THETA INTEGRALS AND GENERALIZED ERROR FUNCTIONS II

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1. Introduction

The theory of theta series attached to integral lattices L in rational quadratic spaces with bilinear form $(\ ,\ )$ of signature (p,q), $pq>0$, has a long history including fundamental work of Hecke, Siegel, Maass, and others. Siegel constructed theta series for such indefinite lattices by using majorants and hence obtained functions depending on both an elliptic modular variable τ and a point $z\in D$, the space of oriented negative q-planes in $V=L\otimes_{\mathbb{Z}}\mathbb{R}$. These Siegel theta series have weight $\frac{p-q}{2}$ in τ, but, unlike the classical theta series for positive definite lattices, they are non-holomorphic. In joint work of the second author and John Millson, [13], [14], and [15], a family of theta series valued in closed differential forms on D was constructed; we will refer to these as theta forms. The series obtained by passing to classes in the cohomology of the locally symmetric space $\Gamma\backslash D$, where Γ is a subgroup of finite index in the isometry group of L, were shown to be holomorphic modular forms of weight $\frac{p+q}{2}$ valued in $H^q(\Gamma\backslash D)$.

The resulting theory provides one analogue of the classical holomorphic theta series in the indefinite case. However, it is still an attractive challenge to define theta series for indefinite lattices more directly by restricting the summation to lattice vectors in suitable subsets $\mathcal{W}$ of V where the quadratic form is positive so that the series

$$\sum_{x\in h+L}\Phi(x;\mathcal{W})\,\mathbf{q}^{Q(x)},\qquad \mathbf{q}=e(\tau)=e^{2\pi i\tau},\ Q(x)=\frac{1}{2}(x,x),\tag{1.1}$$

is termwise absolutely convergence and hence defines a holomorphic function of τ. Here $\Phi(\cdot,\mathcal{W})$ is supported on $\mathcal{W}$, perhaps valued in ±1. Unfortunately, such series are typically not modular.

In his thesis, Zwegers [24] introduced a series of this type for V of signature $(m-1,1)$, where

$$\Phi(x;\mathcal{W})=\frac{1}{2}\,(\operatorname{sgn}(x,C')-\operatorname{sgn}(x,C)\,),$$

for C and $C'\in V$ negative vectors in the same component of the cone of negative vectors in V. He showed that the resulting holomorphic series is not modular in general, but that it can be competed to a (non-holomorphic) modular form of weight $\frac{m}{2}$ by adding a suitable series constructed using the error function.

Recently, Alexandrov, Banerjee, Manschot and Pioline, [1], proposed a generalization of Zwegers' construction to the case of arbitrary signature $(m-q,q)$ where $\Phi(x;\mathcal{W})=\Phi_q^{\square}(x;\mathcal{C})$

is given by

$$\Phi_q^{\square}(x;\mathcal{C}) = 2^{-q}\prod_{j=1}^{q}(\,\mathrm{sgn}(x,C'_j) - \mathrm{sgn}(x,C_j)\,), \tag{1.2}$$

for a collection

$$\mathcal{C} = \mathcal{C}^{\square} = \{\{C_1, C'_1\}, \{C_2, C'_2\}, \dots, \{C_q, C'_q\}\}$$

of pairs of negative vectors satisfying certain incidence relations. They introduced generalized error functions and, in the case $q = 2$, used them to construct a (non-holomorphic) modular completion of the series (1.1). Shortly thereafter, Nazaroglu [18] handled the case of general signature along the lines suggested in [1]. In both [1] and [18], the modularity of the non-holomorphic completion is established by using a result of Vignéras, [21], which asserts the modularity of theta like series built from a certain class of functions. The essential step is to show that suitable combinations of generalized error functions define functions in this class and, at the same time, are suitably linked to the function $\Phi_q^{\square}(\cdot,\mathcal{C})$. Sums of lattice vectors in more general positive polyhedral cones were considered by Westerholt-Raum, [22]; he again uses Vignéras criterion to deduce modularity and also discusses the degenerate case where edges of the cone are allowed to be rational isotropic vectors.

In this paper, we show that the indefinite theta series of [24], [1] and [18] can be obtained by integrating the theta forms for V of signature (p,q) over certain singular q-cubes determined by a collection $\mathcal{C}$ which is in 'good position'. As indicated by the title, this paper is a sequel to [11] where such a result is proved for the case $q = 2$. We also consider the analogous integrals over singular simplices, where the input data is now a collection $\mathcal{C} = \mathcal{C}^{\triangle} = \{C_0, C_1, \dots, C_q\}$ of negative vectors in V in 'good position'. In particular, any q of them span a negative q plane in V and these q-planes give the vertices of a singular simplex in D.

To state the results more precisely, we need some notation. Let L be an even integral lattice in V with dual lattice $L^\vee$. For $\tau = u + iv \in \mathfrak{H}$ and $\mu \in L^\vee/L$, the theta form is the closed Γ_L-invariant q-form on D given by

$$\theta_\mu(\tau,\varphi_{KM}) = \sum_{x\in\mu+L}\varphi_{KM}(\tau,x).$$

Here the Schwartz form

$$\varphi_{KM}(\tau,x) = v^{-\frac{p+q}{4}}\,(\omega(g'_\tau)\varphi_{KM})(x).$$

is obtained by the action $\omega(g'_\tau)$ of the Weil representation on the basic Schwartz form $\varphi_{KM}(x)$, cf., section 2.2. A precise formula for $\varphi_{KM}(x)$ is given in section 5.

First consider the 'cubical' case. For a collection $\mathcal{C} = \mathcal{C}^{\square}$ of q pairs of negative vectors, we can define a q-tuple of vectors

$$B(s) = [(1-s_1)C_1 + s_1C'_1, \dots, (1-s_q)C_q + s_qC'_q] \in V^q,$$

for each $s = [s_1,\dots,s_q] \in [0,1]^q$. We say that $\mathcal{C}$ is in good position if the collection $B(s)$ spans a negative q-plane for all $s \in [0,1]^q$. If $\mathcal{C}$ is in good position, we obtain an oriented singular q-cube

$$\phi_{\mathcal{C}} : [0,1]^q \longrightarrow D, \quad s \mapsto \mathrm{span}\{B_1(s_1),\dots,B_q(s_q)\}_{\text{p.o.}},$$

where the subscript 'p.o.' indicates that the given q-tuple defines the orientation. Let $S^{\square}(\mathcal{C})$ be the resulting singular q-cube.

Next consider the simplicial case. In this case, we suppose that the set of vectors $\mathcal{C} = \mathcal{C}^{\triangle}$ is linearly independent over $\mathbb{R}$ and that any q of them span a negative q-plane. Their span U is an oriented $q+1$-plane of signature $(1, q)$ and the dual basis $\mathcal{C}^{\vee} = \{C_0^{\vee}, \dots, C_q^{\vee}\}$ consists of positive vectors. We say that $\mathcal{C}$ is in good position if, for all

$$s = [s_0, \dots, s_q] \in \Delta^q = \{\, s \in [0,1]^{q+1} \mid \sum_{i=0}^{q} s_i = 1 \,\},$$

the vector

$$C^{\vee}(s) = \sum_i s_i C_i^{\vee}$$

is positive. For example, it suffices to require that all entries of the Gram matrix $((C_i^{\vee}, C_j^{\vee}))$ are non-negative[1]. For $\mathcal{C}$ in good position, we obtain a map

$$\phi_{\mathcal{C}} : \Delta^q \longrightarrow D, \qquad s \mapsto C^{\vee}(s)^{\perp},$$

where the $\perp$ is taken in U and the orientation of $\phi_{\mathcal{C}}(s)$ is determined by the normal vector $C^{\vee}(s)$. We write $S(\mathcal{C})$ for the resulting singular simplex. We also define

$$\Phi_q^{\triangle}(x, \mathcal{C}) = 2^{-q-1} \left(\prod_{j=0}^{q} (1 - \operatorname{sgn}(x, C_j)) + (-1)^q \prod_{j=0}^{q} (1 + \operatorname{sgn}(x, C_j) \right). \tag{1.3}$$

We consider the theta integrals

$$I_{\mu}(\tau, \mathcal{C}) = \int_{S(\mathcal{C})} \theta_{\mu}(\tau, \varphi_{KM}). \tag{1.4}$$

Note that, by construction, $I_{\mu}(\tau, \mathcal{C})$ is a (typically non-holomorphic) modular form of weight $\frac{p+q}{2}$ with transformation law inherited from that of the theta form.

For $1 \le r \le q$ and for a collection of vectors $\boldsymbol{c} = \{c_1, \dots, c_r\}$ spanning an oriented negative r-plane, let $E_r(\boldsymbol{c}, x)$, $x \in V$, be the generalized error function defined by (4.1). Finally, for $x \in V$, $x \neq 0$, let

$$D_x = \{ z \in D \mid x \perp z \, \},$$

and note that, if $Q(x) > 0$, then D_x is a totally geodesic subsymmetric space in D of codimension q. Otherwise, D_x is empty.

Our main result is then the following.

Main Theorem. *Assume that $\mathcal{C}$ is in good position and let $\Phi_q(x, \mathcal{C})$ be $\Phi_q^{\square}(x, \mathcal{C})$ (resp. $\Phi_q^{\triangle}(x, \mathcal{C})$) is the cubical (resp. simplicial) case.*
(i) The series

$$\sum_{x \in \mu + L} \Phi_q(x, \mathcal{C}) \, \mathbf{q}^{Q(x)} \tag{1.5}$$

[1]This was pointed out to the second author by Sander Zwegers at the Dublin Conference in June, 2017.

is termwise absolutely convergent.
(ii) If $\Phi_q(x,\mathcal{C}) \neq 0$, then

$$D_x \cap S(\mathcal{C}) = \phi_{\mathcal{C}}(s(x))$$

for a unique point $s(x) \in [0,1]^q$ (resp. Δ^q), the map $\phi_{\mathcal{C}}$ is immersive at $s(x)$, and

$$\Phi_q(x,\mathcal{C}) = I(S(\mathcal{C}), D_x)$$

is the intersection number[2] *of $S(\mathcal{C})$ and D_x at $\phi_{\mathcal{C}}(s(x))$.*
(iii) In the cubical case, the theta integral is given explicitly by

$$I_\mu(\tau,\mathcal{C}) = \sum_{x\in\mu+L} (-1)^q\, 2^{-q} \sum_I (-1)^{|I|}\, E_q(C^I; x\sqrt{2v})\, \mathbf{q}^{Q(x)}, \tag{1.6}$$

where for a subset $I \subset \{1,\dots,q\}$, C^I is the q-tuple with $C^I_j = C_j$ if $j \notin I$ and $C^I_j = C'_j$ if $j \in I$, ordered by the index j.
Moreover, $I_\mu(\tau,\mathcal{C})$ is the modular completion of the series (1.5).
(iii) In the simplicial case, the theta integral is given by

$$I_\mu(\tau,\mathcal{C}) = \sum_{x\in\mu+L} (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \\ |I|=2r+1}} E_{q-2r}(\mathcal{C}^{(I)}; x\sqrt{2v})\, \mathbf{q}^{Q(x)}. \tag{1.7}$$

where, for a subset $I \subset \{0,1,\dots,q\}$, let $\mathcal{C}^{(I)}$ be the collection of $q+1-|I|$ elements where the C_i with $i \in I$ have been omitted. Here $E_0(\dots) = 1$.

Remark 1.1. (1) The series on the right side of (1.6) coincides with that in [1] and [18], at least when the collection $\mathcal{C}$ satisfies their incidence conditions. The incidence conditions they impose on $\mathcal{C}$, i.e., conditions expressed as requirements on the entries of the Gram matrix of $\mathcal{C}$, imply that $\mathcal{C}$ is in good position. On the other hand, the 'good position' condition, which is a condition on the Gram matrix of the collection $B(s)$ for all $s \in [0,1]^q$, is sufficient for our results. We leave aside the, perhaps subtle, problem of expressing this condition in terms of incidence.
(2) Part (ii) of the theorem provides a geometric interpretation of the coefficients of the holomorphic generating series as intersection numbers. It would be interesting to see if this interpretation has any significance in the physics context which was the original motivation for [1].
(3) The proof of (i) is already given in the general case in [11]. That the right side of (1.6) is the modular completion of the series (1.5) is, of course, a main result of [24], [1], and [18].
(4) It is interesting that generalized error functions for negative r-planes with $r < q$ occur in the explicit formula in the simplicial case. This phenomenon was pointed out by Westerholt-Raum for more general cones, [22]. The indefinite theta series associated to collections $\mathcal{C}^{\triangle}$ were also discussed by Zwegers in his talk at the Dublin conference on Indefinite Theta Functions in June 2017.

