Zagier's conjecture implies the algebraic independence of $\pi, \zeta(3), \zeta(5), \ldots$.

5.1. The upper bound. We now have all that we need to prove Theorem A, the upper bound for the dimension of the $\mathbb{Q}$-vector space generated by multiple zeta values of a given weight.

5.1.1. *Setting.* Recall the construction of the Tannakian category $\mathbf{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$. The simple objects are the Tate motives $\mathbb{Q}(n)$, for all $n \in \mathbb{Z}$, and the structure is determined by the extension groups

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q} & \text{if } n \geq 3 \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

and the vanishing of higher extensions. The fibre functor

$$\omega \colon \mathbf{MT}(\mathbb{Z}) \longrightarrow \mathbf{Vec}_{\mathbb{Q}}$$

from (4.101) makes $\mathbf{MT}(\mathbb{Z})$ into a Tannakian category, hence equivalent to the category of representations of the pro-algebraic $\mathbb{Q}$-group

$$G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega).$$

We have already determined the structure of G_{dR} using the computation of the extension groups. It is a semidirect product

$$G_{\mathrm{dR}} \cong U_{\mathrm{dR}} \rtimes \mathbb{G}_m, \tag{5.2}$$

where U_{dR} is a pro-unipotent algebraic group over $\mathbb{Q}$. The Lie algebra $\mathfrak{u}_{\mathrm{dR}}$ of U_{dR} is (non-canonically) isomorphic to the completion of the free Lie algebra with one generator σ_{2n+1} in each degree $-(2n+1)$ for all $n \geq 1$. Therefore, the graded Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is a free Lie algebra.

Besides, in Section 3.9 we introduced the algebraic groups of symmetries of the de Rham fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which were denoted by $\mathrm{Aut}^0(D^{\mathrm{dR}})$ and $\mathrm{Aut}(D^{\mathrm{dR}})$. Moreover we showed in Lemma 3.259 that there is an isomorphism of $\mathbb{Q}$-schemes

$$\mathrm{Aut}^0(D^{\mathrm{dR}}) \simeq {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}.$$

This led us to define an algebraic group $(\Pi, \circ)$ with underlying scheme ${}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}$ and the multiplication induced by the above isomorphism (Definition 3.261).

The group $(\Pi, \circ)$ acts on ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ and the map $v \mapsto v({}_{\mathbf{1}}1_{\mathbf{0}})$ is an isomorphism. Thus, ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ is a trivial torsor under $(\Pi, \circ)$.

Theorem 4.159 implies that the diagram D^{dR} is the de Rham realization of a diagram of mixed Tate motives over $\mathbb{Z}$. Therefore we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U_{\mathrm{dR}} \arrow[r] \arrow[d] & G_{\mathrm{dR}} \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Pi \arrow[r] & \mathrm{Aut}(D^{\mathrm{dR}}) \arrow[r] & \mathbb{G}_m \arrow[r] & 0. \end{tikzcd} \tag{5.3}$$

and we denote the first vertical arrow by

$$I \colon U_{dR} \longrightarrow \Pi. \tag{5.4}$$

In particular, G_{dR} acts on the pro-scheme ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$.

We introduce the notation

$$\mathcal{A}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}}), \qquad \mathcal{A} = \mathcal{O}(I(U_{\mathrm{dR}})). \tag{5.5}$$

Note that there is an injective morphism of algebras $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{MT}}$.

In (4.131) we introduced

$$\mathcal{H}^{\mathcal{MT}} = \mathcal{A}^{\mathcal{MT}} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2].$$

It is a Hopf module over $\mathcal{A}^{\mathcal{MT}}$, with f_2 in degree two, and its Hilbert-Poincaré series is given by

$$H_{\mathcal{H}^{\mathcal{MT}}}(t) = \sum_{k \geq 0} d_k t^k.$$

5.1.2. *The algebra of motivic multiple zeta values.* From now on we will let $\mathbf{dch}^{\mathrm{dR}}$ denote the image by

$$\mathrm{comp}_{\mathrm{dR,B}} \colon {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{B}}(\mathbb{C}) \longrightarrow {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}(\mathbb{C})$$

of the straight path $\mathbf{dch} \in {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{B}}(\mathbb{Q})$ from $\mathbf{0}$ to $\mathbf{1}$. This is nothing other than what was previously denoted in (3.240) by:

$$L(\mathbf{dch}) = \sum_{\alpha} \zeta_{\sqcup\!\sqcup}(x_\alpha) e_\alpha.$$

In particular, $\mathbf{dch}^{\mathrm{dR}}$, which was a priori only a complex point of ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$, lives actually in ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}(\mathbb{R})$, since all (regularized) multiple zeta values $\zeta_{\sqcup\!\sqcup}(x_\alpha)$ are real numbers.

Recall that the affine ring of ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ is

$$\mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}) = \mathbb{Q}\langle x_0, x_1 \rangle = \mathfrak{H}.$$

Evaluating an element $f \in \mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}})$ at the point $\mathbf{dch}^{\mathrm{dR}}$ yields a map

$$\mathrm{dch} \colon \mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}) \longrightarrow \mathbb{R}$$

given by $\mathrm{dch}(f) = f(\mathbf{dch}^{\mathrm{dR}})$. For a word w in the alphabet x_0, x_1, we get $w(\mathbf{dch}^{\mathrm{dR}}) = \zeta_{\sqcup\!\sqcup}(w)$. Thus, by Corollary 1.177, we obtain a surjective map from $\mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}})$ to the algebra $\mathcal{Z}$ of multiple zeta values.

Remark 5.6. This map is very far from being injective, as all relations between multiple zeta values belong to the kernel. As a result, the algebra $\mathbb{Q}\langle x_0, x_1\rangle$, which has the advantage of being elementary, is too big for the purpose of proving Theorem A. The algebra $\mathcal{O}(G_{\mathrm{dR}})$ looks more promising but it is still too big. In fact $\mathcal{O}(\mathbb{G}_m) = \mathbb{Q}[x, x^{-1}]$ with x in degree 1. Using the splitting of Lemma 4.113 we derive

$$\mathcal{O}(G_{\mathrm{dR}}) \simeq \mathcal{A}^{\mathcal{MT}} \otimes_{\mathbb{Q}} \mathbb{Q}[x, x^{-1}].$$

The presence of x^{-1}, that has degree -1, implies that the dimension of each graded piece of $\mathcal{O}(G_{\mathrm{dR}})$ is infinite, therefore this algebra is also not useful for our purposes. Identifying f_2 with $x^2/24$ we obtain an injective map $\mathcal{H}^{\mathcal{MT}} \to \mathcal{O}(G_{\mathrm{dR}})$. The strategy to prove Theorem A is to prove that the evaluation map "dch" factors through $\mathcal{H}^{\mathcal{MT}}$. This can be done in an ad hoc way or we can use a nice geometric interpretation due to Brown.

Following Brown [**Bro12**, §2.3], we define a closed subvariety $\mathcal{Y} \subseteq {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ as the Zariski closure of the orbit of $\mathbf{dch}^{\mathrm{dR}}$, that is:

$$\mathcal{Y} = \overline{G_{\mathrm{dR}} \cdot \mathbf{dch}^{\mathrm{dR}}}. \tag{5.7}$$

Lemma 5.8. *The subvariety $\mathcal{Y}$ is defined over $\mathbb{Q}$.*

Proof. To see that $\mathcal{Y}$ is defined over $\mathbb{Q}$ we give another interpretation of it. Recall that $P_{\mathrm{dR,B}}$ is the torsor of isomorphisms between the fibre functors ω_{B} and ω_{dR}. Thus there is an action

$$P_{\mathrm{dR,B}} \times {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{B}} \longrightarrow {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}.$$

The point $\mathbf{dch} \in {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{B}}(\mathbb{Q})$ induces a map $\mathrm{dch}\colon P_{\mathrm{dR,B}} \to {}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$. This map is G_{dR}-equivariant and sends $\mathrm{comp}_{\mathrm{dR,B}}$ to $\mathbf{dch}^{\mathrm{dR}}$. Thus, $\mathcal{Y}$ is the Zariski closure of the image of the map dch. The point $\mathbf{dch}$ being rational, we deduce that $\mathcal{Y}$ is defined over $\mathbb{Q}$. $\square$

We consider the $\mathbb{Q}$-algebra

$$\mathcal{H} = \mathcal{O}(\mathcal{Y}).$$

The action of $\mathbb{G}_m$ on $\mathcal{Y}$ induces a grading of $\mathcal{H}$. Since $\mathcal{Y}$ contains $\mathbf{dch}^{\mathrm{dR}}$, the map "dch" factors through $\mathcal{H}$ giving a map

$$\mathrm{per}\colon \mathcal{H} \longrightarrow \mathbb{R}. \tag{5.9}$$

By Corollary 1.177 the image of "per" is $\mathcal{Z}$. Moreover, since the action of $\mathbb{G}_m$ on $\mathcal{Y}$ is compatible with its action on ${}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}}$ and the grading that this action induces on $\mathcal{O}({}_{\mathbf{1}}\Pi_{\mathbf{0}}^{\mathrm{dR}})$ agrees with the natural grading of $\mathbb{Q}\langle x_0, x_1\rangle$, we deduce that the image of $\mathcal{H}_k$ is $\mathcal{Z}_k$.

DEFINITION 5.10. $\mathcal{H}$ is called the *algebra of motivic multiple zeta values* and the map "per" the *period map.*

The map (5.9) is called the period map is because it is compatible with the period map from Definition 4.119. In fact, since

$$\mathrm{dch}(\mathrm{comp}_{\mathrm{dR,B}}) = \mathbf{dch}^{\mathrm{dR}},$$

there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}(\mathcal{Y}) \arrow[r, "\mathrm{dch}^*"] \arrow[dr, "\mathrm{per}"'] & \mathcal{O}(P_{\mathrm{dR,B}}) \arrow[d, "\mathrm{per}"] \\ & \mathbb{C}. \end{tikzcd} \tag{5.11}$$

REMARK 5.12. We can interpret $\mathcal{H}$ as follows. Let $\mathcal{I} \subset \mathbb{Q}\langle x_0, x_1\rangle$ be the ideal of functions vanishing on $\mathbf{dch}$. This is the ideal of rational relations among multiple zeta values. The ideal of $\mathcal{Y}$, denoted by $\mathcal{J}^{\mathcal{MT}}$, is the ideal of motivic relations between multiple zeta values, that is, those explained by geometry. The fact that $\mathcal{J}^{\mathcal{MT}} \subseteq \mathcal{I}$ will imply the upper bound of the dimension of the space of multiple zeta values, while Zagier's conjecture will be equivalent to $\mathcal{J}^{\mathcal{MT}} = \mathcal{I}$, that is, that every rational relation among multiple zeta values comes from geometry.

Now, the strategy to prove Theorem A is to get the inequality $\dim \mathcal{H}_k \leq d_k$ from an injection $\mathcal{H} \to \mathcal{H}^{\mathcal{MT}}$. This injection will come from the study of the geometry of $\mathcal{Y}$. As a consequence of Theorem B to be proved latter, we will see that, in fact, the equality $\dim \mathcal{H}_k = d_k$ holds and the algebras $\mathcal{H}$ and $\mathcal{H}^{\mathcal{MT}}$ are isomorphic.

5.1.3. *The structure of* $\mathcal{Y}$. Recall from Proposition 4.115 that there exists an element

$$a = u_0 \cdot \tau(2\pi i) \in U_{\mathrm{dR}}(\mathbb{R})\tau(2\pi i) \subset G_{\mathrm{dR}}(\mathbb{C}) \tag{5.13}$$

such that $\omega_{\mathrm{B}}(M) = (\mathrm{comp}_{B,\mathrm{dR}} \circ a)(\omega_{\mathrm{dR}}(M))$ for all M in $\mathbf{MT}(\mathbb{Z})$.

LEMMA 5.14. *There exists* $\gamma \in \Pi(\mathbb{Q})$ *such that*

$$\mathbf{dch}^{\mathrm{dR}} = (I(u_0) \circ \tau(2\pi i)(\gamma))({}_{\mathbf{1}}1_{\mathbf{0}}). \tag{5.15}$$

Moreover, for any γ *satisfying* (5.15), *one has* $\tau(-1)(\gamma) = \gamma$.

PROOF. By Proposition 4.115 there is a $\gamma' \in {}_{\mathbf{1}}\Pi^{\mathrm{dR}}_{\mathbf{0}}(\mathbb{Q})$ such that $\mathbf{dch}^{\mathrm{dR}} = a(\gamma')$. Let $\gamma \in \Pi(\mathbb{Q})$ such that $\gamma' = \gamma({}_{\mathbf{1}}1_{\mathbf{0}})$. Then

$$\begin{aligned} \mathbf{dch}^{\mathrm{dR}} &= (u_0 \cdot \tau(2\pi i))(\gamma') \\ &= I(u_0)(\tau(2\pi i)(\gamma({}_{\mathbf{1}}1_{\mathbf{0}}))) \\ &= (I(u_0) \circ \tau(2\pi i)(\gamma))({}_{\mathbf{1}}1_{\mathbf{0}}). \end{aligned}$$

The second assertion follows from the fact that both $\mathbf{dch}$ and u_0 are real, hence so is $\tau(2\pi i)(\gamma)$. Writing $\gamma = \sum c_w w$ in $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$, it follows that $c_w = 0$ for w of odd degree, thus $\tau(-1)(\gamma) = \gamma$. $\square$

Let us write $\gamma = \sum c_w w$ in $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$. Since $\tau(-1)(\gamma) = \gamma$ by the previous lemma, only monomials w of even degree contribute. It follows that the map $\mathbb{G}_m \to \Pi$ defined by $t \mapsto \tau(t)(\gamma)$ depends only on t^2. Indeed, if one defines

$$\rho(t) = \sum t^{\frac{\deg(w)}{2}} c_w w \tag{5.16}$$

one has $\tau(t)(\gamma) = \rho(t^2)$. Observe that ρ extends to $\mathbb{A}^1$ with $\rho(0) = 1$.

