POLYGONAL COMBINATORICS FOR ALGEBRAIC CYCLES

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1. A polygon algebra

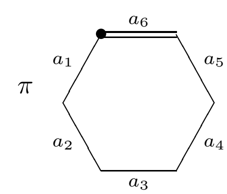
This text is a somewhat vulgarized version of a portion of our joint work [4] with Goncharov and Levin. We want to give evidence for the claim that polygons and their internal structure are very (mixed Tate) motivic, at least if we work over a field.

Definition 1.1. *Let R be a set. An R-***deco polygon** *π is an oriented polygon with a distinguished* **root side** *and a decoration $\{$sides of $\pi\} \to R$.*

We indicate the root side of a polygon by a double line. The orientation is expressed by a bullet at one of the two vertices of the root side, with the convention that the first vertex of the polygon is that bulleted one and the last side is the root side.

The set R that we typically have in mind is the multiplicative group of a field.

Example 1.2: Here is an R-deco 6-gon $\pi = [\,a_1, \ldots, a_6\,]$. The root side is drawn by a double line, the first vertex is marked by a bullet, and the orientation is counterclockwise.



Each such polygon π has a natural **grading** (also called its (Adams) **weight**) $\chi(\pi)$ given by the number of its non-root sides. We form the graded $\mathbb{Q}$-vector space which they generate:

$$V^{\mathrm{pg}} = V_\bullet^{\mathrm{pg}}(R) = \bigoplus_{n \geqslant 0} V_n^{\mathrm{pg}}\,,$$

where for $n \geqslant 1$ we take V_n^{pg} to be the $(n+1)$-polygons and for $n = 0$ we put $V_0^{\mathrm{pg}} = \mathbb{Q}$.

Now consider the exterior algebra on that vector space

$$\mathcal{P}_\bullet^\bullet = \bigwedge^\bullet V_\bullet^{\mathrm{pg}}\,,$$

which we refer to as the polygon algebra (associated to R).

2. The differential on polygons

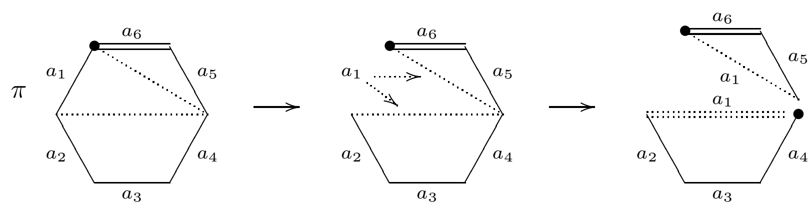
It is on this polygon algebra that we find a differential which mimics the differential on certain algebraic cycles. We only need to give it on generators $\pi \in V^{\mathrm{pg}}$,

$$\partial : V^{\mathrm{pg}} \to \bigwedge^2 V^{\mathrm{pg}}\,,$$

and for the general case $\bigwedge^n V^{\mathrm{pg}} \to \bigwedge^{n+1} V^{\mathrm{pg}}$ we invoke of course linearity and the Leibniz rule.

2.1. **Preparation: a partial differential on pictures.** So let π be an R-deco N-gon, where $N \geqslant 3$. We first give a "partial" differential on π: this consists essentially of cutting out one triangle of the polygon, but since in the process we destroy some of the structure of the polygon, we have to restore it for the resulting two smaller polygons. In particular, they need to be assigned a root and decorations, as well as an orientation.

On a typical picture, where the process is perhaps easier to understand, it looks as follows:

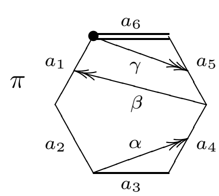


The restoring data come about as follows:

(1) The new sides (one for each of the two new polygons) inherit the decoration from the side which is removed (i.e. the base side of the cut out triangle).
(2) If a new polygon has no root side, the new side becomes its root.
(3) The first vertex (and thus the orientation) is lacking in precisely one of the two polygons, referred to in the following as the **cut-off polygon**. The new first vertex is the one which forms the top of the cut out triangle.

Finally we need to decide which of the two new polygons is on the left and on the right of the wedge product, respectively, and we need to introduce a sign for each such term. A convenient way to explain this is to picture the cut out triangle by inserting an **arrow** from the top vertex to the base side inside the polygon. Then each arrow is either **forward** or **backward**, depending on whether the beginning of the arrow is *before* or *after* in the natural order of the vertices and sides of the polygon.

Example 2.1: The following picture shows two forward arrows α and γ and a backward arrow β in the R-deco 6-gon of the previous example.