Since the theta integral (1.4) can be computed termwise, the formulas of parts (iii) and (iv) follow immediately from the formulas for the integral of $\varphi_{KM}(x)$ over $S(\mathcal{C})$ given in Theorem 4.1 and Theorem 9.3 respectively. These results are, in turn, proved by induction

[2] If $s(x)$ is on the boundary of $[0,1]^q$, this quantity is defined in (11.1) in section 11.

on q, where the case $q = 1$ is an elementary calculation. The key points are the following. First note that both sides of the identities in Theorem 4.1 and Theorem 9.3 are smooth functions of x and $\mathcal{C}$, so that it suffices to consider the case where x is regular with respect to $\mathcal{C}$, i.e., where $(x, C) \neq 0$ for all $C \in \mathcal{C}$. As already noted in [5], the Schwartz form $\varphi_{KM}(x)$ comes equipped with an explicit primitive $\Psi(x)$, defined on the set $D - D_x$. Taking care of the possible singularity, which under the regularity assumption occurs at most at a unique interior point of $S(\mathcal{C})$, we can apply Stokes' theorem. The boundary of $S(\mathcal{C})$ consists of singular $(q-1)$-cubes (resp. simplices) in totally geodesic subsymmetric spaces of the form

$$D'_y = \{\ z \in D \mid y \in z\ \}$$

for $y = C_j$ or C'_j in $\mathcal{C}$. Note that D'_y will then be isomorphic to the space of oriented negative $(q-1)$-planes in the space $V_y = y^\perp$, of signature $(p, q-1)$. Now the crucial (and remarkable!) fact is that the pullback of the primitive $\Psi(x)$ to such a subspace D'_y can be written as an integral transform of the Schwartz $(q-1)$-form $\varphi_{KM}^{V_y}(\mathrm{pr}_{V_y} x)$ for V_y, cf. Proposition 6.2. By induction, we obtain an expression for the boundary integral as a sum of the corresponding signature $(p, q-1)$ theta integrals. Finally, we invoke an inductive identity for generalized error functions from [18], Proposition 7.3, to conclude the proof.

Remark 1.2. (1) One can consider the theta integral $I(\tau, S)$ over any oriented q-chain S in D, and, if S is compact, this can again be computed termwise. If, moreover, the boundary of S consists of $(q-1)$ chains lying in D'_y's, one can proceed by induction. In particular, our result gives an explicit formula for any q-chain written as a sum of simplices of the form $S(\mathcal{C}^{\triangle})$. Moreover, since the theta forms are Γ_L-invariant, their integrals over Γ_L equivalent q-chains coincide.
(2) We can also consider the theta integral $I(\tau, \mathcal{C})$ in the degenerate case, when some of the elements in $\mathcal{C}$ are rational isotropic vectors. Geometrically, this amounts to the q-chain $S(\mathcal{C})$ going out to some of the rational cusps (of the arithmetic quotient) of D. However, while the theta integral over the non-compact region $S(\mathcal{C})$ still is convergent by the results of [8] (for signature $(m-1, 1)$, see [6]), it is in general no longer termwise absolutely convergent (unless one imposes a "non-singularity" condition as in [10], see also [22]). One interesting example is signature $(1, 2)$, where one can realize the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ as a surface $S(\mathcal{C})$ for a certain $\mathcal{C}$, and the associated theta integral $I(\tau, \mathcal{C})$ gives Zagier's non-holomorphic Eisenstein series of weight $3/2$, see [4, 3].
(3) In the companion paper [5], we consider the theta integral $\int_D \eta \wedge \theta_\mu(\tau, \varphi_{KM})$ against a compactly supported $(p-1)q$ differential form η on D. In particular, we establish the properties of the primitive $\Psi(x)$ as a current on D.

Our construction yields a formula for the image of the (typically non-holomorphic) modular form $I_\mu(\tau, \mathcal{C})$ under the lowering operator $-2iv^2 \frac{\partial}{\partial \bar{\tau}}$ or, alternatively, for its shadow given by taking the complex conjugate of this. This formula implies the following, cf. section 8.

Corollary 1.3. *Suppose that $\mathcal{C}$ is rational collection, i.e., that $C \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $C \in \mathcal{C}$. Then the shadow of $I_\mu(\tau, \mathcal{C})$ is a linear combination of products of unary theta series of weight $\frac{3}{2}$ and complex conjugates of indefinite theta series for the spaces $V_C = C^\perp$ for $C \in \mathcal{C}$.*

Here is an outline of the contents of the various sections. Section 2 contains an overview of the construction of theta forms, their modular transformation properties, and their relation to geodesic cycles. There is considerable overlap with the material in [5], although our notation and perspective here differs somewhat. Section 3 explains the singular q-cubes associated to collections $\mathcal{C}$ in good position and their intersection with the cycles D_x in the regular case. It should be noted that the role of the symmetric space D and the singular q-cubes is not so evident in [1] and [18]. The use of the 'good position' condition streamlines the treatment, although the important problem of finding equivalent incidence relations is left open. The explicit formula for the 'cubical' integrals of $\varphi_{KM}(x)$ is given in Theorem 4.1 of Section 4. In Section 5, we give a more detailed discussion of the Schwartz forms φ_{KM} and their primitives. In Section 6 we prove the key formulas for the pullbacks of these forms to the spaces D'_y. Section 7 contains the proof of Theorem 4.1. Section 8 contains the computation of the shadows. Section 9 contains the analogous computations in the simplicial case, where the are several crucial and interesting differences. Some technical details are provided in the Appendix.

1.1. **Thanks.** The second author benefited from the Banff workshop on Modular forms in String Theory in September 2016, as well as from discussions with B. Pioline and S. Zwegers at the conference, Indefinite Theta Functions and Applications in Physics and Geometry, at Trinity College, Dublin in June of 2017.

1.2. **Notation.** For vectors x and y in a non-degenerate inner product space V, $(\ ,\ )$ with $Q(y) \neq 0$, we write

$$x_{\perp y} = x - \frac{(x,y)}{(y,y)}y.$$

Note that

$$(x_{\perp y}, x'_{\perp y}) = (x,x') - \frac{(x,y)(x',y)}{(y,y)}.$$

For a non-degenerate subspace z in V, we write pr_z for the orthogonal projection to z.

We write $e(x) = e^{2\pi i x}$.

2. Theta forms and their integrals

2.1. **Preliminaries.** We begin by reviewing some standard notation and constructions. A good reference is [19]. Suppose that L, $(\ ,\ )$ is a lattice of rank $m = p + q$ with an even integral symmetric bilinear form of signature (p, q) with $pq > 0$. Let $L^\vee \supset L$ be the dual lattice and set $Q(x) = \frac{1}{2}(x, x)$. Let $G = O(L \otimes_{\mathbb{Z}} \mathbb{R})$ be the orthogonal group and let

$$\Gamma_L = \{ \ \gamma \in G \mid \gamma L = L,\ \gamma|_{L^\vee/L} = \mathrm{id}\ \}.$$

Let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and let

$$D = D(V) = \{ \ z \in \mathrm{Gr}_q(V) \mid \ (\ ,\ )|_z < 0,\ z \text{ oriented} \}$$

be the space of oriented negative q-planes in V. For $z \in D$, the associated Gaussian is

$$\varphi_0(x,z) = e^{-\pi(x,x)_z},$$

where

$$R(x,z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x)),$$

and

$$(x,x)_z = (x,x) + 2R(x,z)$$

is the majorant determined by z. For fixed z, $\varphi_0(\cdot, z) = \varphi_0(z) \in \mathcal{S}(V)$ is a Schwartz function on V, while, for fixed $x \in V$, $\varphi_0(x, \cdot) = \varphi_0(x) \in A^0(D)$ is a smooth function on D satisfying the equivariance

$$\varphi_0(gx, gz) = \varphi_0(x,z)$$

for $g \in G$, or equivalently

$$g^*\varphi_0(x) = \varphi_0(g^{-1}x) =: \omega(g)\varphi_0(x),$$

where g^* denotes the pullback of functions on D and $\omega(g)$ denotes the action of g on $\mathcal{S}(V)$. Thus

$$\varphi_0 \in [\,\mathcal{S}(V) \otimes A^0(D)\,]^G. \tag{2.1}$$

The action ω of G on $\mathcal{S}(V)$ commutes with the Weil representation action of the two-fold cover $G' = \mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{S}(V)$, and hence there is a representation of $G \times G'$ on this space, which we also denote by ω. Recall that for $b \in \mathbb{R}$ and $a \in \mathbb{R}^\times$, there are elements $n'(b)$, $m'(a)$, and w' in G' projecting to

$$n(b) = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{R})$ whose Weil representation action is given by

$$\begin{aligned}
\omega(n'(b))\varphi(x) &= e(uQ(x))\,\varphi(x) \\
\omega(m'(a))\varphi(x) &= |a|^{\frac{m}{2}}\varphi(ax) \\
\omega(w')\varphi(x) &= e(\tfrac{p-q}{8})\,\hat{\varphi}(x) = e(\tfrac{p-q}{8}) \int_V \varphi(y)\, e(-(x,y))\, dy.
\end{aligned}$$

Then, for $\tau = u + iv \in \mathfrak{H}$ and $g'_\tau = n'(u)m'(v^{\frac{1}{2}})$, we have

$$\omega(g'_\tau)\varphi_0(x,z) = v^{\frac{p+q}{4}} e^{-2\pi v R(x,z)}\, \mathbf{q}^{Q(x)}, \qquad \mathbf{q} = e(\tau) = e^{2\pi i \tau}.$$

The following invariance property gives rise to the modularity of the theta series. Define a vector valued tempered distribution

$$\Theta_L : \mathcal{S}(V) \longrightarrow \mathbb{C}[L^\vee/L], \qquad \varphi \mapsto \Theta(\varphi; L) = \sum_{\mu \in L^\vee/L} \theta_\mu(\varphi)\, e_\mu,$$

where $e_\mu \in \mathbb{C}[L^\vee/L]$ is the characteristic function of the coset $\mu + L$ and

$$\theta_\mu(\varphi) = \sum_{x \in \mu + L} \varphi(x).$$

Let Γ' be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in G'. Then there is a finite Weil representation ρ_L of Γ' acting on $\mathbb{C}[L^\vee/L]$, and the theta distribution Θ_L satisfies

$$\Theta_L(\omega(\gamma')\varphi) = \rho_L(\gamma')\Theta_L(\varphi).$$

Let K' be the inverse image of $\mathrm{SO}(2)$ in G', and suppose that φ is eigenfunction of weight $\ell \in \frac{1}{2}\mathbb{Z}$ for the Weil representation action of K', i.e.,

$$\omega(k'_\theta)\varphi = e(\ell\theta)\,\varphi, \qquad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

for a suitable preimage k'_θ of k_θ in G'.

Then the invariance of the theta distribution together with a standard calculation, [19], pp. 90–98, implies that the $\mathbb{C}[L^\vee/L]$-valued theta series

$$\sum_{\mu \in L^\vee/L} \theta_\mu(\tau, z; \varphi)\, e_\mu = v^{-\frac{\ell}{2}}\, \Theta_L(\omega(g'_\tau)\varphi)$$

is a (non-holomorphic) vector-valued modular form of weight ℓ and type $(\rho_L, \mathbb{C}[L^\vee/L])$.

The Gaussian φ_0 is an eigenfunction of K' of weight $\frac{p-q}{2}$ so that the Siegel theta series $\theta_\mu(\tau, z; \varphi_0)$ are components of vector valued modular forms and, moreover, via equivariance (2.1), are Γ_L-invariant as functions of z, i.e.,

$$\theta_\mu(\tau; \varphi_0) \in A^0(D)^{\Gamma_L}.$$

For this semi-classical reformulation of Weil's construction of theta functions we are following Shintani [19], cf. also [2].