THEOREM 5.17. *The morphism of schemes*

$$\begin{aligned} \psi\colon I(U_{\mathrm{dR}}) \times \mathbb{A}^1 &\longrightarrow \Pi \\ (u,t) &\longmapsto u \circ \rho(t) \end{aligned}$$

induces an isomorphism $I(U_{\mathrm{dR}}) \times \mathbb{A}^1 \simeq \mathcal{Y}$ *given by* $(u,t) \mapsto \psi(u,t)({}_{\mathbf{1}}1_{\mathbf{0}})$.

PROOF. Recall that the graded Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is positively graded and is zero in degree < 3 by Theorem 4.123. Thus any element $u \in I(U_{\mathrm{dR}})$ can be written as

$$u = 1 + \sum_{\deg(w)\geq 3} e_w w.$$

The coefficients of the monomial $e_0 e_1$ in $\rho(t)$ and $u \circ \rho(t)$ agree. Let us compute the former. Recall that

$$\mathbf{dch}^{\mathrm{dR}} = 1 - \zeta(2) e_0 e_1 + \text{higher degree}.$$

Since $\mathbf{dch}^{\mathrm{dR}} = (u \circ \tau(2\pi i)(\gamma))({}_{\mathbf{1}}1_{\mathbf{0}})$ by Lemma 5.14, one has $(2\pi i)^2 c_{e_0e_1} = -\zeta(2)$, which yields the value $c_{e_0e_1} = \frac{1}{24}$ by Euler's formula. The coefficient of e_0e_1 in $\rho(t)$ is thus equal to $\frac{t}{24}$.

This leads naturally to consider the maps

$$\begin{aligned} c\colon \Pi &\longrightarrow \mathbb{A}^1 \\ x &\longmapsto 24 \cdot \text{coefficient of } e_0e_1 \text{ in } x. \\ \varphi\colon \Pi &\longrightarrow \Pi \\ x &\longmapsto x \circ \rho(c(x))^{-1}. \end{aligned}$$

By the previous discussion, we have $c(\psi(u,t)) = t$ and $\varphi(\psi(u,t)) = u$. In particular, the morphism ψ is injective.

Observe that $x \in \Pi$ is in the image of ψ if and only if $\varphi(x)$ belongs to $I(U_{\mathrm{dR}})$. Therefore, $\operatorname{Im}\psi = \varphi^{-1}(I(U_{\mathrm{dR}}))$. Since $I(U_{\mathrm{dR}})$ is closed in Π, the same holds for $\operatorname{Im}\psi$. By Lemma 5.14, $(\operatorname{Im}\psi)_{\mathbf{1}}1_{\mathbf{0}}$ contains $G_{\mathrm{dR}} \cdot \mathbf{dch}^{\mathrm{dR}}$ as an open dense subset, so it has to be equal to its closure $\mathcal{Y}$. Write $\mathcal{Y}'$ for the preimage of $\mathcal{Y}$ in Π. To conclude, we note that the map $\mathcal{Y}' \to I(U_{\mathrm{dR}}) \times \mathbb{A}^1$ given by $x \mapsto (\varphi(x), c(x))$ is an inverse of ψ. $\square$

Corollary 5.18. *There is an isomorphism of graded algebras*

$$\mathcal{H} \simeq \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[t],$$

where t *sits in degree two. This isomorphism induces an injection* $\mathcal{H} \hookrightarrow \mathcal{H}^{\mathcal{MT}}$ *that sends* t *to* $24f_2$.

Proof. We need to show that the map ψ from the previous theorem is $\mathbb{G}_m$-equivariant provided that one makes $\lambda \in \mathbb{G}_m$ act on $\mathbb{A}^1$ by $t \mapsto \lambda^2 t$. On the one hand, formula (5.16) gives $\rho(\lambda^2 t) = \tau(\lambda)(\rho(t))$. On the other hand, using Proposition 4.113, we get

$$\tau(\lambda)(u \circ \rho(t)) = \tau(\lambda)(u) \circ \tau(\lambda)(\rho(t)),$$

from which the result follows. □

5.1.4. *Proof of Theorem A.* Since the map (5.9) is surjective and respects the weight, it suffices to show that $\dim \mathcal{H}_k \leq d_k$ for each $k \geq 2$. But Corollary 5.18 and Lemma 4.132 yield

$$\dim \mathcal{H}_k \leq \dim(\mathcal{H}^{\mathcal{MT}})_k = d_k.$$

5.2. Motivic multiple zeta values and the motivic coaction. In this section, we define some elements of the algebra $\mathcal{H}$ which will be called motivic multiple zeta values. Thanks to the existence of the coproduct, we can find many relations between them, which will translate into relations for the usual numbers.

5.2.1. *The structure of* $\mathcal{A}^{\mathcal{MT}}$. From the fact that $\mathrm{Lie}(U_{\mathrm{dR}})$ is isomorphic to the completion of the free Lie algebra with one generator in each odd degree ≤ -3, we know that $\mathcal{A}^{\mathcal{MT}}$ is non-canonically isomorphic to the graded Hopf algebra

$$\mathcal{U}' = \mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle$$

whose underlying space is the set of non-commutative words in symbols f_{2i+1}, $i \geq 1$ in degree $2i+1$, whose product is the shuffle product and whose coproduct is the deconcatenation coproduct

$$\Delta(f_{i_1} f_{i_2} \dots f_{i_r}) = \sum_{k=0}^{r} f_{i_1} \dots f_{i_k} \otimes f_{i_{k+1}} \dots f_{i_r}. \tag{5.19}$$

We introduce the commutative graded algebra

$$\mathcal{U} = \mathcal{U}' \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{5.20}$$

with f_2 in degree 2. There is a coaction

$$\Delta \colon \mathcal{U} \longrightarrow \mathcal{U}' \otimes_{\mathbb{Q}} \mathcal{U} \tag{5.21}$$

obtained by declaring $\Delta f_2 = 1 \otimes f_2$, that turns $\mathcal{U}$ into an $\mathcal{U}'$-comodule. Clearly $\mathcal{H}^{\mathcal{MT}}$ is non-canonically isomorphic to $\mathcal{U}$.

For later use, it will also be convenient to introduce the elements f_{2n} for $n \geq 2$. They are defined as $f_{2n} = b_n f_2^n$, where b_n is the rational number satisfying $\zeta(2n) = b_n \zeta(2)^n$ by Euler's Theorem 1.3.

The Hopf algebra $\mathcal{U}'$ and its comodule $\mathcal{U}$ are useful for explicit computations. Later we will fix a convenient isomorphism

$$\phi\colon \mathcal{H}^{\mathcal{MT}} \to \mathcal{U} \tag{5.22}$$

satisfying certain normalization requirements. But for the moment we denote by ϕ any such isomorphism.

For compatibility with the theory of multiple zeta values, the grading in $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{H}$ and the other algebras will be called the weight.

We first present the computational tools we will use at the level of $\mathcal{U}'$. As in Definition 3.69, the Lie coalgebra associated to $\mathcal{U}'$ is

$$L = \mathcal{U}'_{>0}/(\mathcal{U}'_{>0})^2.$$

Since there is a canonical decomposition $\mathcal{U}' = \mathbb{Q} \oplus \mathcal{U}'_{>0}$, we have a projection $q\colon \mathcal{U}' \to L$. The Lie coalgebra L inherits a grading from $\mathcal{U}'$. Let L_N be the subspace of weight N and $p_N\colon L \to L_N$ the projection. We define a map

$$D_{2r+1}\colon \mathcal{U} \longrightarrow L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U} \tag{5.23}$$

as the composition

$$\mathcal{U} \xrightarrow{\Delta - 1\otimes \mathrm{Id}} \mathcal{U}' \otimes_{\mathbb{Q}} \mathcal{U} \xrightarrow{q\otimes \mathrm{Id}} L \otimes_{\mathbb{Q}} \mathcal{U} \xrightarrow{p_{2r+1}\otimes \mathrm{Id}} L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U},$$

where Δ is the extended coproduct (5.21). We will see in Exercise 5.46 that the maps D_{2r+1} are derivations. We put

$$D_{<N} = \bigoplus_{3 \leq 2r+1 < N} D_{2r+1}. \tag{5.24}$$

LEMMA 5.25. *Let $N \geq 2$ be an integer. Then:*

$$(\operatorname{Ker} D_{<N}) \cap \mathcal{U}_N = \mathbb{Q} f_N.$$

PROOF. We first show that $f_N \in \operatorname{Ker} D_{<N}$. When N is even, we already have $\Delta f_N - 1 \otimes f_N = 0$. If N is odd and $2r+1 < N$, then

$$D_{2r+1} f_N = p_{2r+1}(q(f_N)) \otimes 1 = 0.$$

Thus $f_N \in \operatorname{Ker} D_{<N}$. Conversely, let $\xi \in \mathcal{U}_N$. Such an element can be uniquely written as

$$\xi = \alpha f_N + \sum_{3 \leq 2r+1 < N} f_{2r+1} v_r$$

with $v_r \in \mathcal{U}_{N-2r-1}$ and $\alpha \in \mathbb{Q}$. Using the explicit expresion of the coproduct Δ, we see that

$$D_{2r+1}\xi = f_{2r+1} \otimes v_r + \text{other terms},$$

where none of the monomials of $\mathcal{U}'_{2r+1}$ which appear in the extra terms is f_{2r+1}. Thus, if $D_{2r+1}\xi = 0$ we deduce that $v_r = 0$. $\square$

5.2.2. *Motivic multiple zeta values.* Recall that, in formula (3.268) at the end of chapter 3, we introduced, for each binary sequence α, a function on Π which we denoted by

$$I(1;\alpha;0)=x_\alpha.$$

We now let $I^{\mathfrak{m}}(1;\alpha;0)$ denote the restriction of this function to $\mathcal{Y}$, that is, the projection to the quotient

$$I^{\mathfrak{m}}(1;\alpha;0)\in\mathcal{H}=\mathcal{O}(\Pi)/\mathcal{J}^{MT}.$$

Following formulas (3.268), for latter use, we denote

$$I^{\mathfrak{m}}(0;\alpha;1)=x^*_\alpha|_{\mathcal{Y}}.$$

and

$$I^{\mathfrak{m}}(0;\alpha;0)=I^{\mathfrak{m}}(1;\alpha;1)=\begin{cases}1 & \alpha=\emptyset,\\ 0 & \alpha\neq\emptyset.\end{cases}\tag{5.26}$$

We now list some useful properties of the motivic iterated integrals.

LEMMA 5.27.

(1) If $N\geq 1$, then $I^{\mathfrak{m}}(\varepsilon_0;\varepsilon_1\cdots\varepsilon_N;\varepsilon_{N+1})=0$ when $\varepsilon_1=\cdots=\varepsilon_N$.
(2) Reflection formula

$$\begin{aligned}I^{\mathfrak{m}}(1;\varepsilon_1\cdots\varepsilon_N;0)&=(-1)^N I^{\mathfrak{m}}(0;\varepsilon_N\cdots\varepsilon_1;1)\\ &=I^{\mathfrak{m}}(1;1-\varepsilon_N\cdots 1-\varepsilon_1;0)\end{aligned}$$

PROOF. Property (2) follows from Theorem 3.217 (1) and the symmetry $z\mapsto z-1$.

We prove property (1). Since $I^{\mathfrak{m}}(0;\alpha;0)=I^{\mathfrak{m}}(1;\alpha;1)=0$ for a non-empty binary sequence α and, by (2), $I^{\mathfrak{m}}(0;\varepsilon^{\{N\}};1)=(-1)^N I^{\mathfrak{m}}(1;\varepsilon^{\{N\}};0)$, it suffices to show that $I^{\mathfrak{m}}(1;\varepsilon^{\{N\}};0)=0$. For this, we use the identity

$$I^{\mathfrak{m}}(1;\varepsilon^{\{N\}};0)=\frac{1}{N!}I^{\mathfrak{m}}(1;\varepsilon;0)^N$$

and the fact that $I^{\mathfrak{m}}(1;\varepsilon;0)=0$ since the algebra $\mathcal{H}$ has no elements in degree one. $\square$

DEFINITION 5.28. For a positive multi-index $\boldsymbol{s}=(s_1,\dots,s_r)$, the associated *motivic multiple zeta value* is the element of $\mathcal{H}$ given by

$$\zeta^{\mathfrak{m}}(\boldsymbol{s})=I^{\mathfrak{m}}(1;0^{\{s_1-1\}}1\cdots 0^{\{s_r-1\}}1;0).$$

The binary sequence $(0^{\{s_1-1\}},1,\cdots,0^{\{s_r-1\}},1)$ is called the binary sequence associated to $\boldsymbol{s}$ and is denoted in Definition 1.124 as $\mathrm{bs}(\boldsymbol{s})$.

The period map $\mathrm{per}:\mathcal{H}\longrightarrow\mathbb{R}$ from Definition 5.10 satisfies

$$\mathrm{per}(\zeta^{\mathfrak{m}}(\boldsymbol{s}))=\zeta_{\sqcup\!\sqcup}(\boldsymbol{s}).$$

REMARK 5.29. Due to the different convention on the definition of multiple zeta values and of iterated integrals there is a discrepancy between the symbols used here and the symbols used in [**Bro12**], that we summarize here. We denote $\zeta_B^{\mathfrak{m}}(s_1,\dots,s_r)$, $I_B^{\mathfrak{m}}(\varepsilon_0:\varepsilon_1,\dots,\varepsilon_n:\varepsilon_{n+1})$ and $\zeta_B(s_1,\dots,s_r)$ the motivic multiple zeta values, motivic iterated integrals and multiple zeta values used in [**Bro12**]. Then

$$\begin{aligned}\zeta_B^{\mathfrak{m}}(s_1,\dots,s_r) &= \zeta^{\mathfrak{m}}(s_r,\dots,s_1)\\ I_B^{\mathfrak{m}}(\varepsilon_0:\varepsilon_1,\dots,\varepsilon_n:\varepsilon_{n+1}) &= I^{\mathfrak{m}}(\varepsilon_{n+1}:\varepsilon_n,\dots,\varepsilon_1:\varepsilon_0)\\ \zeta_B(s_1,\dots,s_r) &= \zeta(s_r,\dots,s_1).\end{aligned}$$

The map per is the same in [**Bro12**] and in this book because it is the evaluation morphism at a point. The translation from motivic multiple zeta values in [**Bro12**] is given by

$$\zeta_B^{\mathfrak{m}}(s_1,\dots,s_r) = I_B^{\mathfrak{m}}(0:10^{\{s_1-1\}}\dots10^{\{s_r-1\}}:1),$$

while here is given by

$$\zeta^{\mathfrak{m}}(s_1,\dots,s_r) = I^{\mathfrak{m}}(1:0^{\{s_1-1\}}1\dots0^{\{s_r-1\}}1:0).$$

Both equations are compatible via the change of notation.