2.2. **The full differential.** It is necessary to assign a **sign** to each arrow α inside π, by putting

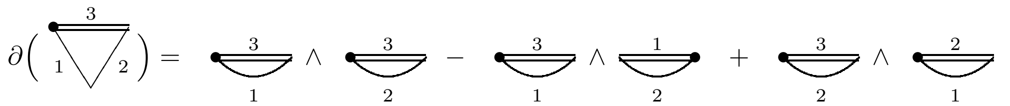
$$\operatorname{sgn}(\alpha) = \begin{cases} +1 & \text{if } \alpha \text{ is forward} \\ (-1)^{\chi(\pi_\alpha^{\sqcup})} & \text{if } \alpha \text{ is backward,} \end{cases}$$

where $\pi_\alpha^{\sqcup}$ denotes the polygon which is cut off by α, while the root polygon of α is denoted by $\pi_\alpha^{\bullet\!=}$. Now we have finally all ingredients ready for the definition of the differential. (We note that an arrow does not connect a vertex to an adjacent side.)

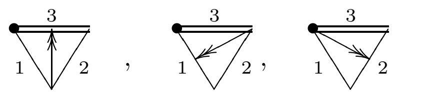
Definition 2.2. *The differential on an R-deco polygon π is given by the formula*

$$\partial(\pi) = \sum_{\alpha \ arrow} \operatorname{sgn}(\alpha)\ \pi_\alpha^{\bullet\!=} \wedge \pi_\alpha^{\sqcup}\,.$$

Example 2.3: The differential applied to a triangle gives three terms:



The summands correspond respectively to the following three dissections:



Lemma 2.4. *The differential ∂ deserves its name, i.e., $\partial \circ \partial = 0$. It makes $\mathcal{P}(R)$ into a differential graded algebra.* □

3. Relating polygons and algebraic cycles

3.1. **A special class of algebraic cycles from polygons.** In the case of a field F, the DGA $\mathcal{P}(F^\times)$ mimics the combinatorics of (a small part of) Bloch's algebraic cycle DGA $\mathcal{N}$ attached to F (cf., e.g., [2], [1]). More precisely, to each polygon we associate the sum over all its triangulations, to each triangulation its dual (rooted) tree, and to each such tree a "graph cycle" (similar to the ones which were assigned by Totaro and by Bloch to polylogarithms, cf., e.g., [1], p.599) which is parametrized by the interior vertices of that tree. For details, we refer to the *forest cycling map* of [4]. The tree structure guarantees the **admissibility** of the graph cycle, at least if the decorations are mutually different. (This admissibility is a crucial property which needs to be satisfied for any given term in an algebraic cycle.)

Theorem 3.1. ([4]) *There is a map of DGAs from $\widetilde{\mathcal{P}}(F^\times)$ to Bloch's algebraic cycle DGA $\mathcal{N}$ for F.* □

Here $\widetilde{\mathcal{P}}$ denotes the *generic* part of $\mathcal{P}$, i.e. the terms where the decorations in each individual polygon are mutually different.

3.2. **Why the interest in $\widetilde{\mathcal{P}}(F^\times)$?** Although $\widetilde{\mathcal{P}}(F^\times)$ is a very small DGA, it should already contain most of the cohomological information of algebraic cycles over F. More precisely, there is a way to slightly extend $\widetilde{\mathcal{P}}(F^\times)$ to incorporate also the decoration $0 \in F$ for a polygon π, in fact possibly several times for the same polygon, demanding a second type of side in π. Furthermore one expects that it is possible to extend the map of DGAs in the theorem such that one associates *admissible* algebraic cycles also to non-generic decorations. Since the polygons can be viewed as corresponding—in a very precise sense—to multiple polylogarithms, and since Goncharov has conjectured that the category of mixed Tate motives over a field F is governed by those multiple polylogarithms (for a more precise statement, we refer to [5]), we expect that $\widetilde{\mathcal{P}}(F^\times)$ captures already a good deal of those mixed Tate motives.

In fact, we can relate the polygons more directly to the latter motives, using work of Bloch-Kriz [1] who gave a candidate for the category of mixed Tate motives over a field in terms of algebraic cycles, more precisely using (a version of) the usual bar construction from algebraic topology. The bar construction promotes a DGA to a coalgebra, and in good cases (like the one at hand) even to a Hopf algebra.

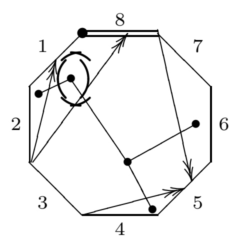
In the following we will try to convince the reader that this construction can in fact be applied to polygons, and we will retrieve a small Hopf subalgebra of the one of Bloch-Kriz.