2.2. **Theta forms.** The basic idea is to replace equivariant families of Schwartz functions by equivariant families of Schwartz forms, i.e., Schwartz functions valued in differential forms on D. Let $A^r(D)$ be the space of smooth r-forms on D. A main result of [13], [14] is the explicit construction of a family of Schwartz forms

$$\varphi_{KM} \in [\,\mathcal{S}(V) \otimes A^q(D)\,]^G. \tag{2.2}$$

Thus, for $x \in V$ and $g \in G$,

$$g^*\varphi_{KM}(x) = \varphi_{KM}(g^{-1}x) \in A^q(D).$$

In particular, for fixed $x \in V$, $\varphi_{KM}(x)$ is a G_x-invariant q-form on D. For example: $\varphi_{KM}(0)$ is a G-invariant form. Under K',

$$\omega(k'_\theta)\varphi_{KM} = e(\tfrac{p+q}{2}\,\theta)\,\varphi_{KM}.$$

Note the shift in weight! Moreover, the q-form $\varphi_{KM}(x)$ is closed,

$$d\varphi_{KM} = 0,$$

where $d : A^q(D) \to A^{q+1}(D)$ is the exterior derivative.

Define the **theta form**

$$\theta_\mu(\tau, \varphi_{KM}) := v^{-\frac{(p+q)}{4}}\,\theta_\mu(\omega(g'_\tau)\varphi_{KM}).$$

Then, by construction, $\theta_\mu(\tau,\varphi_{KM})$ is a closed Γ_L-invariant q-form on D and hence defines a closed q-form on the (orbifold) quotient $M_L = [\Gamma_L\backslash D]$. Moreover, as a function of τ, $\theta_\mu(\tau,\varphi_{KM})$ is a component of a (non-holomorphic) modular form of weight $\frac{p+q}{2}$ and type $(\rho_L, \mathbb{C}[L^\vee/L])$.

2.3. **Relation to geodesic cycles.** The theta forms define cohomology classes for the locally symmetric space M_L which are related to totally geodesic cycles. This was the original motivation for their constuction. We recall briefly the basic facts. For $x \in V$ with $x \neq 0$, let $V_x = x^\perp$, and let

$$D_x = \{z \in D \mid R(x,z) = 0, \text{i.e., } z \subset V_x\}.$$

In particular, $D_x \simeq D(V_x)$ so that D_x is empty if $Q(x) \leq 0$, and is a totally geodesic sub-symmetic space of codimension q if $Q(x) > 0$.

Let $\mathrm{pr}_{\Gamma_L} : D \to \Gamma_L\backslash D = M_L$, and, for x with $Q(x) > 0$, let

$$Z(x) = \mathrm{pr}_{\Gamma_L}(D_x),$$

a totally geodesic codimension q-cycle in M_L with an immersion

$$i_x : \Gamma_x\backslash D_x \longrightarrow Z(x) \subset \Gamma\backslash D.$$

Notice that $Z(x)$ depends only on the Γ_L-orbit of x.

The following results are special cases of those obtained in [13], [14] and [15]:

(i) Suppose that η is a closed and compactly supported $(p-1)q$-form on M_L. Then

$$\int_{M_L} \eta \wedge \theta_\mu(\tau,\varphi_{KM}) = \int_{M_L} \eta \wedge \varphi_{KM}(0) + \sum_{\substack{x\in\mu+L\\ Q(x)>0\\ \bmod \Gamma_L}} \left(\int_{Z(x)} \eta\right) \mathbf{q}^{Q(x)}.$$

(ii) Suppose that S is a compact closed (i.e., $\partial S = 0$) oriented q-cycle on M_L. Then

$$\int_S \theta_\mu(\tau,\varphi_{KM}) = \int_S \varphi_{KM}(0) + \sum_{\substack{x\in\mu+L\\ Q(x)>0\\ \bmod \Gamma_L}} I(S, Z(x))\, \mathbf{q}^{Q(x)},$$

where $I(S, Z(x))$ is the intersection number of the cycles S and $Z(x)$.

In particular, both series are termwise absolutely convergent and define *holomorphic* modular forms of weight $\frac{p+q}{2}$.

Note that these results exactly fit into the framework of (1.1). Additional discussion is given in [5]. Many interesting variations are possible! For example, the case of certain non-compact cycles S in M_L is considered in joint work of the first author with John Millson, [6], [7], [9].

2.4. **Non-closed compact cycles.** Suppose that S is a piecewise smooth oriented q-chain in the symmetric space D. Then, from the general machinery sketched in the previous sections, we obtain (non-holomorphic) modular forms, which we will refer to as indefinite theta series,

$$I_\mu(\tau; S) := \int_S \theta_\mu(\tau; \varphi_{KM}) \tag{2.3}$$

of weight $\frac{m}{2}$. Since S is compact, we can compute such integrals termwise. Define an operator

$$I_S : \mathcal{S}(V) \otimes A^q(D) \longrightarrow \mathcal{S}(V), \qquad \varphi \mapsto \int_S \varphi, \tag{2.4}$$

from Schwartz forms to Schwartz functions by integrating out the form part. This operator commutes with the Weil representation action of G'. Thus, we have

$$\begin{aligned} I_\mu(\tau, S) &= \int_S v^{-\frac{(p+q)}{4}} \Theta_\mu(\omega(g'_\tau)\varphi_{KM}) \\ &= v^{-\frac{(p+q)}{4}} \Theta_\mu(\omega(g'_\tau)\big(I_S(\varphi_{KM})\big), \end{aligned}$$

so that the indefinite theta series (2.3) is just the theta series defined by the Schwartz function $I_S(\varphi_{KM})$. We obtain explicit formulas for the modular forms $I_\mu(\tau, S)$ whenever we can compute the Schwartz function

$$I_S(\varphi_{KM}) \in \mathcal{S}(V) \tag{2.5}$$

for a given q-chain S.

The remainder of this paper is devoted to the computation in the case of the singular q-cubes defined in the next section. The resulting indefinite theta series are those defined by Zwegers [24] in the case of signature $(m-1, 1)$, by Alexandrov, Banerjee, Manschot, and Pioline [1] in the case of signature $(m-2, 2)$ and by Nazaroglu, [18], completing the proposal in [1], in the case of general signature.

3. Singular q-cubes

The data $\mathcal{C}$, cf. (3.1), introduced in [1] section 6 determines a singular q-cube $S(\mathcal{C})$ in D, whose geometry we discuss in this section. We give an explicit formula in terms of generalized error functions for the integral (2.5) in the case when $S = S(\mathcal{C})$ for $\mathcal{C}$ in 'good position'.

3.1. **The singular q-cube $S(\mathcal{C})$ and its faces.** Let

$$\mathcal{C} = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}, \dots, \{C_q, C_{q'}\}\} \tag{3.1}$$

be a collection of q pairs of negative vectors in V. For a subset $I \subset \{1, \dots, q\}$, let C^I be the ordered set $\{C^I_1, \dots, C^I_q\}$ of q vectors where we take $C^I_j = C_j$ if $j \notin I$ and $C^I_j = C_{j'}$ if $j \in I$. The vectors are ordered according to the index j. Thus, $C^\emptyset = \{C_1, \dots, C_q\}$, etc. We would like to have the following 'incidence relations':

(Inc-1) Each collection C^I spans an oriented negative q-plane

$$z^I = \mathrm{span}\{C^I\}_{\mathrm{p.o.}}.$$

(Inc-2) The oriented negative q-planes z^I all lie on the same component of D.

These relations, which can be achieved by imposing conditions on the determinants of minors of Gram matrices, should allow us to construct a singular q-cube with the points z^I as the vertices. However, as already seen in [11], it will be more convenient to work with the following formalism.

For $s = [s_1, \dots, s_q] \in [0,1]^q$, let

$$B(s) = [B_1(s_1), \dots, B_q(s_q)],$$

where

$$B_j(s_j) = (1 - s_j)C_j + s_j C_{j'}.$$

Definition 3.1. A collection $\mathcal{C}$ is said to be in **good position** if for all $s \in [0,1]^q$,

$$\operatorname{span}\{B(s)\}_{\text{p.o.}} = \operatorname{span}\{B_1(s_1), \dots, B_q(s_q)\}_{\text{p.o.}} \in D.$$

If $\mathcal{C}$ is in good position, then relations (Inc-1) and (Inc-2) hold, and we obtain an oriented singular q-cube

$$\rho_{\mathcal{C}} : [0,1]^q \longrightarrow D, \qquad s = [s_1, \dots, s_q] \mapsto \operatorname{span}\{B_1(s_1), \dots, B_q(s_q)\}_{\text{p.o.}} \in D. \tag{3.2}$$

with the z^I as its vertices. Let $S(\mathcal{C}) = \rho_{\mathcal{C}}([0,1]^q)$ be its image in D. Note that the most degenerate case, in which $C_j = C_{j'}$ for all j and $S(\mathcal{C})$ is a point, is allowed.

From now on, unless stated otherwise, we assume that $\mathcal{C}$ is in good position, so that $\rho_{\mathcal{C}}$ and $S(\mathcal{C})$ are defined.

As in [17], we define the front j-face

$$\alpha_j\rho_{\mathcal{C}} : [0,1]^{q-1} \longrightarrow D, \qquad \alpha_j\rho_{\mathcal{C}}(s_1, \dots, s_{q-1}) = \rho_{\mathcal{C}}(s_1, \dots, s_{j-1}, 0, s_j, \dots, s_{q-1}),$$

and back j-face

$$\beta_j\rho_{\mathcal{C}} : [0,1]^{q-1} \longrightarrow D, \qquad \beta_j\rho_{\mathcal{C}}(s_1, \dots, s_{q-1}) = \rho_{\mathcal{C}}(s_1, \dots, s_{j-1}, 1, s_j, \dots, s_{q-1}).$$

We write $\partial_j^+ S(\mathcal{C})$ (resp. $\partial_j^- S(\mathcal{C})$) for the image of $\alpha_j\rho_{\mathcal{C}}$ (resp. $\beta_j\rho_{\mathcal{C}}$), viewed as an oriented $(q-1)$-cube. With this convention, the boundary of the oriented q-cube $S(\mathcal{C})$ is given by

$$\partial S(\mathcal{C}) = \sum_{j=1}^{q} (-1)^j \left(\partial_j^+ S(\mathcal{C}) - \partial_j^- S(\mathcal{C})\right). \tag{3.3}$$

We define collections

$$\mathcal{C}[j] = \{\{C_{1\perp j}, C_{1'\perp j}\}, \dots, \widehat{\{C_j, C_{j'}\}}, \dots, \{C_{q\perp j}, C_{q'\perp j}\}\} \tag{3.4}$$

and

$$\mathcal{C}[j'] = \{\{C_{1\perp j'}, C_{1'\perp j'}\}, \dots, \widehat{\{C_j, C_{j'}\}}, \dots, \{C_{q\perp j'}, C_{q'\perp j'}\}\} \tag{3.5}$$

of $(q-1)$ pairs of negative vectors in $V_j = C_j^\perp$ and $V_{j'} = C_{j'}^\perp$ respectively.