If $\boldsymbol{s}$ is admissible, then $\zeta^{\mathfrak{m}}(\boldsymbol{s})\neq 0$. In particular, $\zeta^{\mathfrak{m}}(2)\neq 0$. In fact $\zeta^{\mathfrak{m}}(2)$ is the function on $\mathcal{Y}$ that sends an element g of $\mathcal{Y}(\mathbb{Q})\subset\mathbb{Q}\langle\langle e_0,e_1\rangle\rangle$ to its coefficient on e_0e_1. It follows that $\zeta^{\mathfrak{m}}(2)$ is sent to $\frac{t}{24}$ under the isomorphism $\mathcal{H}\to\mathcal{A}\otimes\mathbb{Q}[t]$ of Corollary 5.18. Therefore it is sent to f_2 under the injection $\mathcal{H}\to\mathcal{H}^{\mathcal{MT}}$ of the same corollary.

REMARK 5.30. The fact that $\zeta^{\mathfrak{m}}(2)$ is not zero is an important difference between Brown's and Goncharov's approaches to motivic multiple zeta values. Recall that $U_{\mathrm{dR}}\subset G_{\mathrm{dR}}$ and that we had elements $1_{\mathrm{dR}}\in\Pi(\mathbb{Q})$ and $\mathbf{dch}^{\mathrm{dR}}\in\Pi(\mathbb{C})$. Goncharov works with the orbit of 1_{dR} under U_{dR}:

$$\mathcal{X} = U_{\mathrm{dR}}\cdot 1_{\mathrm{dR}}\subset\Pi$$

which is isomorphic, as a variety, to $I(U_{\mathrm{dR}})$. Hence $\mathcal{O}(\mathcal{X})\simeq\mathcal{A}$. However, Brown works with the variety $\mathcal{Y}$ defined as the closure of the orbit of $\mathbf{dch}^{\mathrm{dR}}$ under G_{dR}

$$\mathcal{Y} = \overline{G_{\mathrm{dR}}\cdot\mathbf{dch}^{\mathrm{dR}}}\simeq I(U_{\mathrm{dR}})\times\mathbb{A}^1.$$

Since the leading term of $\mathbf{dch}^{\mathrm{dR}}$ is 1_{dR} we deduce that

$$\lim_{t\to 0}\tau(t)\mathbf{dch}^{\mathrm{dR}} = 1_{\mathrm{dR}}.$$

Therefore $\mathcal{X}\subset\mathcal{Y}$ and the inclusion from $\mathcal{X}$ to $\mathcal{Y}$ is given by $x\mapsto(x,0)$. In other words, the inclusion $\mathcal{X}\hookrightarrow\mathcal{Y}$ is given by the algebra morphism

$$\pi\colon\mathcal{H}\longrightarrow\mathcal{H}/\zeta^{\mathfrak{m}}(2)\mathcal{H}=\mathcal{A}. \tag{5.31}$$

5.2.3. *The motivic coaction.* Goncharov's coproduct from Proposition 3.270 induces a coaction

$$\Delta\colon \mathcal{H} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H} \tag{5.32}$$

given by the formula

$$\Delta I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} \pi\left(\prod_{p=0}^{k} I^{\mathfrak{m}}(\varepsilon_{i_p}; \varepsilon_{i_p+1} \cdots \varepsilon_{i_{p+1}-1}; \varepsilon_{i_{p+1}})\right) \otimes I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_{i_1} \cdots \varepsilon_{i_k}; \varepsilon_{N+1}), \tag{5.33}$$

where π denotes the projection (5.31).

Lemma 5.34. *For all $N \geq 2$,*

$$\Delta\zeta^{\mathfrak{m}}(N) = 1 \otimes \zeta^{\mathfrak{m}}(N) + \pi(\zeta^{\mathfrak{m}}(N)) \otimes 1.$$

Proof. By Definition 5.28, we have $\zeta^{\mathfrak{m}}(N) = I^{\mathfrak{m}}(1; 0^{\{N-1\}}1; 0)$. Using part (1) of Lemma 5.27, we see that the only non-vanishing terms in the coproduct formula (5.33) correspond to the partitions

$$k = 0,\ i_0 = 0,\ i_1 = N+1, \quad k = N,\ i_j = j,\ j = 0, \dots, N+1.$$

The first partition yields the term $\pi(\zeta^{\mathfrak{m}}(N)) \otimes 1$, while the second one gives $1 \otimes \zeta^{\mathfrak{m}}(N)$, thus proving the result. □

The formula (5.33) is rather complicated, so we will use an infinitesimal version of it, which is the analogue of the derivations D_{2r+1} for the algebra of motivic multiple zeta values $\mathcal{H}$. For this, we consider the Lie coalgebra

$$\mathcal{L} = \mathcal{A}_{>0}/(\mathcal{A}_{>0})^2,$$

which inherits a grading from $\mathcal{A}$. Let $\mathcal{L}_N$ be the subspace of degree N and $p_N\colon \mathcal{L} \to \mathcal{L}_N$ the projection. We still have a projection $q\colon \mathcal{A} \to \mathcal{L}$.

Definition 5.35. We define a map

$$D_{2r+1}\colon \mathcal{H} \longrightarrow \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \mathcal{H} \tag{5.36}$$

as the composition

$$\mathcal{H} \xrightarrow{\Delta - 1\otimes \mathrm{Id}} \mathcal{A}_{>0} \otimes_{\mathbb{Q}} \mathcal{H} \xrightarrow{q\otimes \mathrm{Id}} \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{H} \xrightarrow{p_{2r+1}\otimes \mathrm{Id}} \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \mathcal{H}.$$

We put

$$D_{<N} = \bigoplus_{3\leq 2r+1<N} D_{2r+1}. \tag{5.37}$$

For any isomorphism $\phi\colon \mathcal{A}^{\mathcal{MT}} \to \mathcal{U}'$ of Hopf algebras, we extend it to an algebra isomorphism, $\phi\colon \mathcal{H}^{\mathcal{MT}} \to \mathcal{U}$ by sending f_2 to f_2. It is also an isomorphism of comodules. By abuse of notation, we will denote by ϕ the restriction of ϕ to $\mathcal{H}$ and the map $\mathcal{L} \to L$ induced by $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{MT}} \xrightarrow{\phi} \mathcal{U}'$. Then the following diagram commutes:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\phi"] \arrow[d, "D_{2r+1}"'] & \mathcal{U} \arrow[d, "D_{2r+1}"] \\ \mathcal{L}_{2r+1} \otimes \mathcal{H} \arrow[r, "\phi\otimes\phi"] & L_{2r+1} \otimes \mathcal{U}. \end{tikzcd} \tag{5.38}$$

LEMMA 5.39. *There exists an isomorphism* $\phi\colon \mathcal{H}^{\mathcal{MT}} \to \mathcal{U}$ *as before that sends* $\zeta^{\mathfrak{m}}(N)$ *to* f_N *for all* $N \geq 2$.

PROOF. We start with any such isomorphism ϕ. By construction ϕ sends f_2 to f_2. By the discussion before Remark 5.30, ϕ sends $\zeta^{\mathfrak{m}}(2)$ to f_2.

By Lemma 5.34 we deduce that $D_{<N}\zeta^{\mathfrak{m}}(N) = 0$. By the commutativity of the diagram (5.38) we deduce that $D_{<N}\phi(\zeta^{\mathfrak{m}}(N)) = 0$. By Lemma 5.25, $\phi(\zeta^{\mathfrak{m}}(N)) = \alpha_N f_N$ for $\alpha_N \in \mathbb{Q}^{\times}$. For $N = 2r$ even,

$$\phi(\zeta^{\mathfrak{m}}(2r)) = \alpha_{2r} f_{2r} = \alpha_{2r} b_r f_2^r = \phi(\alpha_{2r} b_r \zeta^{\mathfrak{m}}(2)^r).$$

By the injectivity of ϕ we deduce that $\zeta^{\mathfrak{m}}(2r) = \alpha_{2r} b_r \zeta^{\mathfrak{m}}(2)^r$. Taking the period map we see that $\alpha_{2r} = 1$.

By the structure of $\mathcal{U}'$, for any family of non-zero rational numbers α_{2r+1}, $r \geq 1$, there is an automorphism of $\mathcal{U}'$ that sends f_{2r+1} to $\alpha_{2r+1}^{-1} f_{2r+1}$. Therefore we can normalize ϕ as we want. $\square$

As a byproduct of the proof, we have seen that

$$\zeta^{\mathfrak{m}}(2r) = b_r \zeta^{\mathfrak{m}}(2)^r. \tag{5.40}$$

The following projection will appear in the explicit description of the operators D_n.

DEFINITION 5.41. For each $n > 1$, we denote $\varpi_n = p_n \circ q \circ \pi \colon \mathcal{H} \to \mathcal{L}_n$.

The projection ϖ_n kills $\zeta^{\mathfrak{m}}(2)$, all products and all motivic multiple zeta values of weight different from n.

PROPOSITION 5.42. *For $n < N$ odd, the action of D_n is given by*

$$\begin{aligned} D_n I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = \\ \sum_{p=0}^{N-n} \varpi_n \left(I^{\mathfrak{m}}(\varepsilon_p; \varepsilon_{p+1} \cdots \varepsilon_{p+n}; \varepsilon_{p+n+1}) \right) \\ \otimes I^{\mathfrak{m}}(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_p, \varepsilon_{p+n+1}, \dots, \varepsilon_N; \varepsilon_{N+1}). \end{aligned} \tag{5.43}$$

PROOF. The projection q kills all decomposable elements of $\mathcal{A}_{>0}$ and the projection p_n kills all the elements of degree different from n. Taking into account that

$$I^{\mathfrak{m}}(\varepsilon;\alpha;\varepsilon') = 1, \text{ if } \alpha = \emptyset, \text{ while}$$
$$I^{\mathfrak{m}}(\varepsilon;\alpha;\varepsilon') \in \mathcal{A}_{>0}, \text{ if } \alpha \neq \emptyset,$$

it follows that in the sum that runs over partitions

$$0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = N+1$$

only the terms having exactly one gap of length n survive. This gives the desired formula. □

5.2.4. *The kernel of $D_{<N}$.*

THEOREM 5.44. *Let $N \geq 2$ be an integer. Then*

$$\operatorname{Ker} D_{<N} \cap \mathcal{H}_N = \mathbb{Q}\zeta^{\mathfrak{m}}(N).$$

PROOF. Choosing a normalized isomorphism ϕ as in Lemma 5.39, the result follows from the combination of Lemma 5.25 and the commutativity of the diagram (5.38). □

The theorem has the following useful corollary:

COROLLARY 5.45. *Let $N \geq 2$ be an integer and $a^{\mathfrak{m}}$ an element of $\mathcal{H}_N$. Assume that $D_{<N}(a^{\mathfrak{m}}) = 0$ and $\operatorname{per}(a^{\mathfrak{m}}) = \alpha\zeta(N)$ for some rational number α. Then $a^{\mathfrak{m}} = \alpha\zeta^{\mathfrak{m}}(N)$ in $\mathcal{H}$.*

PROOF. Since $a^{\mathfrak{m}} \in \operatorname{Ker} D_{<N} \cap \mathcal{H}_N$, Theorem 5.44 gives the existence of a rational number β such that $a^{\mathfrak{m}} = \beta\zeta^{\mathfrak{m}}(N)$. Applying the period map, one gets $\beta\zeta(N) = \operatorname{per}(a^{\mathfrak{m}}) = \alpha\zeta(N)$, hence $\beta = \alpha$. □

The importance of this corollary is that it allows one to lift relations between classical multiple zeta values to their motivic counterparts.

⋆ ⋆ ⋆

EXERCISE 5.46. Show that the maps $D_{2r+1} \colon \mathcal{U} \to L_{2r+1} \otimes_{\mathbb{Q}} \mathcal{U}$ from (5.23) are derivations, that is, they satisfy

$$D_{2r+1}(\xi_1\xi_2) = (1 \otimes \xi_1)D_{2r+1}(\xi_2) + (1 \otimes \xi_2)D_{2r+1}(\xi_1)$$

for all $\xi_1, \xi_2 \in \mathcal{U}$. The same holds for the maps $D_{2r+1} \colon \mathcal{H} \to \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \mathcal{H}$ introduced in Definition 5.35.

EXERCISE 5.47 (Linear independence of $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$). The goal of this exercise is to prove the linear independence of the motivic multiple zeta values $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$ by exploiting the derivation D_3. Since $\mathcal{H}_5$ has dimension at most $d_5 = 2$ by Theorem A, it will follow that they form a basis. This is the first non-trivial case of Brown's theorem.

(a) Prove that $I^{\mathfrak{m}}(1;010;0) = -2\zeta^{\mathfrak{m}}(3)$ and $I^{\mathfrak{m}}(0;100;1) = -\zeta^{\mathfrak{m}}(3)$.

(b) Use the general formula (5.43) for the action of the derivation D_3 and the identities from part (a) of the exercise to compute

$$D_3\zeta^{\mathfrak{m}}(2,3) = -2\varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2),$$
$$D_3\zeta^{\mathfrak{m}}(3,2) = 3\varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2).$$

(c) Now assume that $\zeta^{\mathfrak{m}}(2,3) = \lambda\zeta^{\mathfrak{m}}(3,2)$ for some rational number λ. By part (b), then one necessarily has $\lambda = -2/3$. Upon application of the period map, argue that this leads to a contradiction.

Exercise 5.48 (Brown's proof in weight 5). The trick from the previous exercise does not generalize to higher weight. Here we present an alternative argument which can be seen as a toy case of Brown's proof.