4. The bar construction for polygons

4.1. **Why bar?** The fact that the differential for each polygon lies in $\bigwedge^2 \mathcal{P}$ can be restated by saying that $\partial(\pi)$ "vanishes modulo (wedge) products". This suggests to lift the differential to a larger "enhanced" complex where the (enhanced) differential vanishes for the (enhanced) polygons on the nose. The standard method to achieve this is the well-known bar construction which, roughly speaking, takes a DGA and makes a double complex out of it, by taking a ("simplicial") differential in a second direction (this is essentially the tensor algebra over the original DGA, where each tensor is written as a "bar" | to avoid confusion with the tensor product for the coproduct structure to be introduced, whence the name), and then taking the associated total complex.

4.2. **Multiple dissections.** In order to define the bar element for an R-deco polygon π, we need to introduce **multiple dissections** as the non-intersecting sets of arrows inside the polygon. (The arrows are allowed to meet at the boundary of the polygon.) We give an example in terms of pictures:

Example 4.1: Here is a polygon with a 5-fold dissection, and its dual tree.



Now the following steps give the idea how to define the bar element associated to a polygon π:

- Each set of r non-intersecting arrows in π defines a dissection D of $r+1$ new polygons $\pi_1^D, \ldots, \pi_{r+1}^D$.
- The dual tree $\tau(D)$ of the (polygon with a) dissection D is a rooted plane tree, each vertex being decorated by a polygon.
- The tree $\tau(D)$ defines a partial order $\prec$ on its vertices.
- For the bar construction, sum over all linear orders λ of the vertices of $\tau(D)$ which are compatible with $\prec$, giving a typical term $[\pi^D_{\lambda(1)} \mid \ldots \mid \pi^D_{\lambda(r+1)}]$, where, as indicated above, the bar | should be interpreted as a tensor product.

Definition 4.2. *The bar element* $\mathrm{B}(\pi)$ *associated to an R-deco polygon* π *is given by*

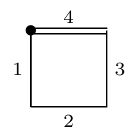
$$\mathrm{B}(\pi) = \sum_D \mathrm{sgn}(D) \sum_\lambda [\pi^D_{\lambda(1)} \,|\, \ldots \,|\, \pi^D_{\lambda(r+1)}] \quad \in \bigoplus_{n \geqslant 1} \underbrace{\mathcal{P} \,|\, \mathcal{P} \,|\, \ldots \,|\, \mathcal{P}}_{n \text{ factors}},$$

where the outer sum runs through all (possibly empty) multiple dissections D *of* π *and the inner sum over all linear orders* λ *on the subpolygons* π^D_i *which are compatible with* $\tau(D)$*, the dual tree of* D*.*

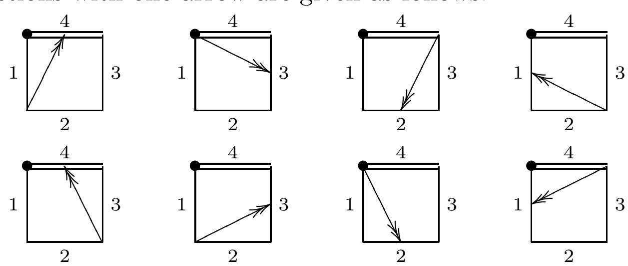
Here the sign $\mathrm{sgn}(D)$ *of a dissection* D *is defined by*

$$\mathrm{sgn}(D) = \prod_{\alpha \text{ backward}} (-1)^{\chi(\text{cut-off polygon } \pi^{\sqcup}_\alpha)}.$$

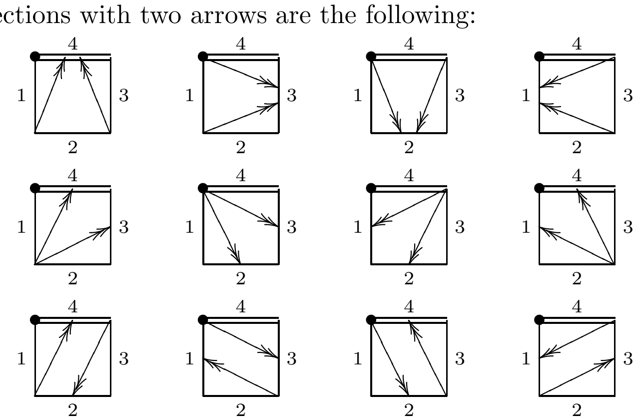
Example 4.3: The bar element associated to a quadrangle consists of $1+8+12$ terms, with $0, 1$ and 2 arrows, respectively:



The dissections with one arrow are given as follows:



The dissections with two arrows are the following:



Proposition 4.4. *For any* π, $\mathrm{B}(\pi)$ *gives a 0-cocycle.* □

5. The coproduct on polygons

5.1. **Closure under the coproduct.** Now the bar construction immediately induces a coproduct Δ on $\mathrm{B}(\mathcal{P})$ by replacing alternatingly any $|$ by a tensor $\otimes$ (note that the two tensors $|$ and $\otimes$ have a completely different meaning!), e.g.,

$$\Delta : [a|b|c] \;\mapsto\; 1 \otimes [a|b|c] \;-\; [a] \otimes [b|c] \;+\; [a|b] \otimes [c] \;-\; [a|b|c] \otimes 1\,.$$

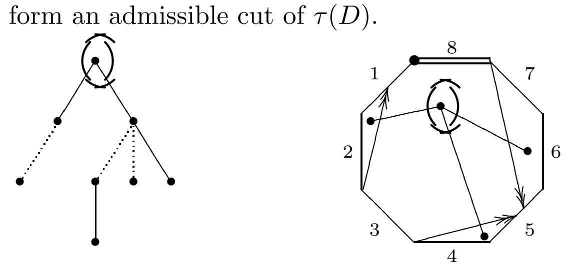
Since this produces many terms, we seem to have gained nothing. But fortunately the terms recombine in a particularly nice way: in order to describe it, we introduce the notion of an **admissible dissection** which is characterized by the property that its dual tree is a star-like tree with $\geqslant 1$ edge (i.e., each leaf is connected to the root by an edge). Equivalently, the dissection is such that there is a root piece and several ($\geqslant 1$) disjoint cut-off pieces (cf. the right hand picture below).

Theorem 5.1. [4] *The restricted coproduct* $\Delta' = \Delta - 1 \otimes \mathrm{id} - id \otimes 1$ *for an R-deco polygon π is given by*

$$\Delta' \mathrm{B}(\pi) = \sum_{D \ \mathit{admissible}} \mathrm{sgn}(D)\ \mathrm{B}(\pi_D^{root}) \otimes \coprod_i \mathrm{B}(\pi_D^{cut\text{-}off_i})\,.$$

5.2. **Analogy with the Connes-Kreimer coproduct.** Note that this gives a description of the coproduct which is a planar decorated version of the Connes-Kreimer coproduct for rooted trees (cf. [3]), and the notion of admissible dissection corresponds to their notion of "admissible cut".

Recall ([3]) that an **admissible cut** in a rooted tree τ is given by a collection C of edges of τ such that each simple path from the root to a leaf vertex contains at most one element of C. We use the same notion for the rooted *plane* trees. We call a dissection D **admissible** if the set of those edges of $\tau(D)$ corresponding to the arrows in D form an admissible cut of $\tau(D)$.



The dotted edges form an admissible cut in the non-planar tree on the left, while the arrows on the right form an admissible dissection of the polygon.

Corollary 5.2. *The shuffle products of* $\mathrm{B}(\pi)$ *with generic decoration form a Hopf subalgebra of* $\mathrm{B}(\mathcal{P}(R))$ *which maps, for* $R = F^\times$*, to the Bloch-Kriz Hopf algebra* $H^0\mathrm{B}(\mathcal{N})$ *over the field* F.

5.3. **Analogy with Goncharov's "semicircle coproduct".** In [6], Goncharov gave a beautiful construction of a coproduct for the Hopf algebra of "motivic iterated integrals". It is somewhat reminiscent of the coproduct given above; for an ordered set of points located, say, on a semicircle, one cuts out a convex piece by using a polygonal path from first to last point (this roughly corresponds to the root piece in the coproduct above) which leaves the remaining disjoint segments of the semicircle (which correspond to combinations of the cut-off polygons in the coproduct above). For a precise relationship, we refer to [4], §8.

5.4. **Conclusion.** The comodules over the Bloch-Kriz Hopf algebra are the "algebraic cycle candidate" for the definition of the category of mixed Tate motives over F. Furthermore we have indicated that the coproduct above is closely related to Goncharov's coproduct on motivic iterated intgerals. Together this provides the evidence, alluded to above, that polygons and their internal structure can be viewed as "motivic".

References

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[2] **Bloch, S.** *Lectures on mixed motives.* Algebraic geometry—Santa Cruz 1995, 329–359, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997.

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[5] **Goncharov, A.B.** *Multiple polylogarithms and mixed Tate motives.* arXiv: math.AG/0103059.

[6] **Goncharov, A.B.** *Galois symmetries of fundamental groupoids and noncommutative geometry.* Duke Math. J., Vol 128, no 2, 2005, 209–284.