Lemma 3.2. *If the collection $\mathcal{C}$ is in good position for V and D, then the collections $\mathcal{C}[j]$ and $\mathcal{C}[j']$ are in good position for V_j, $D(V_j)$ and $V_{j'}$, $D(V_{j'})$ respectively.*

Proof. Note that, if we set $s' = [s_1, \dots, s_{q-1}] \in [0,1]^{q-1}$ and write $\alpha_j s' = [s_1, \dots, s_{j-1}, 0, s_j, \dots, s_{q-1}]$, then, since $\mathcal{C}$ is in good position,

$$\begin{aligned}\alpha_j \rho_{\mathcal{C}}(s') &= \rho_{\mathcal{C}}(\alpha_j s') \\ &= \operatorname{span}\{B_1(s'_1), \dots, B_{j-1}(s'_{j-1}), C_j, B_{j+1}(s'_j), \dots, B_q(s'_{q-1})\}_{\text{p.o.}} \\ &= \operatorname{span}\{B_1(s'_1)_{\perp j}, \dots, B_{j-1}(s'_{j-1})_{\perp j}, C_j, B_{j+1}(s'_j)_{\perp j}, \dots, B_q(s'_{q-1})_{\perp j}\}_{\text{p.o.}} \in D,\end{aligned}$$

which implies that $\mathcal{C}[j]$ is in good position for V_j and $D(V_j)$. Similarly for $\mathcal{C}[j']$. □

We write $S(\mathcal{C}[j])$ and $S(\mathcal{C}[j'])$ for the corresponding oriented singular $(q-1)$-cubes in $D(V_j)$ and $D(V_{j'})$ with parametrizations analogous to (3.2),

$$\rho_{\mathcal{C}[j]} : [0,1]^{q-1} \longrightarrow D(V_j)$$

and

$$\rho_{\mathcal{C}[j']} : [0,1]^{q-1} \longrightarrow D(V_{j'}).$$

In the notation defined in (6.3), we let $\kappa_j = \kappa_{C_j}[j]$ and $\kappa_{j'} = \kappa_{C_{j'}}[j]$ so that

$$\kappa_j \circ \rho_{\mathcal{C}[j]} = \alpha_j \rho_{\mathcal{C}}, \tag{3.6}$$

and

$$\kappa_{j'} \circ \rho_{\mathcal{C}[j']} = \beta_j \rho_{\mathcal{C}}. \tag{3.7}$$

Here the key point to note is that

$$\begin{aligned}&\operatorname{span}\{B_1(s_1), \dots, B_q(s_q)\}_{\text{p.o.}}|_{s_j=0} \\ &= \operatorname{span}\{B_{1\perp j}(s_1), \dots, B_{(j-1)\perp j}(s_{j-1}), \underline{C}_j, B_{(j+1)\perp j}(s_{j+1}), \dots, B_{q\perp j}(s_q)\}_{\text{p.o.}} \\ &= \kappa_j \circ \rho_{\mathcal{C}[j]}(s_1, \dots, \widehat{s_j}, \dots, s_q),\end{aligned}$$

where, for example,

$$B_{1\perp j}(s_1) = (1-s_1)C_{1\perp j} + s_1 C_{1'\perp j}.$$

3.2. **The regular case.** Following (6.5) of [1], for a vector $x \in V$, let $\Phi_q(x, \mathcal{C}) = \Phi_q^{\square}(x, \mathcal{C})$ be as in (1.2). Recall that $\operatorname{sgn}(0) = 0$.

Recall from [11] that a vector $x \in V$ is said to be **regular** with respect to $\mathcal{C}$ if $(x, C) \neq 0$ for all $C \in \mathcal{C}$. Parts (i) and (ii) of the following are an analogue of Lemma 4.2 in loc. cit. and the proofs given there extend immediately to the general case. Part (iii) will be proved in Appendix I, where the definition of the local intersection number will also be reviewed.

Lemma 3.3. *Let $\mathcal{C}$ be a collection in good position.*
(i) If $x \in V$ is regular with respect to $\mathcal{C}$, then $D_x \cap S(\mathcal{C})$ is non-empty if and only if $\Phi_q(x, \mathcal{C}) \neq 0$, and, in this case $D_x \cap S(\mathcal{C}) = \rho_{\mathcal{C}}(s(x))$ for a unique point $s(x) \in (0,1)^q$ given by

$$s(x)_j = \frac{(x, C_j)}{(x, C_j) - (x, C_{j'})}. \tag{3.8}$$

(ii) If $x \in V$ is any vector with $\Phi_q(x, \mathcal{C}) \neq 0$, then $D_x \cap S(\mathcal{C})$ consists of a single point $\rho_{\mathcal{C}}(s(x))$ with $s(x) \in [0,1]^q$ given by (3.8).

(iii) If $x \in V$ is any vector with $\Phi_q(x, \mathcal{C}) \neq 0$ and $s(x)$ is as in (ii), then the map $\rho_{\mathcal{C}}$ is immersive at $s(x)$, and the quantity $\Phi_q(x)$ is the local intersection number of D_x and $S(\mathcal{C})$ at $s(x)$. A precise definition of this quantity is given in (11.1) in section 11.

Again, as in [11], we say that $\mathcal{C}$ is in very good position if it is in good position and $\rho_{\mathcal{C}}$ is an embedding, i.e., an injective immersion. The following is easily checked by the general version of the calculation in section 6.3 of [11].

Lemma 3.4. *If $\mathcal{C}$ is in good position and the 2^q vectors in $\mathcal{C}$ are linearly independent, then $\mathcal{C}$ is in very good position.*

4. Cubical Integrals and generalized error functions

In this section, we state our main result, an explicit expression for the Schwartz function (2.5) defined by the integral

$$I(x; \mathcal{C}) := \int_{S(\mathcal{C})} \varphi_{KM}(x)$$

of the q-form $\varphi_{KM}(x)$ over the singular q-cube $S(\mathcal{C})$ in D in terms of generalized error functions, as suggested in section 5 of [11]. Recall, [1], (6.1), that, for a collection of vectors negative $C = \{C_1, \dots, C_q\}$ spanning an oriented q-plane $z \in D$, and for $x \in V$, the generalized error function is given by the integral

$$E_q(\{C_1, \dots, C_q\}; x) = \int_z e^{\pi(y - \mathrm{pr}_z(x), y - \mathrm{pr}_z(x))} \operatorname{sgn}(C_1, y) \operatorname{sgn}(C_2, y) \dots \operatorname{sgn}(C_q, y)\, dy, \tag{4.1}$$

where the measure dy is normalized so that

$$\int_z e^{\pi(y,y)}\, dy = 1.$$

We will frequently abbreviate this to $E_q(C; x)$ and write

$$\operatorname{sgn}(C; y) = \operatorname{sgn}(C_1, y) \operatorname{sgn}(C_2, y) \dots \operatorname{sgn}(C_q, y). \tag{4.2}$$

Our main result is the following explicit formula for $I(x; \mathcal{C})$.

Theorem 4.1. *Suppose that $\mathcal{C}$ is in good position. Then*

$$I(x; \mathcal{C}) = (-1)^q\, 2^{-q} \sum_I (-1)^{|I|}\, E_q(C^I; x\sqrt{2})\, e^{-\pi(x,x)}, \tag{4.3}$$

where, as in section 3.1, for a subset $I \subset \{1, \dots, q\}$, C^I is the q-tuple with $C^I_j = C_j$ if $j \notin I$ and $C^I_j = C_{j'}$ if $j \in I$, ordered by the index j.

The 2^q terms in the sum on the right side of (4.3) are generalized error functions associated to the vertices $z^I = \mathrm{span}\{C^I\}_{\text{p.o.}}$ of $S(\mathcal{C})$ of the singular q-cube evaluated on the projections of x to those q-planes.

Remark 4.2. In the case $q = 2$, the expression given in Theorem 4.1 is the negative of the expression found in [11]. But there is a simple explanation, namely the orientation of $S(\mathcal{C})$ used there is defined by the 'loop' in (3.11), but this is the opposite of the orientation we use here, defined by the singular square $\rho_{\mathcal{C}}$.

The proof of Theorem 4.1 by induction on q is given in section 7.

5. Review of the Schwartz form φ_{KM} and its relatives

In this section, we review the basic facts about the Schwartz forms $\varphi_{KM}(x)$ which we need.

5.1. **Local formulas.** We fix a base point $z_0 \in D$ and an orthonormal basis $\{e_1, \dots, e_m\}$, $m = p + q$, $(e_r, e_s) = \epsilon_r \delta_{rs}$, $\epsilon_r = +1$ for $1 \le r \le p$ and $\epsilon_r = -1$ for $r > p$, with

$$z_0 = \mathrm{span}\{e_{p+1}, \dots, e_m\}_{\text{p.o.}}.$$

In particular $V \simeq \mathbb{R}^m$, and the Gaussian is given by

$$\varphi_0(x) = \varphi_0(x, z_0) = e^{-\pi \sum_j x_j^2} \in \mathcal{S}(V), \qquad x = \sum_i x_i e_i. \tag{5.1}$$

Let K be the stabilizer of z_0 in G and write $\mathfrak{g}_o = \mathrm{Lie}(G) = \mathfrak{k}_o + \mathfrak{p}_o$ where $\mathfrak{k}_o = \mathrm{Lie}(K)$ and $\mathfrak{p}_o$ are the $+1$ and -1 eigenspace for the Cartan involution at z_0. There is a canonical isomorphism $T_{z_0}(D) \simeq \mathfrak{p}_0$. Under the idenitfication

$$V \otimes V \xrightarrow{\sim} \mathrm{End}(V), \quad (v_1 \otimes v_2)(v) = (v_2, v)v_1,$$

a basis for $\mathfrak{p}_0$ is given by

$$X_{\alpha\mu} = e_\alpha \otimes e_\mu + e_\mu \otimes e_\alpha, \qquad 1 \le \alpha \le p < \mu \le p + q.$$

Let $\omega_{\alpha\mu}$ be the dual basis for $\mathfrak{p}_o^*$.

By the equivariance property (2.2), $\varphi_{KM}(x)$ is determined by the element of the complex

$$[\mathcal{S}(V) \otimes \textstyle\bigwedge^\bullet(\mathfrak{p}_o^*)]^K$$

obtained by restriction to the point z_0.

For $1 \le s, t \le q$, let

$$\omega(s) = \sum_{j=1}^p x_j \, \omega_{j,p+s} \in \mathfrak{p}_o^*,$$

and

$$\Omega(s,t) = \sum_{j=1}^p \omega_{j,p+s} \wedge \omega_{j,p+t} \in \textstyle\bigwedge^2(\mathfrak{p}_o^*).$$

For λ with $0 \le \lambda \le [q/2]$, we define q-forms

$$\mathbf{AO}_\lambda(q) = A\big[\,\omega(1) \wedge \cdots \wedge \omega(q-2\lambda) \wedge \Omega(q-2\lambda+1, q-2\lambda+2) \wedge \cdots \wedge \Omega(q-1,q)\,\big], \tag{5.2}$$

where A is the alternation

$$A\big[\,\omega(1)\wedge\cdots\wedge\Omega(t-1,t)\,\big] = \frac{1}{t!}\sum_{\sigma\in S_t}\operatorname{sgn}(\sigma)\,\omega(\sigma(1))\wedge\cdots\wedge\Omega(\sigma(t-1),\sigma(t)). \tag{5.3}$$

Note that these are homogeneous of degree $q-2\lambda$ in x, and it will sometimes be useful to write $\mathbf{AO}_\lambda(q)(x)$ to indicate this dependence. With this notation, we have the following formula for the restriction of $\varphi_{KM}(x)$ at the point z_0, cf. [13] p. 371,

$$\varphi_{KM}(x) = 2^{q/2}\sum_{\lambda=0}^{[q/2]} C(q,\lambda)\,\mathbf{AO}_\lambda(q)(x)\,\varphi_0(x), \tag{5.4}$$

where

$$C(t,\lambda) = (-\frac{1}{4\pi})^\lambda\frac{t!}{2^\lambda\lambda!(t-2\lambda)!}. \tag{5.5}$$

There are two auxiliary $q-1$ forms associated to $\varphi_{KM}(x)$ which will play a fundamental role in our calculations. We will recall their relation to $\varphi_{KM}(x)$ in a moment. The first of these is given by

$$\psi_{KM}(x) = 2^{q/2-1}\sum_{\lambda=0}^{[(q-1)/2]}\sum_{s=1}^{q}(-1)^s x_{p+s}\,C(q-1,\lambda)\,\mathbf{AO}_\lambda(q;s)(x)\,\varphi_0(x), \tag{5.6}$$

where the $(q-1)$-form $\mathbf{AO}_\lambda(q;s)$ is defined by the alternation analogous to $\mathbf{AO}_\lambda(q-1)$ but for the index set $\{1,\dots,\hat{s},\dots,q\}$ replacing $\{1,\dots,q-1\}$. For example, $\mathbf{AO}_\lambda(q,q) = \mathbf{AO}_\lambda(q-1)$.