(a) Prove the equality $D_3(\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)) = \varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2)$. Together with the computations in Exercise 5.47 and Theorem 5.44, this implies that there exist rational numbers $\alpha, \beta \in \mathbb{Q}$ such that

$$\zeta^{\mathfrak{m}}(2,3) + 2\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) = \alpha\zeta^{\mathfrak{m}}(5),$$
$$\zeta^{\mathfrak{m}}(3,2) - 3\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) = \beta\zeta^{\mathfrak{m}}(5).$$

(b) By virtue of Corollary 5.45, the stuffle product and the first identity in (1.68), deduce that $\alpha = 9/2$ and $\beta = -11/2$.

(c) Let $\mathrm{gr}_1^F\mathcal{H}_5^{2,3} \subset \mathcal{H}_5$ be the subspace spanned by $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$ (the reason for this notation will become apparent later). We define a linear map $(f, g)\colon \mathrm{gr}_1^F\mathcal{H}_5^{2,3} \to \mathbb{Q}^2$ by requiring

$$D_3(a) = f(a)\varpi_3(\zeta^{\mathfrak{m}}(3)) \otimes \zeta^{\mathfrak{m}}(2),$$
$$D_5(a) = g(a)\varpi_5(\zeta^{\mathfrak{m}}(5)) \otimes 1$$

for all $a \in \mathrm{gr}_1^F\mathcal{H}_5^{2,3}$. Use parts (a) and (b) to show that this map has rank two, hence $\zeta^{\mathfrak{m}}(2,3)$ and $\zeta^{\mathfrak{m}}(3,2)$ form a basis of $\mathcal{H}_5$.

5.3. Two families of motivic multiple zeta values and Zagier's theorem.

5.3.1. *Certain relations among motivic multiple zeta values.*

Lemma 5.49. *For each $n \geq 1$, the following equality holds:*

$$\zeta^{\mathfrak{m}}(2^{\{n\}}) = \frac{6^n}{(2n+1)!}\zeta^{\mathfrak{m}}(2)^n.$$

Proof. Recall that

$$\zeta^{\mathfrak{m}}(2^{\{n\}}) = I^{\mathfrak{m}}(1; 01.\overset{n}{\cdot\cdot}.01; 0).$$

Then $D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{n\}}) = 0$ for all $3 \leq 2r+1 < 2n$, because in formula (5.43) every sequence of the form $\varepsilon_p, \ldots, \varepsilon_{p+2r+2}$ will start and end with the same value. By (5.26) the corresponding motivic iterated integral is zero. Hence $\zeta^{\mathfrak{m}}(2^{\{n\}}) \in \operatorname{Ker} D_{<2n}$. By Theorem 5.44 and equation (5.40), we deduce that

$\zeta^{\mathsf{m}}(2^{\{n\}})$ is a rational multiple of $\zeta^{\mathsf{m}}(2)^n$. To get the precise multiple we use the period map and Example 1.29. □

In order to simplify notations, we write

$$\zeta_1^{\mathsf{m}}(\boldsymbol{s}) = I^{\mathsf{m}}(1; 0^{\{s_1-1\}}1 \cdots 0^{\{s_r-1\}}10; 0).$$

LEMMA 5.50. *For $n \geq 1$ the following equalities hold:*

$$\zeta_1^{\mathsf{m}}(2^{\{n\}}) = -2\sum_{i=0}^{n-1} \zeta^{\mathsf{m}}(2^{\{i\}}32^{\{n-i-1\}}), \tag{5.51}$$

$$\zeta_1^{\mathsf{m}}(2^{\{n\}}) = 2\sum_{i=1}^{n} (-1)^i \zeta^{\mathsf{m}}(2i+1)\zeta^{\mathsf{m}}(2^{\{n-i\}}). \tag{5.52}$$

PROOF. Recall from (5.26) that $I^{\mathsf{m}}(1;0;1) = 0$. Since the multiplication in $\mathcal{H}$ is given by the shuffle product, we have

$$0 = I^{\mathsf{m}}(1; 01.\overset{n}{\cdot\cdot}.01; 1)I^{\mathsf{m}}(1;0;1) = \zeta_1^{\mathsf{m}}(2^{\{n\}}) + 2\sum_{i=0}^{n-1} \zeta_1^{\mathsf{m}}(2^{\{i\}}32^{\{n-i-1\}}),$$

from which the identity (5.51) follows.

To prove equation (5.52) we first show the equality of multiple zeta values

$$-\sum_{i=0}^{n-1} \zeta(2^{\{i\}}32^{\{n-i-1\}}) = \sum_{i=1}^{n} (-1)^i \zeta(2i+1)\zeta(2^{\{n-i\}}) \tag{5.53}$$

using the stuffle product. Indeed, by Exercise 1.46, we have

$$\begin{aligned}
\zeta(3)\zeta(2^{\{n-1\}}) &= \sum_{i=0}^{n-1} \zeta(2^{\{i\}}32^{\{n-1-i\}}) + \sum_{i=0}^{n-2} \zeta(2^{\{i\}}52^{\{n-2-i\}})\\
\zeta(5)\zeta(2^{\{n-2\}}) &= \sum_{i=0}^{n-2} \zeta(2^{\{i\}}52^{\{n-2-i\}}) + \sum_{i=0}^{n-3} \zeta(2^{\{i\}}72^{\{n-3-i\}})\\
&\vdots\\
\zeta(2n-1)\zeta(2) &= \zeta(2n-1,2) + \zeta(2,2n-1) + \zeta(2n+1)\\
\zeta(2n+1) &= \zeta(2n+1).
\end{aligned}$$

Taking the alternate sum of each row we obtain equation (5.53).

We now prove equation (5.52) by induction on n. The case $n = 1$ is contained in Exercise 5.47. By Exercise 5.77, for $3 \leq 2r+1 < 2n$,

$$D_{2r+1}\zeta_1^{\mathsf{m}}(2^{\{n\}}) = \varpi_{2r+1}(\zeta_1^{\mathsf{m}}(2^{\{r\}})) \otimes \zeta^{\mathsf{m}}(2^{n-r}), \tag{5.54}$$

by induction hypothesis and the fact that ϖ_{2r+1} kills products

$$D_{2r+1}\zeta_1^{\mathsf{m}}(2^{\{n\}}) = 2(-1)^r\varpi_{2r+1}(\zeta^{\mathsf{m}}(2r+1)) \otimes \zeta^{\mathsf{m}}(2^{n-r}).$$

Moreover, using the fact that D_{2r+1} is a derivation,

$$D_{2r+1}(\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}})) = \varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1))) \otimes \zeta^{\mathfrak{m}}(2^{\{n-r\}}) \quad (5.55)$$

and for $r \neq i$

$$D_{2r+1}(\zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(2^{\{n-i\}})) = 0. \quad (5.56)$$

Therefore, if Θ is the difference of the two terms of equation (5.52), then for $3 \leq 2r+1 < 2n$,

$$D_{2r+1}\Theta = 0.$$

Hence, by Theorem 5.44, Θ is a multiple of $\zeta^{\mathfrak{m}}(2n+1)$, and formula (5.52) follows from Corollary 5.45 and equations (5.53) and (5.51). □

Given two integers r and s, we let $\mathbb{I}(r \geq s)$ denote the indicator function

$$\mathbb{I}(r \geq s) = \begin{cases} 1 & r \geq s, \\ 0 & \text{else.} \end{cases}$$

LEMMA 5.57. *Let $a, b \geq 0$ be integers. For each $1 \leq r \leq a+b$, one has*

$$D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = \varpi_{2r+1}(\xi^r_{a,b}) \otimes \zeta^{\mathfrak{m}}(2^{\{a+b+1-r\}}),$$

where the element $\xi^r_{a,b} \in \mathcal{H}$ is given by

$$\xi^r_{a,b} = \sum_{\substack{\alpha \leq a \\ \beta \leq b \\ \alpha+\beta=r-1}} \zeta^{\mathfrak{m}}(2^{\{\beta\}}32^{\{\alpha\}}) - \sum_{\substack{\alpha \leq a \\ \beta \leq b \\ \alpha+\beta=r-1}} \zeta^{\mathfrak{m}}(2^{\{\alpha\}}32^{\{\beta\}}) + \Big(\mathbb{I}(b \geq r) - \mathbb{I}(a \geq r)\Big)\zeta^{\mathfrak{m}}_1(2^{\{r\}}). \quad (5.58)$$

PROOF. To prove the result it is enough to check which non-zero terms appear in formula (5.43). These terms are given by consecutive subsequences of $2r+1$ entries and can be of the following types:

(1) subsequences containing 001 and starting with 1, these contribute to the first sum;
(2) subsequences containing 001 and starting with 0, after applying the reflection formula of Lemma 5.27, these subsequences contribute to the second sum;
(3) when $b \geq r$ there is exactly one sequence ending with 00 this gives the term $\mathbb{I}(b \geq r)\zeta^{\mathfrak{m}}_1(2^{\{r\}})$;
(4) when $a \geq r$ there is exactly one sequence starting with 00. After applying the reflection formula we obtain the term $-\mathbb{I}(a \geq r)\zeta^{\mathfrak{m}}_1(2^{\{r\}})$.

Using equation (5.26) is easy to check that all the other subsequences do not contribute to the result. □

Proposition 5.59. *Given $a, b \geq 0$, write $n = a + b + 1$. There exists a unique n-tuple of rational numbers $(\alpha^r_{a,b})_{r=1,\dots,n}$ such that*

$$\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = \sum_{r=1}^{n} \alpha^r_{a,b}\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}}). \tag{5.60}$$

Proof. The proof proceeds by induction on n. Assume that the result holds for all integers smaller than n. In particular, all the numbers $\alpha^r_{a,b}$ are defined for $a + b + 1 < n$. Consider a, b such that $a + b + 1 = n$. We compute $D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}})$ for all $r < n$. Let $\xi^r_{a,b}$ be the term that appears in Lemma 5.57. By induction hypothesis and equation (5.52) we can write

$$\xi^r_{a,b} \equiv \alpha^r_{a,b}\zeta^{\mathfrak{m}}(2r+1) \quad \text{mod products}$$

for a rational number $\alpha^r_{a,b}$. Therefore

$$\varpi_{2r+1}(\xi^r_{a,b}) = \alpha^r_{a,b}\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1)), \tag{5.61}$$

from which it follows that

$$D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = \alpha^r_{a,b}\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1)) \otimes \zeta^{\mathfrak{m}}(2^{\{n-r\}}). \tag{5.62}$$

Using equations (5.55) and (5.56) we deduce that both sides of the equation to be proved have the same image under $D_{<2n+1}$. By Theorem 5.44, they differ by a rational multiple of $\zeta^{\mathfrak{m}}(2n+1)$ and one defines $\alpha^n_{a,b}$ in such a way that this difference is zero. $\square$

We have here a remarkable example of both the strength and the limits of the motivic formalism. Applying the period map (5.9), the motivic identity (5.60) implies that the same holds for usual multiple zeta values, something which would have been difficult to predict working only with numbers, where the coaction is invisible. However, the motivic formalism alone does not allow us to compute the precise value of the constants $\alpha^r_{a,b}$. For this one needs to prove the corresponding identity of numbers first, then show that it is motivic. The first task was accomplished by Zagier in [**Zag12**].

5.3.2. *Zagier's theorem.* Define, for each $a, b, r \geq 0$, rational numbers

$$A^r_{a,b} = \binom{2r}{2a+2}, \quad B^r_{a,b} = (1 - 2^{-2r})\binom{2r}{2b+1}. \tag{5.63}$$

As in the previous paragraph, we set $n = a + b + 1$.

Theorem 5.64 (Zagier, [**Zag12**]). *The following equality holds:*

$$\zeta(2^{\{b\}}32^{\{a\}}) = 2\sum_{r=1}^{n}(-1)^r\left(A^r_{a,b} - B^r_{a,b}\right)\zeta(2r+1)\zeta(2^{\{n-r\}}). \tag{5.65}$$

Remark 5.66. The original proof of Zagier's theorem has been simplified in [**Li13**]. It is also worth mentioning that Terasoma [**Ter13**] showed that the relation (5.65) holds for any associator.

5.3.3. *Lifting Zagier's theorem to a motivic identity.* The first non-trivial case of Zagier's theorem are the identities

$$\begin{aligned}\zeta(2,3) &= -2\zeta(3)\zeta(2) + \tfrac{9}{2}\zeta(5),\\ \zeta(3,2) &= 3\zeta(3)\zeta(2) - \tfrac{11}{2}\zeta(5).\end{aligned}$$

In Exercise 5.48 we show that they lift to motivic equalities.

THEOREM 5.67. *For $a, b \geq 0$ and $1 \leq r \leq a+b+1$, the numbers $\alpha^r_{a,b}$ from the statement of Proposition 5.59 are equal to*

$$\alpha^r_{a,b} = (-1)^r 2\left(A^r_{a,b} - B^r_{a,b}\right). \tag{5.68}$$

In other words, writing $n = a+b+1$, the following identity of motivic multiple zeta values holds:

$$\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) = 2\sum_{r=1}^{n}(-1)^r\left(A^r_{a,b} - B^r_{a,b}\right)\zeta^{\mathfrak{m}}(2r+1)\zeta^{\mathfrak{m}}(2^{\{n-r\}}). \tag{5.69}$$

PROOF. We first note that, for any $a, b \geq 0$ and $1 \leq r \leq a+b+1$, the following identities are satisfied:

$$A^r_{a,b} = \sum_{\substack{\alpha\leq a\\ \beta\leq b\\ \alpha+\beta=r-1}} A^r_{\alpha,\beta} - \sum_{\substack{\alpha\leq a\\ \beta< b\\ \alpha+\beta=r-1}} A^r_{\beta,\alpha} + \mathbb{I}(b\geq r) - \mathbb{I}(a\geq r), \tag{5.70}$$

$$B^r_{a,b} = \sum_{\substack{\alpha\leq a\\ \beta\leq b\\ \alpha+\beta=r-1}} B^r_{\alpha,\beta} - \sum_{\substack{\alpha\leq a\\ \beta< b\\ \alpha+\beta=r-1}} B^r_{\beta,\alpha}. \tag{5.71}$$

This can be proved using that $A^r_{a,b}$ does not depend on b, that $B^r_{a,b}$ does not depend on a, and the symmetries $A^{\alpha+\beta+1}_{\alpha,\beta} = A^{\alpha+\beta+1}_{\beta-1,\alpha+1}$ and $B^{\alpha+\beta+1}_{\alpha,\beta} = B^{\alpha+\beta+1}_{\beta,\alpha}$. For instance, the second equality is clear because by symmetry each term of the second sum cancels one term of the first sum; the only remaining term in the first sum is $B^r_{r-1-b,b}$ that agrees with $B^r_{a,b}$ because it is independent of a. To prove the first equality we may distinguish different cases according to whether a and b are bigger or equal to r or not. For instance if $a < r$ and $b \geq r$, the term $A^r_{a,b}$ is different from zero. In this cases both sums range from $(\alpha,\beta) = (a, r-1-a)$ to $(0, r-1)$. By the symmetry of the A's all terms cancel except $A^r_{a,r-1-a}$ from the first sum, that agrees with $A^r_{a,b}$ and $-A^r_{r-1,0} = -1$ that cancels with $\mathbb{I}(b \geq r)$. The remaining cases are similar.

We now prove the result by induction on $n = a+b+1$. So we assume that equation (5.68) is true for all a', b' with $a'+b' < n-1$ and all $1 \leq r' \leq a'+b'+1$ and fix a, b with $a+b+1 = n$. We conpute $D_{2r+1}\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}})$ in two ways and compare the results. The first way is equation (5.62), while the second is to apply Lemma 5.57, then use Lemma 5.50 to get rid of the terms $\zeta^{\mathfrak{m}}_1(2^{\{r\}})$

and apply equation (5.61) to the terms $\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2^{\{?\}}32^{\{?\}}))$. Comparing both results we obtain

$$\alpha^r_{a,b} = \sum_{\substack{\alpha\leq a\\ \beta\leq b\\ \alpha+\beta=r-1}} \alpha^r_{\alpha,\beta} - \sum_{\substack{\alpha\leq a\\ \beta< b\\ \alpha+\beta=r-1}} \alpha^r_{\beta,\alpha} + 2(-1)^r\big(\mathbb{I}(b\geq r) - \mathbb{I}(a\geq r)\big).$$

By the induction hypothesis and equations (5.70) and (5.71), we deduce the equality (5.68) for $1\leq r\leq a+b$.

To treat the remaining case $r=a+b+1$, let Θ be the difference between the left and right hand sides of equation (5.69); it is a motivic zeta value of weight $2a+2b+3$. The identities we already proved and equation (5.60) yield $D_{<2a+2b+3}(\Theta)=0$. By Zagier's Theorem 5.64, we obtain $\operatorname{per}(\Theta)=0$. Finally, Corollary 5.45 implies $\Theta=0$, thus proving the result. □

5.3.4. *The coefficients $c_{\boldsymbol{s}}$.* Among the coefficients $\alpha^r_{a,b}$, the leading one $\alpha^{a+b+1}_{a,b}$ will play a special role, so we single it out.

Definition 5.72. Let $\boldsymbol{s}=2^{\{b\}}32^{\{a\}}$ be an admissible multi-index with only one entry equal to 3 and all the remaining entries equal to 2. We set

$$c_{\boldsymbol{s}} = \alpha^{a+b+1}_{a,b}.$$

We will also write

$$c_{12^{\{n\}}} = 2(-1)^n.$$

Lemma 5.50 and Proposition 5.59 are then rephrased as follows:

Corollary 5.73. *For positive integers n,a,b with $n=a+b+1$, the following equalities hold:*

(1) $\varpi_{2n+1}(\zeta^{\mathfrak{m}}_1(2^{\{n\}})) = c_{12^{\{n\}}}\zeta^{\mathfrak{m}}(2n+1)$,
(2) $\varpi_{2n+1}(\zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}})) = c_{2^{\{b\}}32^{\{a\}}}\zeta^{\mathfrak{m}}(2n+1)$.

Moreover,

$$c_{12^{\{n\}}} = -2\sum_{i=0}^{n-1} c_{2^{\{i\}}32^{\{n-i-1\}}}. \tag{5.74}$$

Recall that, given a prime number p, the p-adic valuation of a non-zero rational number x is the only integer $v_p(x)$ such that x can be written in the form $x=p^{v_p(x)}\frac{a}{b}$ with a and b relatively prime to p. We set $v_p(0)=\infty$.

As a consequence of Theorem 5.67, the coefficients c_w have the following 2-adic properties.

Lemma 5.75. *Let w a word of the form $w=2^{\{b\}}32^{\{a\}}$ and denote by w^* the word written in reverse order, i.e. $w^*=2^{\{a\}}32^{\{b\}}$. Then*

(1) $c_w\in\mathbb{Z}[\frac{1}{2}]$,
(2) $c_w - c_{w^*}$ *is an even integer,*

(3) $v_2(c_{2^{\{n-1\}}3}) = v_2(c_{32^{\{n-1\}}}) \le v_2(c_w) \le 0.$

PROOF. Set $n = a+b+1$. Recall the formula from Theorem 5.67:

$$c_w = (-1)^n 2\left(A^n_{a,b} - B^n_{a,b}\right).$$

Since $A^n_{a,b}$ is an integer and $B^n_{a,b}$ belongs to $\mathbb{Z}[\frac{1}{2}]$, the first claim follows.

Property (2) is obtained from the symmetry $B^n_{a,b} = B^n_{b,a}$. Indeed,

$$c_w - c_{w^*} = (-1)^n 2[A^n_{a,b} - A^n_{b,a}] \in 2\mathbb{Z}.$$

To prove (3), we first observe that $v_2((2n)!) < 2n$, hence

$$v_2(2^{-2n}\tbinom{2n}{2b+1}) < 0.$$

Using the triangle inequality, it follows that

$$v_2(c_w) = v_2(2\cdot 2^{-2n}\tbinom{2n}{2b+1}) = 1 + v_2(2^{-2n}\tbinom{2n}{2b+1}) \le 0.$$

For the remaining inequality, we write

$$\tbinom{2n}{2b+1} = \frac{2n}{2b+1}\tbinom{2n-1}{2b}.$$

Therefore,

$$v_2(c_w) = 2 - 2n + v_2(n) + v_2(\tbinom{2n-1}{2b}).$$

Since $v_2(\binom{2n-1}{2b}) \ge 0$, the right-hand side attains its minimum for $b = n-1$ and $b = 0$, which correspond to the cases $w = 2^{\{n-1\}}3$ and $w = 32^{\{n-1\}}$. $\square$

⋆ ⋆ ⋆

EXERCISE 5.76. Show that one may replace $\zeta(2^{\{n-r\}})$ by either $\zeta(2n-2r)$ or $\zeta(2)^{n-r}$ in the right-hand side of Zagier's theorem 5.64 without losing the rationality of the coefficients $\alpha^r_{a,b}$.

EXERCISE 5.77. Prove equation (5.54).

5.4. The subspaces $\mathcal{H}^{2,3}$.

DEFINITION 5.78. We denote by $\widetilde{\mathcal{H}}^{2,3} \subset \mathcal{O}(\Pi)$ the subspace generated by the functions $I(1;\alpha;0)$, where α is the binary sequence associated to an admissible multi-index containing only 2 and 3 as entries, and by $\mathcal{H}^{2,3} \subseteq \mathcal{H}$ the image of $\widetilde{\mathcal{H}}^{2,3}$ under the restriction map

$$\mathrm{res}\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{H}.$$

Clearly, $\mathcal{H}^{2,3}$ is the $\mathbb{Q}$-vector space spanned by the motivic multiple zeta values $\zeta^{\mathfrak{m}}(s_1, \ldots, s_r)$ with $s_i \in \{2, 3\}$.

From now on, we identify the set of words in the alphabet $\{2, 3\}$ with the set of admissible multi-indices with only 2 and 3 as entries.

We filter $\widetilde{\mathcal{H}}^{2,3}$ by the number of entries equal to 3 in the admissible multi-index. Precisely, for each integer $\ell \geq 0$, consider

$$F_\ell \widetilde{\mathcal{H}}^{2,3} = \langle I(1; \mathrm{bs}(\boldsymbol{s}); 0) \mid \boldsymbol{s} \text{ contains} \leq \ell \text{ entries equal to } 3\rangle_{\mathbb{Q}}.$$

This defines an increasing *level filtration*

$$0 \subseteq F_0 \widetilde{\mathcal{H}}^{2,3} \subseteq F_1 \widetilde{\mathcal{H}}^{2,3} \subseteq \cdots$$

By restriction, we deduce an increasing filtration on $\mathcal{H}^{2,3}$ with

$$F_\ell \mathcal{H}^{2,3} = \langle \zeta^{\mathfrak{m}}(s_1, \ldots, s_r) \in \mathcal{H}^{2,3} \mid \text{number of } s_i = 3 \leq \ell\rangle_{\mathbb{Q}}.$$

The associated graded pieces $\mathrm{gr}_\ell^F \mathcal{H}^{2,3}$ are the $\mathbb{Q}$-linear spans of motivic multiple zeta values with exactly ℓ entries equal to 3. In particular,

$$\begin{aligned}\mathrm{gr}_0^F \mathcal{H}^{2,3} &= \langle \zeta^{\mathfrak{m}}(2^{\{n\}}) \mid n \geq 1\rangle_{\mathbb{Q}},\\ \mathrm{gr}_1^F \mathcal{H}^{2,3} &= \langle \zeta^{\mathfrak{m}}(2^{\{b\}}32^{\{a\}}) \mid a, b \geq 0\rangle_{\mathbb{Q}}.\end{aligned}$$

Note that these are precisely the two families of motivic multiple zeta values that we studied in the previous section.

REMARK 5.79. The $\mathbb{Q}$-vector space $\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3}$ is non-empty if and only if the weight N and the level ℓ have the same parity. When this is the case, writing $N = 2m + 3\ell$, the dimensions are given by

$$\dim_{\mathbb{Q}} \mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} = \binom{m+\ell}{\ell}.$$

5.4.1. *The level lowering operator.* Recall that in Section 3.9.6 we introduced Goncharov's coproduct as a morphism

$$\Delta\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{O}(\Pi) \otimes_{\mathbb{Q}} \mathcal{O}(\Pi). \tag{5.80}$$

From this we obtained the motivic coaction (5.32)

$$\Delta\colon \mathcal{H} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$$

that we have been using in the last pages. In what follows, we will also use an intermediate version

$$\Delta\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{A} \otimes \mathcal{O}(\Pi)$$

which is simply obtained from (5.80) via the projection $\mathcal{O}(\Pi) \to \mathcal{A}$ (recall that this corresponds to restricting a function on Π to the subvariety $\mathcal{X}$ of Remark 5.30). This is nothing else but the coaction associated to the action of $I(U_{\mathrm{dR}})$ on Π. As in Definition 5.35, there are maps

$$D_{2r+1}\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{L}_{2r+1} \otimes \mathcal{O}(\Pi).$$

Following the proof of Proposition 5.42 we see that, for all odd integers $n < N$, the analogue of (5.43) also holds:

$$D_n I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_N; \varepsilon_{N+1}) = \sum_{p=0}^{N-n} \varpi_n \left(I^{\mathfrak{m}}(\varepsilon_p; \varepsilon_{p+1} \cdots \varepsilon_{p+n}; \varepsilon_{p+n+1}) \right) \otimes I(\varepsilon_0; \varepsilon_1 \cdots \varepsilon_p, \varepsilon_{p+n+1}, \dots, \varepsilon_N; \varepsilon_{N+1}). \tag{5.81}$$

We now study how the filtered subspace $\widetilde{\mathcal{H}}^{2,3} \subset \mathcal{O}(\Pi)$ behaves with respect to the coproduct and the infinitesimal coaction.

LEMMA 5.82. *The subspace $\widetilde{\mathcal{H}}^{2,3}$ is stable under the coaction:*

$$\Delta \colon \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \widetilde{\mathcal{H}}^{2,3}.$$

PROOF. Let $I(1; \alpha; 0)$ be an element of $\widetilde{\mathcal{H}}^{2,3}$. Then α is a binary sequence obtained by successive concatenation of the subsequences 01 and 001. From the explicit formula for the coproduct (5.33) and the fact that the iterated integrals $I(\varepsilon; \alpha'; \varepsilon')$ vanish when $\varepsilon = \varepsilon'$ and $\alpha' \neq \emptyset$, we deduce that each non-trivial term appearing in $\Delta I(1; \alpha; 0)$ has, in the right hand side of the coaction, a factor of the form $I(1; \beta; 0)$, where β is again a concatenation of the subsequences 01 and 001. □

REMARK 5.83. In [**Del13**, §6.3], the above result is rephrased by saying that the subspace $\widetilde{\mathcal{H}}^{2,3}$ is "motivic", thus invariant under the action of U_{dR}.