Now we include the parameter $\tau = u+iv$. Writing

$$\varphi_{KM}(x) = \varphi^0_{KM}(x)\,e^{-\pi(x,x)},$$
$$\psi_{KM}(x) = \psi^0_{KM}(x)\,e^{-\pi(x,x)},$$

we have, for $\mathbf{q}=e(\tau)$ and $Q(x)=\frac{1}{2}(x,x)$,

$$\varphi_{KM}(\tau,x) = \varphi^0_{KM}(v^{\frac12}x)\,\mathbf{q}^{Q(x)} = v^{-\frac{p+q}{4}}\omega(g'_\tau)\varphi_{KM}(x), \tag{5.7}$$

and

$$\psi_{KM}(\tau,x) = v\,\psi^0_{KM}(v^{\frac12}x)\,\mathbf{q}^{Q(x)}. \tag{5.8}$$

Note that

$$-2i\,\frac{\partial}{\partial\bar\tau}\varphi_{KM}(\tau,x) = \frac{\partial}{\partial v}\big\{\varphi^0_{KM}(v^{\frac12}x)\big\}\,\mathbf{q}^{Q(x)}.$$

On the set of x such that $R(x,z_0)\ne0$, let

$$\Psi^0_{KM}(x) = -\int_1^\infty\psi^0(t^{\frac12}x)\,t^{-1}\,dt. \tag{5.9}$$

The point here is that

$$\psi^0(t^{\frac12}x) = \big(\text{form valued polynomial in } t^{\frac12}x\,\big)\cdot e^{-2\pi tR(x,z_0)},$$

so that the integral only makes sense when $R(x,z_0) > 0$. For x with $R(x,z_0) > 0$, let

$$\Psi_{KM}(\tau,x) := \Psi^0_{KM}(v^{\frac{1}{2}}x)\,\mathbf{q}^{Q(x)} = -\int_v^\infty \psi^0(t^{\frac{1}{2}}x)\,t^{-1}\,dt\;\mathbf{q}^{Q(x)}. \tag{5.10}$$

The following basic relations between the primitives $\psi_{KM}(\tau,x)$, $\Psi_{KM}(\tau,x)$ and the form $\varphi_{KM}(\tau,x)$ are given in [5], Section 3, Proposition 3.2, cf. also, [15], section 8.

Lemma 5.1. *(i)*

$$-2iv^2\frac{\partial}{\partial\bar\tau}\varphi_{KM}(\tau,x) = d\psi(\tau,x) = v\,d\psi^0_{KM}(v^{\frac{1}{2}}x)\,\mathbf{q}^{Q(x)}.$$

(ii)

$$d\Psi_{KM}(\tau,x) = \varphi_{KM}(\tau,x), \qquad R(x,z_0)\neq 0,$$

and

$$d\Psi^0_{KM}(x) = \varphi^0_{KM}(x), \qquad R(x,z_0)\neq 0.$$

Taking homogeneity in x of various terms into account and writing $R = R(x,z_0)$, we have the explicit formulas

$$\Psi_{KM}(\tau,x) = 2^{\frac{q}{2}-1}\sum_{\lambda=0}^{[(q-1)/2]}\sum_{s=1}^{q}(-1)^{s-1}x_{p+s}\,C(q-1,\lambda)\,\mathbf{AO}_\lambda(q;s) \tag{5.11}$$
$$\times(2\pi R)^{-\frac{1}{2}(q-2\lambda)}\,\Gamma\big(\frac{1}{2}(q-2\lambda),2\pi Rv\big)\,\mathbf{q}^{Q(x)}.$$

and

$$\varphi_{KM}(\tau,x) = 2^{q/2}\sum_{\lambda=0}^{[q/2]}C(q,\lambda)\,\mathbf{AO}_\lambda(q)\,v^{\frac{1}{2}(q-2\lambda)}\,e^{-2\pi vR}\,\mathbf{q}^{Q(x)}. \tag{5.12}$$

5.2. **Global formulas.** We now explain how the formulas of the previous section define global differential forms on D. We will use the notation and conventions explained in [11], especially the Appendix, which we now briefly recall.

Let

$$\mathrm{FD} = \{\,\zeta = [\zeta_1,\dots,\zeta_q]\in V^q \mid (\zeta,\zeta) := ((\zeta_i,\zeta_j)) < 0\,\},$$

be the bundle of oriented negative frames, and let

$$\mathrm{OFD} = \{\,\zeta = [\zeta_1,\dots,\zeta_q]\in V^q \mid (\zeta,\zeta) = -1_q\,\},$$

be the bundle of oriented orthonormal negative frames. Let $\pi : \mathrm{FD}\to D$ be the natural projection, taking ζ to its oriented span. Then, for $\zeta\in\mathrm{OFD}$, we have an identification of tangent spaces

$$V^q \simeq T_\zeta(\mathrm{FD}) \supset T_\zeta(\mathrm{OFD}) = \{\,\eta = [\eta_1,\dots,\eta_q]\in V^q \mid (\eta,\zeta)+(\zeta,\eta) = 0\,\}.$$

For $z\in D$, we let $U(z) = z^\perp$. Then the 'horizontal' subspace $U(z)^q\subset T_\zeta(\mathrm{OFD})$ is identified with $T_z(D)$ under $d\pi_\zeta$. Note that, while the space $U(z)^q$ depends only on z, the identification with $T_z(D)$ depends on ζ. The identifications for different choices of ζ differ by the action of $\mathrm{SO}(q)$.

A priori, the expressions given in (5.12) and (5.11) are elements of $S(V) \otimes \bigwedge^r(\mathfrak{p}_o^*)$ with $r = q$ and $q-1$ respectively, where $\mathfrak{p}_o$ is identified with the tangent space to D at the base point

$$z_0 = \mathrm{span}\{e_{p+1}, \dots, e_{p+q}\}_{\text{p.o.}} \in D$$

determined by our chosen orthonormal basis. They yield global formulas as follows. For any $\zeta \in \mathrm{OFD}$, the function $R(x,z)$ is defined by $R(x,z) = (x,\zeta)(\zeta,x)$. For vectors $\eta = [\eta_1, \dots, \eta_q]$ and $\mu = [\mu_1, \dots, \mu_q]$ in $U(z)^q$, define

$$\omega(s)(\eta) = (x, \eta_s), \qquad \Omega(s,t)(\eta,\mu) = (\eta_s, \mu_t) - (\eta_t, \mu_s). \tag{5.13}$$

Also note that, in the global version of (5.11),

$$x_{p+s} = -(x, \zeta_s). \tag{5.14}$$

Lemma 5.2. *With these definitions, the q-forms $\mathbf{AO}_\lambda(q)$ and $q-1$-forms $\mathbf{AO}_\lambda(q;s)$ on $U(z)^q$ are invariant under* $\mathrm{SO}(q)$ *and hence define forms on $T_z(D)$.*

Proof. We observe that for some non-zero constant c,

$$\mathbf{AO}_\lambda(q)(\eta^1, \dots, \eta^q) = c \det \begin{pmatrix} (x,\eta_1^1) & \dots & (x,\eta_{q-2\lambda}^1) & \eta_{q-2\lambda+1}^1 & \dots & \eta_q^1 \\ \vdots & & \vdots & \vdots & & \vdots \\ (x,\eta_1^q) & \dots & (x,\eta_{q-2\lambda}^q) & \eta_{q-2\lambda+1}^q & \dots & \eta_q^q \end{pmatrix}$$

where, in expanding the determinant, the product of vectors is taken using (,). □

Thus (5.12) (resp. (5.11)) defines a global q form $\varphi_{KM}(\tau,x)$ on D (resp. a global $q-1$-form $\Psi_{KM}(\tau,x)$ on $D - D_x$) and these forms satisfy

$$d\Psi_{KM}(\tau,x) = \varphi_{KM}(\tau,x)$$

on $D - D_x$.

Remark 5.3. The formula for the pullback for these forms to OFD involves additional terms determined by the requirement that the forms vanish if one of the input tangent vectors is vertical, i.e., in the kernel of $d\pi_\zeta$. We will not need these expressions.

6. The pullback to certain sub-symmetric spaces

Suppose that $y \in V$ is a negative vector, and let

$$V_y = y^\perp,$$

$$D'_y = \{\, z \in D \mid y \in z\},$$

and

$$D(V_y) = \{z = \text{oriented neg. } (q-1)\text{-plane in } V_y\}.$$

For the properly oriented orthogonal frame bundle $\mathrm{OFD}(V_y) \to D(V_y)$, there is an embedding

$$\kappa_y : \mathrm{OFD}(V_y) \hookrightarrow \mathrm{OFD}, \qquad \zeta \mapsto [\underline{y}, \zeta], \tag{6.1}$$

where $\underline{y} = y|(y,y)|^{-\frac{1}{2}}$, and a resulting embedding

$$\kappa_y : D(V_y) \xrightarrow{\sim} D'_y \subset D. \tag{6.2}$$

A fundamental result is the following pullback formula, which we find rather striking.

Proposition 6.1. *For $x \in V$, write $x = -(x,\underline{y})\,\underline{y} + x_{\perp y}$, so that $x_{\perp y}$ is the V_y-component of x. Then*
(i)

$$\kappa_y^* \psi_{KM}^0(x) = 2^{-\frac{1}{2}}\,(x,\underline{y})\,e^{-2\pi(x,\underline{y})^2}\,\varphi_{KM}^{V_y,0}(x_{\perp y}).$$

(ii)

$$\kappa_y^*(\psi_{KM}(\tau,x)) = 2^{-\frac{1}{2}}\,v^{\frac{3}{2}}\,(x,\underline{y})\,e^{-2\pi v(x,\underline{y})^2}\mathbf{q}^{-\frac{1}{2}(x,\underline{y})^2}\,\varphi_{KM}^{V_y}(\tau,x_{\perp y}).$$

Here $\varphi_{KM}^{V_y,0}(\tau,\cdot)$ is the φ_{KM}^0 Schwartz $(q-1)$-form on $D(V_y)$.

Proof. The map on tangent spaces is given by

$$d\kappa_y : T_\zeta(\mathrm{OFD}(V_y)) \longrightarrow T_{\kappa_y(\zeta)}(\mathrm{OFD}), \qquad \eta = [\eta_1,\dots,\eta_{q-1}] \mapsto [0,\eta_1,\dots,\eta_{q-1}],$$

and this map is compatible with the 'horizontal' subspaces. It follows that any term in $\psi_{KM}^0(x)$ involving an index $s=1$ in the differential form will vanish under pullback. Thus, by (5.6), we have

$$\begin{aligned}\kappa_y^*\psi_{KM}^0(x) &= 2^{\frac{q}{2}-1}\,(x,\underline{y})\,e^{-2\pi(x,\underline{y})^2}\sum_{\lambda=0}^{[(q-1)/2]} C(q-1,\lambda)\,\mathbf{AO}_\lambda(q-1)(x_{\perp y})\,e^{-2\pi R(x_{\perp y},\zeta)}\\ &= 2^{-\frac{1}{2}}\,(x,\underline{y})\,e^{-2\pi(x,\underline{y})^2}\,\varphi_{KM}^{V_y,0}(x_{\perp y}).\end{aligned}$$

Passing to $\psi_{KM}(\tau,x)$ via (5.8) and noting that

$$Q(x) = -(x,\underline{y})^2 + Q(x_{\perp y}),$$

we obtain the claimed formula. □

Next consider the $(q-1)$-form $\Psi_{KM}^0(x)$. Using the expressions just found and Lemma 5.1, we have the following.

Corollary 6.2. *On the subset of $D(V_y)$ for which $\kappa_y(z) \notin D_x$,*

$$\kappa_y^*\Psi_{KM}^0(x) = -2^{\frac{1}{2}}\,(x,\underline{y})\int_1^\infty e^{-2\pi t^2(x,\underline{y})^2}\,\varphi_{KM}^{V_y,0}(tx_{\perp y})\,dt.$$

In the next section, it will be useful to have the following variant, which involves a shift in the orientations. For an index j, $1 \le j \le q$, define

$$\kappa_y[j] : \mathrm{OFD}(V_y) \hookrightarrow \mathrm{OFD}, \qquad \zeta \mapsto [\zeta_1,\dots,\zeta_{j-1},\underline{y},\zeta_j,\dots,\zeta_{q-1}], \tag{6.3}$$

and write $\kappa_y[j] : D(V_y) \longrightarrow D$ for the corresponding embedding of symmetric spaces. Of course, $\kappa_y = \kappa_y[1]$ and, the embeddings of symmetric spaces only depend on the parity of j.