From this we immediately deduce:

COROLLARY 5.84. *For each $r \geq 1$, the derivation D_{2r+1} restricts to a map*

$$D_{2r+1} \colon \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} \widetilde{\mathcal{H}}^{2,3}.$$

In fact, more is true:

LEMMA 5.85. *The derivations D_{2r+1} are compatible with the level filtration, in the sense that:*

$$D_{2r+1} \colon F_\ell \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} F_{\ell-1} \widetilde{\mathcal{H}}^{2,3}.$$

PROOF. Given a word $\boldsymbol{s}$ in the alphabet $\{2,3\}$ of level ℓ, then $\mathrm{bs}(\boldsymbol{s})$ contains exactly ℓ subsequences 00. Any subsequence of odd length of $(1; \mathrm{bs}(\boldsymbol{s}); 0)$ that begins and ends with the same symbol will be killed by $I^{\mathfrak{m}}$ and will not contribute to D_{2r+1}. Otherwise it must contain at least a subsequence 00. Thus the complementary quotient sequence will contain at most $\ell - 1$ subsequences 00. Hence will have level at most $\ell - 1$. □

The above lemma yields a map

$$\mathrm{gr}_\ell^F D_{2r+1} \colon \mathrm{gr}_\ell^F \widetilde{\mathcal{H}}^{2,3} \longrightarrow \mathcal{L}_{2r+1} \otimes \mathrm{gr}_{\ell-1}^F \widetilde{\mathcal{H}}^{2,3}. \tag{5.86}$$

LEMMA 5.87. *For all $r, \ell \geq 1$, one has*

$$\mathrm{gr}_\ell^F D_{2r+1}(\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}^{2,3}) \subseteq \mathbb{Q}\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1)) \otimes_{\mathbb{Q}} \mathrm{gr}_{\ell-1}^F \widetilde{\mathcal{H}}^{2,3}.$$

PROOF. Let $\boldsymbol{s}$ be a word in the alphabet $\{2,3\}$ of level ℓ, and let $I^{\mathfrak{m}}(1; \mathrm{bs}(\boldsymbol{s}); 0)$ be the corresponding motivic iterated integral. From the definition of D_{2r+1}, we have

$$\mathrm{gr}_\ell^F D_{2r+1}(\zeta^{\mathfrak{m}}(\boldsymbol{s})) = \sum_{\gamma} \varpi_{2r+1}(I^{\mathfrak{m}}(\gamma)) \otimes \zeta^{\mathfrak{m}}(\boldsymbol{s}_\gamma), \tag{5.88}$$

where the sum runs over all subsequences γ of $(1; \mathrm{bs}(\boldsymbol{s}); 0)$ of length $2r+1$, and $\boldsymbol{s}_\gamma$ is obtained by removing the internal part of γ.

If γ contains more than one subsequence 00, then $\boldsymbol{s}_\gamma$ has level $< \ell - 1$, hence does not contribute. If γ begins and ends in the same symbol, then $I^{\mathfrak{m}}(\gamma)$ is zero. One checks that $I^{\mathfrak{m}}(\gamma)$ can be of four remaining types:

(1) $I^{\mathfrak{m}}(1; 01 \ldots 01001 \ldots 01; 0) = \zeta^{\mathfrak{m}}(2^{\{\beta\}}32^{\{\alpha\}})$,
(2) $I^{\mathfrak{m}}(0; 10 \ldots 10010 \ldots 10; 1) = -\zeta^{\mathfrak{m}}(2^{\{\beta\}}32^{\{\alpha\}})$,
(3) $I^{\mathfrak{m}}(1; 01 \ldots 10; 0) = \zeta_1^{\mathfrak{m}}(2^{\{r\}})$,
(4) $I^{\mathfrak{m}}(0; 01 \ldots 10; 1) = -\zeta_1^{\mathfrak{m}}(2^{\{r\}})$.

By Corollary 5.73, in all cases one has $\varpi_{2r+1}(I^{\mathfrak{m}}(\gamma)) \in \mathbb{Q}\zeta^{\mathfrak{m}}(2r+1)$. $\square$

REMARK 5.89. Lemma 5.87 says that the map (5.86) factors through the one-dimensional subspace $[(\mathfrak{u}_{\mathrm{dR}})^{\mathrm{ab}}_{2r+1}]^\vee$ of $\mathcal{L}_{2r+1}$.

The above lemma justifies the following definition:

DEFINITION 5.90. For all $N, \ell \geq 1$, the *level lowering operator* $\widetilde{\partial}_{N,\ell}$ is the $\mathbb{Q}$-linear map

$$\widetilde{\partial}_{N,\ell} \colon \mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} \longrightarrow \bigoplus_{3 \leq 2r+1 \leq N} \mathrm{gr}_{\ell-1}^F \widetilde{\mathcal{H}}_{N-2r-1}^{2,3} \tag{5.91}$$

obtained by first applying

$$\bigoplus_{3 \leq 2r+1 \leq N} \mathrm{gr}_\ell^F D_{2r+1}|_{\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3}}$$

and then sending $\varpi_{2r+1}(\zeta^{\mathfrak{m}}(2r+1))$ to 1.

5.4.2. *A pair of bases.* We next describe bases of the source and the target of the map (5.91). For $\ell \geq 1$ and $N \geq 3$, we define

$B_{N,\ell}$ =set of words in the alphabet $\{2,3\}$ of weight N and level ℓ.

$B'_{N,\ell}$ =set of words in the alphabet $\{2,3\}$ of weight $\leq N-3$ and level $\ell - 1$ (this includes the empty word if $\ell = 1$).

Clearly, $B_{N,\ell}$ gives a basis $\mathcal{B}_{N,\ell}$ of $\mathrm{gr}^F_\ell \widetilde{\mathcal{H}}^{2,3}_N$, while $B'_{N,\ell}$ determines a basis $\mathcal{B}'_{N,\ell}$ of $\bigoplus_{3\leq 2r+1\leq N} \mathrm{gr}^F_{\ell-1} \widetilde{\mathcal{H}}^{2,3}_{N-2r-1}$. Write $N = 3\ell + 2m$, so m is the number of 2s in an element of $B_{N,\ell}$. Then

$$|B_{N,\ell}| = \binom{\ell+m}{\ell} = \sum_{m'=0}^{m} \binom{\ell-1+m'}{\ell-1} = |B'_{N,\ell}|.$$

We provide $B_{N,\ell}$ with the lexicographic order for the ordering $2 < 3$ and $B'_{N,\ell}$ with the order $\boldsymbol{s} \leq \boldsymbol{s}'$ if and only if $\mathrm{wt}(\boldsymbol{s}) < \mathrm{wt}(\boldsymbol{s}')$ or $\mathrm{wt}(\boldsymbol{s}) = \mathrm{wt}(\boldsymbol{s}')$ and $\boldsymbol{s}$ is smaller than or equal to $\boldsymbol{s}'$ in the lexicographic order.

LEMMA 5.92. *The map $B'_{N,\ell} \to B_{N,\ell}$ that sends an element $\boldsymbol{s} \in B'_{N,\ell}$ to $2^{\{r-1\}}3\boldsymbol{s} \in B_{N,\ell}$, where $2r = N - 1 - \mathrm{wt}(\boldsymbol{s})$ is an order preserving bijection.*

PROOF. Denote by υ the map in the statement. If $\mathrm{wt}(\boldsymbol{s}) < \mathrm{wt}(\boldsymbol{s}')$, then $r > r'$, hence $\upsilon(\boldsymbol{s}) = 2^{\{r-1\}}3\boldsymbol{s} < 2^{\{r'-1\}}3\boldsymbol{s}' = \upsilon(\boldsymbol{s}')$. If $\mathrm{wt}(\boldsymbol{s}) = \mathrm{wt}(\boldsymbol{s}')$ but $\boldsymbol{s}$ is smaller than $\boldsymbol{s}'$ in the lexicographic order, then $\upsilon(\boldsymbol{s}) = 2^{\{r-1\}}3\boldsymbol{s} < 2^{\{r-1\}}3\boldsymbol{s}' = \upsilon(\boldsymbol{s}')$. Therefore, υ is injective and order preserving. Since the sets $B'_{N,\ell}$ and $B_{N,\ell}$ have the same cardinality, υ is a bijection. □

5.5. Brown's theorem.

5.5.1. *Statement.* The goal of this section is to prove the following result, which directly implies Theorem B:

THEOREM 5.93 (Brown). *The set of elements*

$$\{\zeta^{\mathfrak{m}}(s_1, \dots, s_r) \mid s_i \in \{2,3\}\}$$

forms a basis of the $\mathbb{Q}$-vector space of motivic multiple zeta values.

Before going into the proof, let us mention the immediate corollary:

COROLLARY 5.94 (Theorem B). *Every multiple zeta value is a $\mathbb{Q}$-linear combination of MZVs with only 2 and 3 as entries.*

PROOF. Apply the period map (5.9). □

REMARKS 5.95.

(1) Unfortunately, the proof does not give an algorithm to compute such a linear combination.
(2) The missing information to deduce that such multiple zeta values furnish a basis, as it is conjectured, is to know that all relations among multiple zeta values have motivic origin.

5.5.2. *Strategy of the proof.* The key point to prove Theorem 5.93 is the following lemma.

LEMMA 5.96. *For all $N, \ell \geq 1$, the level lowering operator $\widetilde{\partial}_{N,\ell}$ is an isomorphism of $\mathbb{Q}$-vector spaces.*

We show how to deduce Theorem 5.93 from Lemma 5.96. This amounts to proving the following:

LEMMA 5.97. *The map $\widetilde{\mathcal{H}}^{2,3} \to \mathcal{H}^{2,3}$ is an isomorphism.*

PROOF. We first prove by induction on the level that, for every weight N and level ℓ, the restriction map $\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} \to \mathrm{gr}_\ell^F \mathcal{H}_N^{2,3}$ is an isomorphism.

The initial step is $\ell = 0$. If $N = 2r$ is even, the space $\mathrm{gr}_0^F \widetilde{\mathcal{H}}_N^{2,3}$ is one-dimensional generated by the symbol $I(1; \mathrm{bs}(2^{\{r\}}); 0)$ while the space $\mathrm{gr}_0^F \mathcal{H}_N^{2,3}$ is generated by $\zeta^{\mathfrak{m}}(2^{\{r\}}) \neq 0$. Thus the restriction map

$$\mathrm{gr}_0^F \widetilde{\mathcal{H}}_N^{2,3} \longrightarrow \mathrm{gr}_0^F \mathcal{H}_{N,0}^{2,3} \tag{5.98}$$

is an isomorphism. If N is odd, then both spaces are zero and therefore the map (5.98) is also an isomorphism.

We now consider the commutative diagram

$$\begin{array}{ccc} \mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} & \xrightarrow{\widetilde{\partial}_{N,\ell}} & \bigoplus_{3\leq 2r+1\leq N} \mathrm{gr}_{\ell-1}^F \widetilde{\mathcal{H}}_{N-2r-1}^{2,3} \\ \downarrow & & \downarrow \\ \mathrm{gr}_\ell^F \mathcal{H}^{2,3} & \xrightarrow{\partial_{N,\ell}} & \bigoplus_{3\leq 2r+1\leq N} \mathrm{gr}_{\ell-1}^F \mathcal{H}_{N-2r-1}^{2,3}. \end{array} \tag{5.99}$$

By definition, the left vertical arrow is an epimorphism. By the induction hypothesis, the right vertical map is an isomorphism and by Lemma 5.96 the upper horizontal map is injective. We conclude that the left vertical arrow is an isomorphism.

Once we now that all the restriction maps $\mathrm{gr}_\ell^F \widetilde{\mathcal{H}}_N^{2,3} \to \mathrm{gr}_\ell^F \mathcal{H}_N^{2,3}$ are isomorphisms, we deduce that the restriction map $\widetilde{\mathcal{H}}_N^{2,3} \to \mathcal{H}_N^{2,3}$ is an isomorphism by using the fact that the filtration F is bounded below and the five lemma. Finally, since the weight is a grading in both $\widetilde{\mathcal{H}}^{2,3}$ and $\mathcal{H}^{2,3}$ we obtain that the map $\widetilde{\mathcal{H}}^{2,3} \to \mathcal{H}^{2,3}$ is an isomorphism. □

5.5.3. *Proof of Lemma 5.96.* The proof is based on the study of the 2-adic valuation of the coefficients of the matrix of $\widetilde{\partial}_{N,\ell}$ with respect to the bases introduced in Section 5.4.2. We shall use the following lemma:

LEMMA 5.100. *Let $A = (a_{ij})_{i,j}$ be a square matrix of size n with rational coefficients. Assume that there exists a prime number p such that the following conditions hold:*

(a) $v_p(a_{ij}) \geq 1$ *for all* $i > j$,

(b) $v_p(a_{ii}) = \min_j\{v_p(a_{ij})\} \leq 0$ *for all* i.

Then A *is invertible.*

Proof. Consider the matrix A' obtained by multiplying the i-th row of A by $p^{-v_p(a_{ii})}$. By condition (b), the coefficients of A' are p-integral, so we can reduce modulo p. Since we still have $v_p(a'_{ij}) \geq 1$ for $i > j$ but now $v_p(a'_{ii}) = 0$, the reduction is upper triangular with non-zero elements in the diagonal. It follows that the determinant of A', and hence the determinant of A, is non-zero. $\square$

We next see that, up to terms with even coefficients, the map $\widetilde{\partial}_{N,\ell}$ acts by deconcatenation.

Theorem 5.101. *Let* $\boldsymbol{s}$ *be a word in the alphabet* $\{2,3\}$ *of weight* N *and level* ℓ. *Then*

$$\widetilde{\partial}_{N,\ell} I(1; \mathrm{bs}(\boldsymbol{s}); 0)) = \sum_{\substack{\boldsymbol{s}=\boldsymbol{uv} \\ \deg_3 \boldsymbol{u}=1}} c_{\boldsymbol{u}} I(1; \mathrm{bs}(\boldsymbol{v}); 0) + \text{terms with } 2\mathbb{Z} \text{ coefficients},$$

where $\deg_3 \boldsymbol{u}$ *is the numbers of* 3 *in the word* $\boldsymbol{u}$.

Proof. Following the proof of Lemma 5.87, there are four types of terms in $\widetilde{\partial}_{N,\ell} I(1; \mathrm{bs}(\boldsymbol{s}); 0))$. We start with (3) and (4). Since $c_{12^{\{n\}}} = 2(-1)^n$, these terms contribute with even coefficients. Besides, almost all terms of types (1) and (2) can be grouped in pairs. Choose four positions as follows

$$I(\dots 0\underset{a}{1}\underset{b}{0} \dots 01001 \dots 0\underset{c}{1}\underset{d}{0} \dots),$$

that is, a and b (resp. c and d) are consecutive, a (resp. d) contains a 0 and b (resp. c) contains a 1. Combining Lemma 5.27 (2) Lemma 5.75 (2), the sum of the contributions of the subsequences ac and bd has again coefficients in $2\mathbb{Z}$. The only terms that cannot be paired this way are the leftmost subsequences appearing in the sum of the statement. $\square$

Corollary 5.102. *With respect to the bases* $\mathcal{B}_{N,\ell}$ *and* $\mathcal{B}'_{N,\ell}$, *ordered as in paragraph 5.4.2, the matrix* $M_{N,\ell}$ *of the operator* $\widetilde{\partial}_{N,\ell}$ *satisfies the assumptions of Lemma 5.100 for the prime* $p = 2$.

Proof. Let $\boldsymbol{v}$ be an admissible multi-index with only 2 and 3 as entries, of weight $\leq N-3$ and level $\ell - 1$. Put $2r = N - 1 - \mathrm{wt}(\boldsymbol{v})$ and $\boldsymbol{s} = 2^{\{r-1\}}3\boldsymbol{v}$. Then $\boldsymbol{s}$ is the multi-index corresponding to $\boldsymbol{v}$ under the order-preserving bijection from Lemma 5.92. Consider any admissible multi-index with only 2 and 3 as entries, of weight N and level ℓ that can be written as $\boldsymbol{uv}$ with $\deg_3 \boldsymbol{u} = 1$. If $\boldsymbol{s} \neq \boldsymbol{uv}$, then the number on 2s before the first 3 in $\boldsymbol{u}$ is smaller than $r - 1$. Hence $\boldsymbol{uv} > \boldsymbol{s}$. By Theorem 5.101, this implies that any term in $M_{N,\ell}$ that is not an even integer is above the diagonal. Moreover, by the same theorem and the statement (3) of Lemma 5.75, the coefficient of $\boldsymbol{v}$ in $\widetilde{\partial}_{N,\ell}\boldsymbol{s}$ sitting at the diagonal of $M_{N,\ell}$ has 2-adic valuation smaller than or equal to zero and it realizes the minimum of this valuation within its row. Therefore, the assumptions of Lemma 5.100 are satisfied. $\square$

Clearly, Lemma 5.96 is a consequence of Corollary 5.102 and Lemma 5.100, thus finishing the proof of Theorem 5.93.

5.5.4. *Some consequences of Brown's theorem.* We conclude these notes with some corollaries of Brown's theorem:

COROLLARY 5.103. *The map $U_{\mathrm{dR}} \to I(U_{\mathrm{dR}})$ is a group isomorphism.*

PROOF. Recall from (5.5) that $\mathcal{A}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}})$ and $\mathcal{A} = \mathcal{O}(I(U_{\mathrm{dR}}))$. We want to show that the injective map $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{MT}}$ induced by $U_{\mathrm{dR}} \to I(U_{\mathrm{dR}})$ is surjective. In Corollary 5.18 we proved that this map extends to an injection $\mathcal{H} \hookrightarrow \mathcal{H}^{\mathcal{MT}}$ compatible with the gradings on both sides. Brown's theorem implies that the dimension of the graded pieces of $\mathcal{H}$ agree with those of $\mathcal{H}^{\mathcal{MT}}$, hence the algebras are isomorphic. $\square$

Let $\mathbf{MT}'(\mathbb{Z})$ be the full Tannakian subcategory of $\mathbf{MT}(\mathbb{Z})$ generated by the objects ${}_{\boldsymbol{x}}U_{\boldsymbol{y}}^{\mathrm{Mot},N}$ for $N \geq 0$ and $\boldsymbol{x}, \boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$ and let ω'_{dR} be the fibre functor ω_{dR} restricted to $\mathbf{MT}'(\mathbb{Z})$.

COROLLARY 5.104. *The quotient*

$$\underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}(\mathbb{Z})}(\omega_{\mathrm{dR}}) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}'(\mathbb{Z})}(\omega'_{\mathrm{dR}})$$

is an isomorphism of affine group schemes. It follows that the inclusion

$$\mathbf{MT}'(\mathbb{Z}) \to \mathbf{MT}(\mathbb{Z})$$

is an equivalence of Tannakian categories, so that every mixed Tate motive over $\mathbb{Z}$ is a subquotient of a tensor construction on one of the finite-dimensional pieces of the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

PROOF. The Tannaka group $\mathrm{Aut}^{\otimes}(\omega'_{\mathrm{dR}})$ is $I(U_{\mathrm{dR}}) \rtimes \mathbb{G}_m$. Thus the fact that the morphism of Tannaka groups is an isomorphism follows from Corollary 5.103. As a consequence, both $\mathbf{MT}(\mathbb{Z})$ and $\mathbf{MT}'(\mathbb{Z})$ are equivalent to the category of finite dimensional representations of G_{dR}. $\square$

COROLLARY 5.105. *The periods of every mixed Tate motive over $\mathbb{Z}$ are linear combinations with $\mathbb{Q}[\frac{1}{2\pi i}]$ coefficients of multiple zeta values. In other words, the ring of periods of mixed Tate motives over $\mathbb{Z}$ is $\mathcal{Z}[\frac{1}{2\pi i}]$.*

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
U_{\mathrm{dR}} \times \mathbb{G}_m \arrow[r, "f_1", "\simeq"'] \arrow[d, "g_1"'] & G_{\mathrm{dR}} \arrow[r, "f_2", "\simeq"'] & P_{\mathrm{dR,B}} \arrow[d, "g_2"] \\
U_{\mathrm{dR}} \times \mathbb{A}^1 \arrow[rr, "f_3", "\simeq"'] & & \mathcal{Y}
\end{tikzcd}$$

where

$$f_1(u,s) = u \cdot \tau(s), \qquad g_1(u,s) = (u, s^2)$$
$$f_2(g) = g \cdot a^{-1} \cdot \mathrm{comp}_{\mathrm{dR,B}}$$
$$g_2(p) = p \cdot \mathbf{dch}, \qquad f_3(u,t) = \psi(u,t)({}_{\mathbf{1}}1_{\mathbf{0}}),$$

where a is defined in Proposition 4.115. The commutativity of the above diagram follows from the definition of ψ in Theorem 5.17. The upper horizontal arrows are clearly isomorphisms and the lower horizontal arrow is an isomorphism by Corollary 5.103.

By (5.13), $f_1(u_0, 2\pi i) = a$. Clearly

$$f_2(a) = \mathrm{comp}_{\mathrm{dR,B}}, \quad g_2(\mathrm{comp}_{\mathrm{dR,B}}) = \mathbf{dch}^{\mathrm{dR}}, \quad g_1(u_0, 2\pi i) = (u_0, (2\pi i)^2).$$

By the commutativity of the diagram $f_3(u_0, (2\pi i)^2) = \mathbf{dch}^{\mathrm{dR}}$. All the morphisms on the diagram are defined over $\mathbb{Q}$.

The algebra of periods of $\mathbf{MT}(\mathbb{Z})$ is

$$\mathrm{ev}_{\mathrm{comp}_{\mathrm{dR,B}}}(\mathcal{O}(P_{\mathrm{dR,B}})) = \mathrm{ev}_{(u_0, 2\pi i)}(\mathcal{O}(U_{\mathrm{dR}} \times \mathbb{G}_m)).$$

The algebra of multiple zeta values is

$$\mathrm{ev}_{\mathbf{dch}^{\mathrm{dR}}}(\mathcal{O}(\mathcal{Y})) = \mathrm{ev}_{(u_0, (2\pi i)^2)}(\mathcal{O}(U_{\mathrm{dR}} \times \mathbb{A}^1))$$

Since $\mathcal{O}(U_{\mathrm{dR}} \times \mathbb{G}_m) = \mathcal{O}(U_{\mathrm{dR}} \times \mathbb{A}^1)[s^{-1}]$ and $s(u_0, 2\pi i) = 2\pi i$ we deduce the result. $\square$

Corollary 5.106. *Zagier's conjecture 1.71 implies that the numbers* $\pi, \zeta(3), \zeta(5), \dots$ *are algebraically independent.*

Proof. The key ingredient is a structure theorem for Hopf algebras due to Milnor and Moore [**MM65**]:

Theorem 5.107 (Milnor-Moore). *Let k be a field of characteristic zero and $A = \bigoplus_{n \geq 0} A_n$ a graded connected commutative Hopf algebra over k with* $\dim A_n < \infty$ *for all n. Then A is the symmetric algebra*

$$A = \mathrm{Sym}[A_{>0}/(A_{>0})^2].$$

We will use it through the following straightforward corollary: if $x_1, x_2, \dots$ are elements of $A_{>0}$ whose classes on the quotient $A_{>0}/(A_{>0})^2$ are linearly independent, then $x_1, x_2, \dots$ are algebraically independent.

We apply this to the Hopf algebra $\mathcal{A} = \mathcal{O}(U_{\mathrm{dR}})$ and the elements $\zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \dots$ These elements lie in different degrees and are not zero in the Lie coalgebra of indecomposable elements $\mathcal{L} = \mathcal{A}_{>0}/(\mathcal{A}_{>0})^2$. Hence, they are linearly independendent in $\mathcal{L}$. By the corollary of the Milnor-Moore Theorem, they are algebraically independent in $\mathcal{A}$. Since $\mathcal{H}$ is equal to $\mathcal{A}[\zeta^{\mathfrak{m}}(2)]$, we deduce that $\zeta^{\mathfrak{m}}(2), \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5) \dots$ are algebraically independent in $\mathcal{H}$.

Now Zagier's conjecture implies that the period map $\mathrm{per}\colon \mathcal{H} \to \mathcal{Z}$ is an isomorphism. Since $\mathrm{per}(\zeta^{\mathfrak{m}}(n)) = \zeta(n)$ and $\zeta(2) = \frac{\pi^2}{6}$, it follows from the previous discussion that Zagier's conjecture implies that $\pi, \zeta(3), \zeta(5), \dots$ are algebraically independent. $\square$

COROLLARY 5.108. *Zagier's conjecture 1.71 is equivalent to Grothendieck's period conjecture for mixed Tate motives 4.122.*

PROOF. Zagier's conjecture is equivalent to the map $\mathrm{per}\colon \mathcal{H} \to \mathbb{C}$ being injective. Since $\mathcal{O}(P_{\mathrm{dR,B}}) = \mathcal{H}[s^{-1}]$ with $s^2 = -24\zeta^{\mathfrak{m}}(2)$, this is equivalent to the injectivity of the period map $\mathrm{per}\colon \mathcal{O}(P_{\mathrm{dR,B}}) \to \mathbb{C}$ which is the content of Conjecture 4.122. $\square$

References

[AK11] D. Arapura and S-J. Kang. Kähler-de Rham cohomology and Chern classes. *Comm. Algebra*, 39(4):1153–1167, 2011.

[And04] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2004.

[Apé79] R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BBV10] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren. The multiple zeta value data mine. *Comput. Phys. Comm.*, 181(3):582–625, 2010.

[Beĭ86] A. A. Beĭlinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

[Bei12] A. Beilinson. Remarks on Grothendieck's standard conjectures. In *Regulators*, volume 571 of *Contemp. Math.*, pages 25–32. Amer. Math. Soc., Providence, RI, 2012.

[BEK06] S. Bloch, H. Esnault, and D. Kreimer. On motives associated to graph polynomials. *Comm. Math. Phys.*, 267(1):181–225, 2006.

[Beu79] F. Beukers. A note on the irrationality of $\zeta(2)$ and $\zeta(3)$. *Bull. London Math. Soc.*, 11(3):268–272, 1979.

[BG02] J. I. Burgos Gil. *The regulators of Beilinson and Borel*, volume 15 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2002.

[BGSV90] A.A. Beilinson, A.A. Goncharov, V.V. Schechtman, and A.N. Varchenko. Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of a pair of triangles in the plane. In *The Grothendieck Festschrift*, number 86 in Progress in Mathematics, pages 135–171. Birkhäuser, 1990.

[BK97] D. J. Broadhurst and D. Kreimer. Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. *Phys. Lett. B*, 393:403–412, 1997.

[Blo86] S. Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[Bor74] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bor77] A. Borel. Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 4(4):612–636, 1977.

[Bou09] A. Bouthier. Réalisation de de Rham des motifs de Voevodsky. http://arxiv.org/pdf/0909.5376.pdf, 2009.

[BR01] K. Ball and T. Rivoal. Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs. *Invent. Math.*, 146(1):193–207, 2001.

[Bro96] D. Broadhurst. On the enumeration of irreducible k-fold Euler sums and their roles in knot theory and field theory. 1996.

[Bro09] F. Brown. Multiple zeta values and periods of moduli spaces $\overline{M}_{0,n}$. *Ann. Sci. Éc. Norm. Supér.*, 42(3):371–489, 2009.

[Bro12] F. Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math.*, 175(2):949–976, 2012.

[Bro13a] F. Brown. Depth-graded motivic multiple zeta values. http://arxiv.org/abs/1301.3053, 2013.

[Bro13b] F. Brown. Iterated integrals in quantum field theory. In *Geometric and topological methods for quantum field theory*, pages 188–240. Cambridge Univ. Press, Cambridge, 2013.

[Bro16] F. Brown. Irrationality proofs for zeta values, moduli spaces and dinner parties. *Mosc. J. Comb. Number Theory*, 6(2–3):102–165, 2016.

[Bro17] F. Brown. Notes on motivic periods. *Commun. Number Theory Phys.*, 11(3):557–655, 2017.

[BS01] P. Balmer and M. Schlichting. Idempotent completion of triangulated categories. *J. Algebra*, 236(2):819–834, 2001.

[BS12] F. Brown and O. Schnetz. A K3 in ϕ^4. *Duke Math. J.*, 161(10):1817–1862, 2012.

[BT82] R. Bott and L. W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[Car80] J. A. Carlson. Extensions of mixed Hodge structures. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 107–127. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Car88] P. Cartier. Jacobiennes généralisées, monodromie unipotente et intégrales itérées. *Astérisque*, (161–162):Exp. No. 687, 3, 31–52 (1989), 1988. Séminaire Bourbaki, Vol. 1987/88.

[Car02] P. Cartier. Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents. *Astérisque*, (282):Exp. No. 885, viii, 137–173, 2002. Séminaire Bourbaki, Vol. 2000/2001.

[Car07] P. Cartier. A primer of Hopf algebras. In *Frontiers in number theory, physics, and geometry. II*, pages 537–615. Springer, Berlin, 2007.

[Che77] K-T. Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83(5):831–879, 1977.

[CK98] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.*, 199(1):203–242, 1998.