Corollary 6.3. *(i) On the subset of D'_y for which $\kappa_j(z) \notin D_x$,*

$$\kappa_y[j]^*\Psi^0_{KM}(x) = (-1)^j\, 2^{\frac{1}{2}}\,(x,\underline{y}) \int_1^\infty e^{-2\pi t^2(x,\underline{y})^2}\,\varphi^{V_y,0}_{KM}(tx_{\perp y})\,dt.$$

(ii) On D'_y,

$$\kappa_y^*(\psi_{KM}(\tau,x)) = (-1)^{j-1}2^{-\frac{1}{2}}\,v^{\frac{3}{2}}\,(x,\underline{y})\,e^{-2\pi v(x,\underline{y})^2}\mathbf{q}^{-\frac{1}{2}(x,\underline{y})^2}\,\varphi^{V_y}_{KM}(\tau,x_{\perp y}).$$

7. Proof of Theorem 4.1

For convenience, we remove a factor independent of z and write

$$\varphi_{KM}(x) = \varphi^0_{KM}(x)\,e^{-\pi(x,x)}.$$

In this section, we compute the cubical integrals

$$I^0(x;\mathcal{C}) = \int_{S(\mathcal{C})} \varphi^0_{KM}(x).$$

7.1. **The regular case.** First suppose that x is regular with respect to $\mathcal{C}$, so that, by Lemma 3.3, the intersection $D_x \cap S(\mathcal{C})$ is either empty or consists of a single interior point $\rho_{\mathcal{C}}(s(x))$ depending on whether $\Phi_q(x,\mathcal{C})$ vanishes or not. If $\Phi_q(x,\mathcal{C}) \neq 0$ and for $\epsilon > 0$ sufficiently small, define a collection

$$\mathcal{C}^\epsilon(x) = \{\{B_1(s(x)_1 - \epsilon), B_1(s(x)_1+\epsilon)\},\dots,\{B_q(s(x)_q-\epsilon), B_q(s(x)_q+\epsilon)\}\}.$$

For simplicity, we will abbreviate this as

$$\mathcal{C}^\epsilon = \mathcal{C}^\epsilon(x) = \{\{C^\epsilon_1, C^\epsilon_{1'}\},\dots,\{C^\epsilon_1, C^\epsilon_{q'}\}\}.$$

The following result illustrates the convenience of the 'good position' formulation.

Lemma 7.1. *The collection $\mathcal{C}^\epsilon(x)$ is in good position.*

Proof. We note that, for $t \in [0,1]$,

$$(1-t)C^\epsilon_j + tC^\epsilon_{j'} = (1 - s(x)_j + \epsilon - 2t\epsilon)C_j + (s(x)_j - \epsilon + 2t\epsilon)C_{j'}$$

so that, for $t \in [0,1]^q$,

$$\rho_{\mathcal{C}^\epsilon(x)}(t) = \rho_{\mathcal{C}}(s(x) - \epsilon + 2\epsilon t) \in D,$$

i.e., $\mathcal{C}^\epsilon(x)$ is in good position. □

By construction, the singular q-cube $S(\mathcal{C}^\epsilon(x))$ contains the point $D_x \cap S(\mathcal{C})$. For x regular with respect to $\mathcal{C}$ and $\Phi_q(x,\mathcal{C}) = 0$, we let $S(\mathcal{C}^\epsilon(x))$ be the empty set. In general, we let

$$S^\epsilon(x;\mathcal{C}) = S(\mathcal{C}) - \mathrm{int}\ S(\mathcal{C}^\epsilon).$$

Then Stokes' Theorem and the inductive relation of Corollary 6.2 imply the following inductive formula.

Proposition 7.2. *Suppose that x is regular with respect to $\mathcal{C}$. Then the set D_x does not meet $\partial S(\mathcal{C})$, the integral*

$$I^{00}(x;\mathcal{C}) := \int_{\partial S(\mathcal{C})} \Psi^0_{KM}(x)$$

is well defined, and

$$I^0(x;\mathcal{C}) = I^{00}(x;\mathcal{C}) - \lim_{\epsilon\downarrow 0} I^{00}(x;\mathcal{C}^{\epsilon}(x)).$$

Moreover,

$$I^{00}(x;\mathcal{C}) = 2^{\frac{1}{2}} \sum_{j=1}^{q} (x,\underline{C}_j)\bigg(\int_1^\infty e^{-2\pi t^2 (x,\underline{C}_j)^2}\, I^0(tx_{\perp j};\mathcal{C}[j])\,dt\bigg) - (x,\underline{C}_{j'})\bigg(\int_1^\infty e^{-2\pi t^2 (x,\underline{C}_{j'})^2}\, I^0(tx_{\perp j'};\mathcal{C}[j'])\,dt \bigg). \tag{7.1}$$

where $\mathcal{C}[j]$ and $\mathcal{C}[j']$ are given by (3.4) and (3.5).

Proof. Combining (3.3), (3.6), (3.7), and Corollary 6.3, we obtain

$$\begin{aligned} I^{00}(x;\mathcal{C}) &= \sum_{j=1}^{q} (-1)^j \bigg(\int_{\partial_j^+ S(\mathcal{C})} \Psi^0_{KM}(x) - \int_{\partial_j^- S(\mathcal{C})} \Psi^0_{KM}(x) \bigg) \\ &= \sum_{j=1}^{q} (-1)^j \bigg(\int_{S(\mathcal{C}[j])} \kappa_j^* \Psi^0_{KM}(x) - \int_{S(\mathcal{C}[j'])} \kappa_{j'}^* \Psi^0_{KM}(x) \bigg) \\ &= 2^{\frac{1}{2}} \sum_{j=1}^{q} (x,\underline{C}_j)\bigg(\int_1^\infty e^{-2\pi t^2 (x,\underline{C}_j)^2}\, I^0(tx_{\perp j};\mathcal{C}[j])\,dt\bigg) \\ &\qquad - (x,\underline{C}_{j'})\bigg(\int_1^\infty e^{-2\pi t^2 (x,\underline{C}_{j'})^2}\, I^0(tx_{\perp j'};\mathcal{C}[j'])\,dt \bigg), \end{aligned}$$

as claimed. □

7.2. **The case $q = 1$.** As a basis for the inductive proof of Theorem 4.1, we first suppose that $q = 1$, so that $\operatorname{sig}(V) = (m-1,1)$. This case is discussed in several places, [12], [5], [16], etc., but we give the calculation for convenient reference. We have

$$D \simeq \{\zeta \in V \mid Q(\zeta) = -1\}, \qquad z = \operatorname{span}\{\zeta\}_{\text{p.o.}},$$

and the tangent space at $z \in D$ is

$$T_z(D) \simeq U(z) := z^{\perp}.$$

For any $x \in V$ the 1-form $\omega(1)$ on D is defined by

$$\omega(1)_z(\eta) = (x,\eta), \qquad \eta \in U(z) \simeq T_z(D),$$

and the Schwartz form is given by

$$\varphi^0_{KM}(x) = 2^{\frac{1}{2}}\, \omega(1)\, e^{-2\pi R(x,z)},$$

with $R(x,z) = (x,\zeta)^2$. Take C, $C' \in V$ such that

$$Q(C) < 0, \quad Q(C') < 0, \quad (C, C') < 0,$$

where the third condition insures that

$$\{C\}_{\text{p.o.}} \simeq \underline{C} = C\,|(C,C)|^{-\frac{1}{2}}, \qquad \{C'\}_{\text{p.o.}} \simeq \underline{C}'$$

lie on the same component of D. For $s \in [0,1]$, we define

$$B(s) = (1-s)C + sC',$$

and note that

$$(B(s), B(s)) = (1-s)^2(C,C) + 2s(1-s)(C,C') + s^2(C',C') < 0,$$

so that the collection $\mathcal{C} = \{\{C, C'\}\}$ is in good position. Writing

$$\zeta = \zeta(s) = B(s)|(B(s), B(s))|^{-\frac{1}{2}},$$

we obtain a geodesic curve

$$\phi_{\mathcal{C}} : [0,1] \longrightarrow D, \qquad s \mapsto \{B(s)\}_{\text{p.o.}} \simeq \zeta(s)$$

joining $\underline{C}$ and $\underline{C}'$. The tangent vector to this curve will be $\dot{\zeta} = \frac{d}{ds}\zeta$, and

$$\begin{aligned} I^0(x;\mathcal{C}) &= 2^{\frac{1}{2}} \int_0^1 (x, \dot{\zeta}(s))\, e^{-2\pi(x,\zeta(s))^2}\, ds \\ &= 2^{\frac{1}{2}} \int_0^1 \frac{\partial}{\partial s}\left(- \int_{(x,\zeta(s))}^{\infty} e^{-2\pi t^2}\, dt \right) ds \\ &= 2^{\frac{1}{2}} \left(\int_{(x,\underline{C})}^{\infty} e^{-2\pi t^2}\, dt - \int_{(x,\underline{C}')}^{\infty} e^{-2\pi t^2}\, dt \right). \end{aligned}$$

Since

$$\int_u^{\infty} e^{-2\pi t^2}\, dt = 2^{-\frac{3}{2}}(1 - E(u\sqrt{2})),$$

for

$$E(u) = 2\int_0^u e^{-\pi t^2}\, dt = 2\,\text{sgn}(u) \int_0^{|u|} e^{-\pi t^2}\, dt,$$

as in [23], we obtain the expression

$$\begin{aligned} I^0(x;\mathcal{C}) &= \frac{1}{2}\left(E((x,\underline{C}')\sqrt{2}) - E((x,\underline{C})\sqrt{2}) \right) \\ &= \frac{1}{2}\left(E_1(C', x\sqrt{2}) - E_1(C, x\sqrt{2}) \right), \end{aligned}$$

which is the $q = 1$ case of Theorem 4.1. Here we use the fact that, for $C \in V$ with $Q(C) < 0$, a simple calculation shows that $E_1(C;x) = E((x,\underline{C}))$. Note that in this calculation we have not used the Stokes' theorem argument. However, it is instructive to note that

$$\psi^0_{KM}(x) = 2^{-\frac{1}{2}}\,(x,\zeta)\, e^{-2\pi(x,\zeta)^2},$$

so that, for $z = \operatorname{span}\{C\}_{\text{p.o.}} \in D - D_x$, the primitive is given by

$$\begin{aligned}
\Psi^0_{KM}(x) &= -2^{-\frac{1}{2}}\,(x,\underline{C}) \int_1^\infty e^{-2\pi t (x,\underline{C})^2}\, t^{-\frac{1}{2}}\, dt \\
&= -2^{\frac{1}{2}}\,(x,\underline{C}) \int_1^\infty e^{-2\pi t^2 (x,\underline{C})^2}\, dt \\
&= -\operatorname{sgn}(x,\underline{C}) \int_{\sqrt{2}|(x,\underline{C})|}^\infty e^{-\pi t^2}\, dt \\
&= \frac{1}{2}\operatorname{sgn}(x,\underline{C})\Big(2\int_0^{\sqrt{2}|(x,\underline{C})|} e^{-\pi t^2}\, dt - 1\Big) \\
&= \frac{1}{2}\big(\, E_1(C;x\sqrt{2}) - \operatorname{sgn}(x,C)\,\big).
\end{aligned}$$

Thus the Stokes' theorem calculation gives

$$I^{00}(x;\mathcal{C}) = \int_{\partial S(\mathcal{C})} \Psi^0_{KM}(x) = \frac{1}{2}\big(\, E_1(C_{1'};x\sqrt{2}) - \operatorname{sgn}(x,C_{1'}) - E_1(C_1;x\sqrt{2}) + \operatorname{sgn}(x,C_1)\,\big),$$

so that the basis for Zwegers 'completion' construction emerges.