[Del71] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del74] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[Del90] P. Deligne. *Grothendiek Festschrift, Vol II*, chapter Catégories Tannakiennes. Number 87 in Progr. Math. Birkhäuser, 1990.

[Del13] P. Deligne. Multizêtas, d'après Francis Brown. *Astérisque*, (352):Exp. No. 1048, viii, 161–185, 2013. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[DG05] P. Deligne and A. B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[DK90] S.K. Donaldson and P.B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Univ. Press, Oxford, 1990.

[DM82] P. Deligne and J. S. Milne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, volume 900 of *LNM*, pages 101–228. Springer-Verlag, 1982.

[Dri90] V. G. Drinfel'd. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Leningrad Math. J.*, 2(4):829–860, 1990.

[Dup17] C. Dupont. Relative cohomology of bi-arrangements. *Trans. Amer. Math. Soc.*, 369(11):8105–8160, 2017.

[Dup18] C. Dupont. Odd zeta motive and linear forms in odd zeta values. *Compos. Math.*, 154:342–379, 2018.

[Dur83] A. H. Durfee. A naive guide to mixed Hodge theory. In *Singularities, Part 1 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 313–320. Amer. Math. Soc., Providence, RI, 1983.

[Eul76] L. Euler. Meditationes circa singulare serierum genus. *Novi Comm. Acad. Sci. Petropol.*, 20:140–186, 1776. Reprinted in Opera Omnia ser. I, vol. 15, Teubner, Berlin (1927), p. 217–267. Downloadable from the website http://eulerarchive.maa.org/pages/E477.html.

[Fis04] S. Fischler. Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...). *Astérisque*, (294):vii, 27–62, 2004.

[Foi] L. Foissy. An introduction to Hopf algebras of trees. Available at http://loic.foissy.free.fr/pageperso/preprint3.pdf.

[Fur10] H. Furusho. Pentagon and hexagon equations. *Ann. of Math. (2)*, 171(1):545–556, 2010.

[Fur11] H. Furusho. Double shuffle relation for associators. *Ann. of Math. (2)*, 174(1):341–360, 2011.

[GD71] A. Grothendieck and J. Dieudonné. *Éléments de Géométrie Algébrique I*, volume 166 of *Grunlehren der Math. Wissenschaften.* 1971.

[GKZ06] H. Gangl, M. Kaneko, and D. Zagier. Double zeta values and modular forms. In *Automorphic forms and zeta functions*, pages 71–106. World Sci. Publ., Hackensack, NJ, 2006.

[GM04] A. B. Goncharov and Yu. I. Manin. Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$. *Compos. Math.*, 140(1):1–14, 2004.

[Gon98] A. B. Goncharov. Multiple polylogarithms, cyclotomy and modular complexes. *Math. Res. Lett.*, 5(4):497–516, 1998.

[Gon01] A. Goncharov. Multiple polylogarithms and mixed Tate motives. ArXiv AG/0103059, 2001.

[Gon05] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[Had] M. Hadian. On a geometric interpretation of the (cyclotomic) pentagon relation. http://www.its.caltech.edu/ hadian/gipr.pdf.

[Hai87a] R. M. Hain. The geometry of the mixed Hodge structure on the fundamental group. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46, pages 247–282. Amer. Math. Soc., Providence, RI, 1987.

[Hai87b] R.M. Hain. The de Rham homotopy theory of complex algebraic varieties. I. *K-Theory*, 1(3):271–324, 1987.

[Hai93] R. M. Hain. Completions of mapping class groups and the cycle $C - C^{-}$. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 75–105. Amer. Math. Soc., Providence, RI, 1993.

[Han99] M. Hanamura. Mixed motives and algebraic cycles. III. *Math. Res. Lett.*, 6(1):61–82, 1999.

[Har77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har17] R. Harada. On Euler's formulae for double zeta values. *Kyushu J. Math.*, 2017.

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[HMS17] A. Huber and S. Müller-Stach. *Periods and Nori motives*, volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2017. With contributions of Benjamin Friedrich and Jonas von Wangenheim.

[Hof97] M. E. Hoffman. The algebra of multiple harmonic series. *J. Algebra*, 194(2):477–495, 1997.

[Hof00] M. E. Hoffman. Quasi-shuffle products. *J. Algebraic Combin.*, 11:49–68, 2000.

[IK04] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[IKZ06] K. Ihara, M. Kaneko, and D. Zagier. Derivation and double shuffle relations for multiple zeta values. *Compos. Math.*, 142(2):307–338, 2006.

[Jan92] U. Jannsen. Motives, numerical equivalence and semi-simplicity. *Invent. Math.*, 107:447–452, 1992.

[Knu83] F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2), 161–199 1983.

[KS06] M. Kashiwara and P. Schapira. *Categories and Sheaves*, volume 332 of *Grundelehren der mathematischen Wissenschaften*. Springer Verlag, Berlin, 2006.

[KV07] J. Kock and I. Vainsencher. *An invitation to quantum cohomology. Kontsevich's formula for rational plane curves*, volume 249 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2007.

[KZ01] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[Lev93] M. Levine. Tate motives and the vanishing conjectures for algebraic K-theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–188. Kluwer Acad. Publ., Dordrecht, 1993.

[Lev94] M. Levine. Bloch's higher Chow groups revisited. *Astérisque*, (226):10, 235–320, 1994. K-theory (Strasbourg, 1992).

[Li13] Z-H. Li. Another proof of Zagier's evaluation formula of the multiple zeta values $\zeta(2,\dots,2,3,2,\dots,2)$. *Math. Res. Lett.*, 20(5):947–950, 2013.

[Lin82] F. Lindemann. Über die zahl π. *Math. Ann.*, 20(2):213–225, 1882.

[Mil12] J. Milne. Basic theory of affine group schemes. http://www.jmilne.org/math/CourseNotes/AGS.pdf, 2012. V 1.00.

[MM65] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[Mor78] J. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, 48:137–204, 1978.

[MVW06] C. Mazza, V. Voevodsky, and C. Weibel. *Lectures notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[NA87] V. Navarro Aznar. Sur la theorie de Hodge-Deligne. *Invent. Math.*, 90:11–76, 1987.

[Nie65] N. Nielsen. *Die Gammafunktion. Band I. Handbuch der Theorie der Gammafunktion. Band II. Theorie des Integrallogarithmus und verwandter Transzendenten, Teubner 1906.* Reprint by Chelsea Publishing Co., New York, 1965.

[PS08] C. A. M. Peters and J. H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, 2008.

[Qui69] D. Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[Qui72] D. Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math.*, 96:552–586, 1972.

[Reu93] C. Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[Riv00] T. Rivoal. La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(4):267–270, 2000.

[Ser94] J. P. Serre. *Cohomologie Galoisienne*. Number 5 in LNM. Springer Verlag, 5 ed edition, 1994.

[Sou10] C. Soulé. Higher K-theory of algebraic integers and the cohomology of arithmetic groups. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 503–517. Int. Press, Somerville, MA, 2010. Notes by Marco Varisco.

[SR72] N. Saavedra Rivano. *Catégories Tannakiennes*. Number 265 in Lecture Notes in Mathematics. Springer Verlag, 1972.

[Sza09] T. Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2009.

[Tan83] D. Tanré. *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, volume 1025 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[Ter02] T. Terasoma. Mixed Tate motives and multiple zeta values. *Invent. Math.*, 149(2):339–369, 2002.

[Ter13] T. Terasoma. Brown-Zagier relations for associators. http://arxiv.org/abs/1301.7474, 2013.

[Tor50] L. Tornheim. Harmonic double series. *Amer. J. Math.*, 72:303–314, 1950.

[vdP79] A. van der Poorten. A proof that Euler missed. . . Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer*, 1(4):195–203, 1978/79. An informal report.

[Voe00] V. Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*, pages 188–238. Princeton University Press, Princeton, NJ, 2000.

[Voe02] V. Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, 2002(7):351–355, 2002.

[Voi02] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[Wal12] M. Waldschmidt. Lectures on Multiple Zeta Values, imsc 2011. http://www.math.jussieu.fr/~miw/articles/pdf/MZV2011IMSc.pdf, 2012.

[Wat79] W.C. Waterhouse. *Introduction to Affine Group Schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

[Wei89] A. Weil. Prehistory of the zeta-function. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pages 1–9. Academic Press, Boston, MA, 1989.

[Wei94] C. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wil09] J. Wildeshaus. f-catégories, tours et motifs de Tate. *C. R. Math. Acad. Sci. Paris*, 347(23-24):1337–1342, 2009. Cf. also Erratum à la note "f-catégories, tours et motifs de Tate", avec un appendice de Q. Liu et de J. Vitória. *C. R. Math. Acad. Sci. Paris*, 350(3-4):129–131, 2012.

[Zag93] D. Zagier. Periods of modular forms, traces of Hecke operators, and multiple zeta values. *Sūrikaisekikenkyūsho Kōkyūroku*, (843):162–170, 1993. Research into automorphic forms and L functions (Japanese) (Kyoto, 1992).

[Zag94] D. Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 497–512. Birkhäuser, Basel, 1994.

[Zag08] D. Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008.

[Zag12] D. Zagier. Evaluation of the multiple zeta values $\zeta(2,\ldots,3,2,\ldots,2)$. *Ann. of Math. (2)*, 175:977–1000, 2012.

[Zha16] J. Zhao. *Multiple zeta functions, multiple polylogarithms and their special values*, volume 12 of *Series on Number Theory and its Applications*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[Zud01] V. V. Zudilin. One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Uspekhi Mat. Nauk*, 56(4(340)):149–150, 2001.

[Zud03] V. V. Zudilin. Algebraic relations for multiple zeta values. *Uspekhi Mat. Nauk*, 58(1(349)):3–32, 2003.

List of symbols

G^{ab}	the abelianization of a group, page 124
$H^*_B(X)$	Betti cohomology of an algebraic variety over a subfield of $\mathbb{C}$, page 64
$\mathrm{comp}_{\mathrm{B,dR}}$	comparison isomorphism from de Rham to Betti cohomology, page 81
$\mathrm{comp}_{\mathrm{dR,B}}$	the inverse of $\mathrm{comp}_{\mathrm{B,dR}}$, page 81
$E^*(M,k)$	the dg-algebra of smooth k-valued differential forms on a manifold M, page 123
$(d_k)_{k\geq 0}$	the sequence of integers defined by $d_0 = d_2 = 1$, $d_1 = 0$ and $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$, page 4
$\mathbf{DM}(k)$	Voevodsky's derived category of mixed motives over k with rational coefficients, page 245
$H^*_{\mathrm{dR}}(X)$	algebraic de Rham cohomology, page 74
$\mathbf{dch}$	the straight path from 0 to 1, page 193
$\mathbf{SmCor}(k)$	the category of finite correspondences, page 246
$\pi_1(M,x)$	the fundamental group based at x, page 121
G_{dR}	The Tannaka group of $\mathbf{MT}(\mathbb{Z})$ with respect to the de Rham fibre functor, page 264
$\mathbf{GrVec}_k$	the category of finite-dimensional graded vector spaces over k, page 227
Id_n	the $n \times n$ identity matrix, page 130
$\mathbb{I}(r \geq s)$	the indicator function of the property $r \geq s$, page 299
$\int \omega_1 \ldots \omega_r$	the iterated integrals of the 1-forms $\omega_1, \ldots, \omega_r$, page 125
$\ell(\boldsymbol{s})$	length of a multi-index, page 16
$M_{0,n}$	the moduli space of n ordered distinct points in $\mathbb{P}^1$, page 88
$\overline{M}_{0,n}$	the Deligne-Mumford compactification of $M_{0,n}$, page 88
$\widetilde{M}(X)$	the reduced motive of a variety X, page 247
$\mathbf{MT}(\mathbb{Z})$	the Tannakian category of mixed Tate motives over $\mathbb{Z}$, page 260
$\zeta(\boldsymbol{s})$	multiple zeta value associated to the multi-index $\boldsymbol{s}$, page 16
$\mathcal{P}(M)$	the space of paths in a differentiable manifold M, page 120
${}_y\mathcal{P}(M)_x$	the subset of $\mathcal{P}(M)$ consisting of paths from x to y, page 120
$\pi_1(M;y,x)$	the set of homotopy classes of paths from x to y, page 121
$ш(r,s)$	the set of shuffles of type (r,s), page 43

$\varepsilon(I, J)$	signs in the definition of the cohomology of normal crossings divisors, page 76
Δ^n_{st}	standard simplex of dimension n, page 62
$\boldsymbol{\Delta}$	the simplicial category, page 183
$\mathbf{Sm}(k)$	the category of smooth varieties over a field k, page 245
$s^{\{n\}}$	the multi-index $(s, \dots, s)$ of length n, page 20
U_{dR}	The pro-unipotent part of G_{dR}, page 264
$\mathfrak{u}_{\mathrm{dR}}$	The Lie algebra of U_{dR}, page 267
$\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$	The graded Lie algebra obtained from u_{dR}, page 267
$\mathrm{wt}(\boldsymbol{s})$	weight of a multi-index, page 16
$\mathcal{Z}$	the $\mathbb{Q}$-algebra of multiple zeta values, page 17
$F_\ell \mathcal{Z}$	the vector subspace of $\mathcal{Z}$ generated by multiple zeta values of length $\leq \ell$, page 17
$F_\ell \mathcal{Z}_k$	the vector subspace of $\mathcal{Z}$ generated by multiple zeta values of weight k and length $\leq \ell$, page 17
$\mathcal{Z}_k$	the vector subspace of $\mathcal{Z}$ generated by multiple zeta values of weight k, page 17
$\zeta(s)$	zeta value, page 9

Index

Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UC3M), Calle Nicolás Cabrera 15, Campus UAM, Cantoblanco, 28049 Madrid, Spain.

E-mail address: burgos@icmat.es

URL: http://www.icmat.es/miembros/burgos/

CMLS, École Polytechnique, F-91128 Palaiseau, France.

E-mail address: javier.fresan@polytechnique.edu

URL: http://javier.fresan.perso.math.cnrs.fr

Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg, Germany.

E-mail address: kuehn@math.uni-hamburg.de

URL: http://www.math.uni-hamburg.de/home/kuehn/