7.3. **Induction.** Next we consider the inductive step. Note that we are assuming that x is regular with respect to $\mathcal{C}$ so that (7.1) holds, and we suppose that the identity (4.3) holds for all $q' < q$ and all $\mathcal{C}'$ in good position. Let $I[j]$ and $I[j']$ be subsets of $\{1,\dots,\widehat{j},\dots,q\}$ and let $C[j]^{I[j]}$ (resp. $C[j']^{I[j']}$) be obtained by the recipe defining C^I in Theorem 4.1, starting with the set $\mathcal{C}[j]$ defined in (3.4) (resp. the set $\mathcal{C}[j']$ defined in (3.5)). Then (7.1) becomes

$$\begin{aligned}
I^{00}(x/\sqrt{2};\mathcal{C}) &= \sum_{j=1}^q (x,\underline{C}_j)\Big(\int_1^\infty e^{-\pi t^2 (x,\underline{C}_j)^2}\, I^0(tx_{\perp j}/\sqrt{2};\mathcal{C}[j])\, dt\Big) \\
&\qquad - (x,\underline{C}_{j'})\Big(\int_1^\infty e^{-\pi t^2 (x,\underline{C}_{j'})^2}\, I^0(tx_{\perp j'}/\sqrt{2};\mathcal{C}[j'])\, dt\Big) \\
&= (-1)^{q-1}2^{1-q}\sum_{j=1}^q \Big(\sum_{I[j]} (-1)^{|I[j]|}\Big((x,\underline{C}_j)\int_1^\infty e^{-\pi t^2 (x,\underline{C}_j)^2}\, E_{q-1}(C[j]^{I[j]};tx_{\perp j})\, dt\Big) \qquad (7.2) \\
&\qquad - \sum_{I[j']} (-1)^{|I[j']|}\Big((x,\underline{C}_{j'})\int_1^\infty e^{-\pi t^2 (x,\underline{C}_{j'})^2}\, E_{q-1}(C[j']^{I[j']};tx_{\perp j'})\, dt\Big)\Big).
\end{aligned}$$

We want to compare this to the expression

$$-2^{-q}\sum_I (-1)^{|I|}\, E_q(C^I;x).$$

The key is to relate the individual quantities $E_q(C^I;x)$ in this sum and the terms on the right side of (7.2) where $I = I[j]$ or $I = \{j\}\cup I[j']$. Note that, if $I = I[j]$ then the collection $C[j]^{I[j]}$ spans a negative $q-1$-plane in V_j which maps to z^I under κ_j. Similarly, if $I = \{j\}\cup I[j']$, then the collection $C[j']^{I[j']}$ spans a negative $q-1$-plane in $V_{j'}$ which maps to z^I under $\kappa_{j'}$. Thus, we are collecting all of the terms which 'correspond to' a given vertex of the q-cube $S(\mathcal{C})$.

The required identities are all consequences of that for $I = \emptyset$, and thus the main identity needed is the following.

Proposition 7.3. *Suppose that x is regular with respect to C. Then*

$$E_q(C;x) - \operatorname{sgn}(C;x) = -2\sum_{j=1}^{q}(x,\underline{C}_j)\int_1^\infty e^{-\pi t^2(x,\underline{C}_j)^2}\, E_{q-1}(C[j];tx_{\perp j})\,dt, \tag{7.3}$$

where $C = \{C_1,\dots,C_q\}$, $C[j] = \{C_{1\perp j},\dots,\widehat{C_j},\dots,C_{q\perp j}\}$, and $\operatorname{sgn}(C;x)$ is defined in (4.2).

Remark 7.4. This result is just an integrated version of equation (25) in Proposition 3.6 in [18]. For convenience, we give the proof, taken from [18], in our notation.

Proof. Let z be the negative q-plane spanned by $C = \{C_1,\dots,C_q\}$, and, for $y,\ y' \in z$, let $(\!(y,y')\!) = -(y,y')$. We also suppose that $x = \operatorname{pr}_z(x)$. If f is a smooth function on z, then

$$\begin{aligned} -\int_1^\infty (\!(\nabla f(t\,x),x)\!)\,dt &= -\int_1^\infty \frac{d}{dt}\{f(tx)\}\,dt \\ &= f(x) - \lim_{t\to\infty} f(tx). \end{aligned} \tag{7.4}$$

Here ∇ is the gradient operator and we assume that the radial limit of f exists. On the other hand, by (25) Proposition 3.6 of [18],

$$-(\!(\nabla E_q(C;x),x)\!) = 2\sum_j (\!(x,\underline{C}_j)\!)\, e^{-\pi\,(\!(x,\underline{C}_j)\!)^2}\, E_{q-1}(C[j];x_{\perp j}). \tag{7.5}$$

Moreover, for x regular with respect to C, we have, [1] and [18], Remark p.7,

$$\lim_{t\to\infty} E_q(C;tx) = \operatorname{sgn}(C;x). \tag{7.6}$$

For convenience, we will give the proof of (7.5) in the Appendix. Combining them and noting that the identity (7.4) is valid for the function $f(x) = E_q(C;x)$ when x is regular with respect to C, we have

$$E_q(C;x) - \operatorname{sgn}(C;x) = -2\sum_j (x,\underline{C}_j)\int_1^\infty e^{-\pi\,t^2(x,\underline{C}_j)^2}\, E_{q-1}(C[j];tx_{\perp j})\,dt,$$

as required. $\square$

Corollary 7.5.

$$I^{00}(x;\mathcal{C}) = (-1)^q 2^{-q}\sum_I (-1)^{|I|}\big(\ E_q(C^I;x\sqrt{2}) - \operatorname{sgn}(C^I;x)\ \big),$$

and

$$I^0(x;\mathcal{C}) = I^{00}(x;\mathcal{C}) + (-1)^q\Phi_q(x;\mathcal{C}) = (-1)^q 2^{-q}\sum_I (-1)^{|I|} E_q(C^I;x\sqrt{2}).$$

Note that the second identity in Corollary 7.5 follows from Proposition 7.2, since the first identity implies that

$$\lim_{\epsilon\downarrow 0} I^{00}(x;\mathcal{C}^\epsilon(x)) = -(-1)^q\Phi_q(x;\mathcal{C}).$$

The identity of Theorem 4.1 follows immediately from this and the continuity of $E(C^I;x)$ with respect to C^I.

Corollary 7.6.

$$I(\tau,x,\mathcal{C}) := \int_{[0,1]^q} \phi^*_{\mathcal{C}}(\varphi_{KM}(\tau,x)) \tag{7.7}$$
$$= \mathbf{q}^{Q(x)}\,(-1)^q 2^{-q} \sum_I (-1)^{|I|} E_q(C^I; x\sqrt{2v}).$$

8. Shadows of indefinite theta series

In this section, we compute the shadow of $I_\mu(\tau,\mathcal{C})$, i.e., the complex conjugate of its image under the lowering operator $-2iv^2\frac{\partial}{\partial\bar\tau}$. The crucial facts are the relation (i) of Lemma 5.1,

$$-2iv^2\frac{\partial}{\partial\bar\tau}\varphi_{KM}(\tau,x) = d\psi(\tau,x),$$

and the pullback identity (ii) of Lemma 6.3. Then, as in the proof of Proposition 7.2, we have

$$-2iv^2\frac{\partial}{\partial\bar\tau}\{I_\mu(\tau,\mathcal{C})\} = \sum_{x\in\mu+L}\int_{\partial S(\mathcal{C})}\psi(\tau,x), \tag{8.1}$$

and

$$\int_{\partial S(\mathcal{C})}\psi(\tau,x) = 2^{-\frac12}\,v^{\frac32}\sum_{j=1}^q\Bigg(\,(x,\underline{C}_{j'})\,e^{-2\pi v(x,\underline{C}_{j'})^2}\,\mathbf{q}^{-\frac12(x,\underline{C}_{j'})^2}\;I(\tau,x_{\perp j'},\mathcal{C}[j']) \tag{8.2}$$
$$-\,(x,\underline{C}_j)\,e^{-2\pi v(x,\underline{C}_j)^2}\,\mathbf{q}^{-\frac12(x,\underline{C}_j)^2}\,I(\tau,x_{\perp j},\mathcal{C}[j])\,\Bigg).$$

where we use the notation introduced in Corollary 7.6. The combination of (8.1), (8.2) and (7.7) yields an explicit formula for the shadow of $I_\mu(\tau,\mathcal{C})$, a (typically non-holomorphic) modular form of weight $2-\frac{p+q}{2}$.

Now suppose that the collection $\mathcal{C}$ is rational. For each j, write

$$L^0_j = L\cap\mathbb{Q}C_j,\quad L^1_j = L\cap V_j$$

so that, for suitable coset representatives $\mu^0_{j,r}\in(L^0_j)^\vee$ and $\mu^1_{j,r}\in(L^1_j)^\vee$,

$$\mu+L = \bigsqcup_r\Big((\mu^0_{j,r}+L^0_j)\oplus(\mu^1_{j,r}+L^1_j)\Big).$$

Then, writing $I_\mu(\tau,\mathcal{C},L)$ to make explicit the dependence on the lattice L,

$$-2iv^2\frac{\partial}{\partial\bar\tau}\{I_\mu(\tau,\mathcal{C},L)\} = 2^{-\frac12}\sum_{j=1}^q\sum_r v^{\frac32}\,\overline{\theta_{\mu^0_{j,r}}(\tau,L^0_j)}\,I_{\mu^1_{j,r}}(\tau,\mathcal{C}[j],L^1_j)$$
$$-\sum_{r'} v^{\frac32}\,\overline{\theta_{\mu^0_{j',r'}}(\tau,L^0_{j'})}\,I_{\mu^1_{j',r'}}(\tau,\mathcal{C}[j'],L^1_{j'}),$$

where

$$\theta_{\mu^0_{j,r}}(\tau, L^0_j) = \sum_{x^0 \in \mu^0_{j,r} + L^0_j} (x^0, \underline{C}_j)\, \mathbf{q}^{\frac{1}{2}(x^0, \underline{C}_j)^2}$$

is a unary theta series of weight $\frac{3}{2}$. Thus, the image of $I_\mu(\tau, \mathcal{C})$ under the ξ-operator is a linear combination of products of unary theta series of weight $\frac{3}{2}$ and the conjugates of indefinite theta series for spaces of signature $(p, q-1)$, as asserted in Corollary 1.3.

Remark 8.1. While we have only discussed the shadows of 'cubical' integrals, the case of simplical integrals can the treated in the same way.

9. The case of a simplex

In this section, we work out the theta integral over a simplex. The general inductive procedure is the same as in the cubical case, but some interesting differences arise.

9.1. **Some geometry.** For V of signature (p, q), we consider a collection of vectors

$$\mathcal{C} = [C_0, \dots, C_q]$$

$C_i \in V$ with $(C_i, C_i) < 0$. We suppose that, for all j,

$$P_j = \operatorname{span}\{C_0, \dots, \widehat{C_j}, \dots, C_q\}$$

is a negative q-plane. We assume that the collection $\mathcal{C}$ is linearly independent and let $U = \operatorname{span}(\mathcal{C})$. Note that $\operatorname{sig}(U) = (1, q)$, and let $D(U)$ be the space of oriented negative q-planes in U.

Let

$$\mathcal{C}^\vee = [C_0^\vee, \dots, C_q^\vee] = \mathcal{C}\,(\mathcal{C}, \mathcal{C})^{-1}$$

be the dual basis of U with respect to $(\ ,\ )$. Since $C_j^\vee$ then spans $P_j^\perp$, we have $(C_j^\vee, C_j^\vee) > 0$. Let

$$\Delta_q = \{s = [s_0, \dots, s_q] \in \mathbb{R} \mid \ s_j \geq 0, \text{ for all } j, \sum_j s_j = 1\},$$

and, for $s \in \Delta_q$, let

$$C^\vee(s) = \sum_{j=0}^q s_j C_j^\vee = \mathcal{C}^\vee\, {}^t s.$$

Note that $s_j = (C^\vee(s), C_j)$. We say that $\mathcal{C}$ is in good position if

$$0 < (C^\vee(s), C^\vee(s)) = s(\mathcal{C}^\vee, \mathcal{C}^\vee)^t s = s(\mathcal{C}, \mathcal{C})^{-1 t} s$$

for all $s \in \Delta_q$. For example, if all entries of $(\mathcal{C}^\vee, \mathcal{C}^\vee) = (\mathcal{C}, \mathcal{C})^{-1}$ are non-negative, then $\mathcal{C}$ is in good position.

Given $\mathcal{C}$ in good position, we define

$$z(s) = C^\vee(s)^\perp \in D,$$

with orientation $\nu_{z(s)} \in \bigwedge^q z(s)$ defined by

$$C^\vee(s) \wedge \nu_{z(s)} = \nu_U, \tag{9.1}$$

where we have fixed an orientation

$$\nu_U = C_0 \wedge C_1 \wedge \cdots \wedge C_q$$

in $\bigwedge^{q+1} U$. For example,

$$z_j = z(0, \dots, 1, \dots, 0) = (C_j^\vee)^\perp = \operatorname{span}\{C_0, \dots, \widehat{C_j}, \dots, C_q\}$$

with orientation given as follows. Let R_j be the jth column of the matrix $(C, C)^{-1}$, so that

$$C_j^\vee = \mathcal{C}\, R_j = \sum_{i=0}^{q} R_{ij} C_i. \tag{9.2}$$

Then

$$C_j^\vee \wedge C_0 \wedge \cdots \wedge \widehat{C_j} \wedge \cdots \wedge C_q = (-1)^j R_{jj}\, C_0 \wedge C_1 \wedge \cdots \wedge C_q.$$

Since $R_{jj} = (C_j^\vee, C_j^\vee) > 0$,

$$z_j = \operatorname{span}\{C_0, \dots, \widehat{C_j}, \dots, C_q\}[j] \tag{9.3}$$

where the 'twist' $[j]$ indicates that the given basis gives $(-1)^j \nu_{z(s)}$.

For example, for $q = 1$ we have

$$z_0 = \operatorname{span}\{C_1\}_{\text{p.o.}}, \qquad z_1 = \operatorname{span}\{-C_0\}_{\text{p.o.}}. \tag{9.4}$$

In particular, good position requires $(C_0, C_1) > 0$ in this case! For $q = 2$, we have

$$z_0 = \operatorname{span}\{C_1, C_2\}_{\text{p.o.}}, \quad z_1 = \operatorname{span}\{-C_0, C_2\}_{\text{p.o.}}, \qquad z_2 = \operatorname{span}\{C_0, C_1\}_{\text{p.o.}}. \tag{9.5}$$

By construction, all the z_j's lie in the same component of D and, by linear independence, the map

$$\phi_{\mathcal{C}} : \Delta_q \longrightarrow D, \qquad s \mapsto z(s)$$

is an embedding. Let $S(\mathcal{C}) = \phi_{\mathcal{C}}(\Delta_q)$ be its image. The jth face of this tetrahedron is given by restricting to the subset of s with $s_j = 0$, so that it is given as

$$\{z \in S(\mathcal{C}) \mid (C^\vee(s), C_j) = 0\} = \{z \in S \mid C_j \in z\}.$$

Moreover, in the image U_j of U under the projection to $V_j = C_j^\perp$, we have

$$[C_0^\vee, \dots, \widehat{C_j^\vee}, \dots, C_q^\vee]$$

is the dual basis to

$$\mathcal{C}_{\perp j} := [C_{0\perp j}, \dots, C_{q\perp j}].$$

Thus, up to orientation, to be discussed in a moment, the restriction of $\phi_{\mathcal{C}}$ to a face of Δ_q is again a simplex $\phi_{\mathcal{C}_{\perp j}}$ in $D(V_j)$! Note that, in particular, $\mathcal{C}$ in good position implies that $\mathcal{C}_{\perp j}$ is in good position for all j.

Next consider $S(\mathcal{C}) \cap D_x$. This set depends only on $\mathrm{pr}_U(x)$ and is given by

$$S(\mathcal{C}) \cap D_x = \begin{cases} S(\mathcal{C}) \cap D(U)_{\mathrm{pr}_U(x)} & \text{if } Q(\mathrm{pr}_U(x)) > 0, \\ \emptyset & \text{if } \mathrm{pr}_U(x) \neq 0 \text{ and } Q(\mathrm{pr}_U(x)) \leq 0, \\ S(\mathcal{C}) & \text{if } \mathrm{pr}_U(x) = 0. \end{cases}$$

Here, when $Q(\mathrm{pr}_U(x)) > 0$ so that $\mathrm{pr}_U(x)$ is a positive vector in U, $D(U)_{\mathrm{pr}_U(x)}$ is a pair of oriented negative q-planes in U given by the orthogonal complement to $\mathrm{pr}_U(x)$ with its two orientations. One of these has orientation determined by $\mathrm{pr}_U(x)$ by the analogue of the recipe (9.1). Then $S(\mathcal{C}) \cap D_x = \phi_{\mathcal{C}}(s(x))$ is the same q-plane with orientation shifted by

$$\mathrm{sgn}(\mathrm{pr}_U(x), C^\vee(s(x))^q = \mathrm{sgn}(x, C^\vee(s(x)))^q.$$

To determine $s(x)$, we solve

$$\mathrm{pr}_U(x) = \lambda\, C^\vee(s), \qquad s \in \Delta_q, \quad \lambda \in \mathbb{R}^\times$$

i.e.,

$$(x, C_j) = \lambda\, s_j, \qquad 0 \leq j \leq q.$$

The existence of a solution implies that $\mathrm{sgn}(x, C_j)$, if non-zero, is independent of j and that

$$\sum_{j=0}^{q} (x, C_j) = \lambda. \tag{9.6}$$

Thus we have the following simple description.

Lemma 9.1. *Suppose that $Q(\mathrm{pr}_U(x)) > 0$. If $\mathrm{sgn}(x, C_j)$ is independent of j when it is non-zero, then*

$$S(\mathcal{C}) \cap D_x = \phi_{\mathcal{C}}(s(x)),$$

where

$$s(x)_j = (x, C_j)\lambda(x; \mathcal{C})^{-1}$$

with

$$\lambda(x, \mathcal{C}) = \sum_j (x, C_j).$$

Otherwise $S(\mathcal{C}) \cap D_x = \phi_{\mathcal{C}}(s(x))$ is empty.

When $S(\mathcal{C}) \cap D_x$ is non-empty, we determine the intersection number of the oriented q-simplex $S(\mathcal{C})$ with the oriented codimension q cycle D_x. The claim is that this is determined by the sign of the inner product of $\mathrm{pr}_U(x)$ with $C^\vee(s(x))$.

Proposition 9.2. *Let $\Phi_q^{\triangle}(x, \mathcal{C})$ be as in (1.3). Then, if x is regular with respect to $\mathcal{C}$,*

$$I(S(\mathcal{C}), D_x) = \Phi_q^{\triangle}(x, \mathcal{C}). \tag{9.7}$$

Suppose that $\mathrm{pr}_U(x) \neq 0$. Then $\Phi_q^{\triangle}(x, \mathcal{C})$ is non-zero precisely when all of the non-zero $\mathrm{sgn}(x, C_i)$'s coincide. Suppose further that $s(x)$ lies on r 'walls', i.e., that r of the inner products (x, C_j) vanish. Then

$$\Phi_q^{\triangle}(x, \mathcal{C}) = 2^{-r}\,(-1)^q \mathrm{sgn}(\lambda(x, \mathcal{C}))^q.$$

When $\mathrm{pr}_U(x)=0$, then $\Phi_q^{\triangle}(x,\mathcal{C})=2^{-q}$ for q even and vanishes for q odd. Note that, if x is not regular with respect to $\mathcal{C}$, then the intersection number is not defined.

Proof. Recall that, if $\zeta\in\mathrm{OFD}$ is a properly oriented q-frame projecting to $z\in D$, then $T_z(D)\simeq U(z)^q$, where $U(z)=z^{\perp}$ in V. Also note that, under this isomorphism, the natural metric on $T_z(D)$ is given by $(\!(\eta,\eta')\!)=-\mathrm{tr}((\eta_i,\eta'_j))$ where $\eta=[\eta_1,\dots,\eta_q]$ and $\eta'=[\eta'_1,\dots,\eta'_q]$. For our fixed collection $\mathcal{C}$ with $U=\mathrm{span}\{\mathcal{C}\}$, we have an embedding $D(U)\longrightarrow D$, where $D(U)$ is the space of oriented negative q-planes in U. Recall that $\mathrm{sig}(U)=(1,q)$. For $z\in D(U)$, write $W(z)$ for its orthogonal complement in U. Again supposing that $\zeta\in\mathrm{OFD}$ with projection z is given, we have

$$T_z(D(U))\simeq W(z)^q.$$

Note that $\dim W(z)=1$, and suppose that $w=w(z)$ is a properly oriented basis vector. Then $T_z(D(U))$ is spanned by the vectors $\tau_1(w)=[w,0,\dots,0]$, $\tau_2(w)=[0,w,0,\dots,0]$, etc. Similarly, if $z\in D_x$, then the normal subspace to $T_z(D_x)$ is spanned by the vectors $\tau_i(x)$, $1\le i\le q$. For $z=\phi_{\mathcal{C}}(s(x))$, we have $w=C^{\vee}(s(x))$, and the intersection number of these two cycles is then given by

$$\begin{aligned}\mathrm{sgn}(\!(\,\tau_1(x)\wedge\cdots\wedge\tau_q(x),\tau_1(w)\wedge\cdots\wedge\tau_q(w)\,)\!)&=(-1)^q\det((\tau_i(x),\tau_j(w)))\\&=(-1)^q\mathrm{sgn}(x,C^{\vee}(s(x)))^q.\end{aligned}$$

But now

$$C^{\vee}(s(x))=\lambda(x,\mathcal{C})^{-1}\sum_j(x,C_j)\,C_j^{\vee},$$

and, recalling (9.2),

$$(x,C^{\vee}(s(x)))=\lambda(x,\mathcal{C})^{-1}\sum_j(x,C_j)(x,C_j^{\vee})=\lambda(x,\mathcal{C})^{-1}\sum_{i,j}(x,C_j)R_{i,j}(x,C_i).$$

If we assume that all of the non-zero (x,C_i)'s have the same sign, and recalling that $R_{i,j}\ge 0$, we see that

$$\mathrm{sgn}(x,C^{\vee}(s(x)))=\mathrm{sgn}(\lambda(x,\mathcal{C})).$$

$\square$

For $q=1$, and x regular with respect to $\mathcal{C}$,

$$I(S(\mathcal{C}),D_x)=-\frac{1}{2}(\mathrm{sgn}(x,C_0)+\mathrm{sgn}(x,C_1)).$$

Note that, due to the 'twist' occurring in (9.3), our negative lines are $z_0=\mathrm{span}\{C_1\}_{\mathrm{p.o.}}$ and $z_1=\mathrm{span}\{-C_0\}_{\mathrm{p.o.}}$ Thus the 'cubical' data is $\mathcal{C}^{\square}=\{C_1,-C_0\}$, and $I(S(\mathcal{C}),D_x)$ coincides with

$$\Phi_1^{\square}(x,\mathcal{C}^{\square})=\frac{1}{2}(\mathrm{sgn}(x,-C_0)-\mathrm{sgn}(x,C_1)).$$

9.2. **The integral of the theta form.** We would like to compute

$$I^0(x;\mathcal{C}) = \int_{S(\mathcal{C})} \varphi^0_{KM}(x).$$

The case $q=1$, coincides with the Zwegers case for $\mathcal{C}^\square = \{C_1, -C_0\}$, and we have

$$I^0(x/\sqrt{2};\mathcal{C}) = -\frac{1}{2}(\,E_1(C_0;x) + E_1(C_1;x)). \tag{9.8}$$

As a check on signs, note that, since

$$\lim_{t\to\infty} E_1(C;tx) = \operatorname{sgn}(x,C),$$

this is consistent with the value of $I(S(\mathcal{C}),D_x)$ for $q=1$ above.

For the general case, we suppose that x is regular with respect to $\mathcal{C}$ and proceed by induction. Due to regularity, $S(\mathcal{C})\cap D_x$ is either empty or is a single point $\phi_{\mathcal{C}}(s(x))$ on the interior of $S(\mathcal{C})$. Recall that, by (9.3),

$$\partial S(\mathcal{C}) = \sum_{j=0}^{q} (-1)^j S(\mathcal{C}_{\perp j}).$$

Then by Remark 3.4 of [5], we have

$$I^0(x;\mathcal{C}) = \int_{S(\mathcal{C})} \varphi^0_{KM}(x) = I(S(\mathcal{C}),D_x) + \int_{\partial S(\mathcal{C})} \Psi^0_{KM}(x). \tag{9.9}$$

Since $\lim_{t\to\infty}\Psi^0_{KM}(tx) = 0$, this identity gives the limiting value

$$\lim_{t\to\infty} I^0(tx;\mathcal{C}) = \lim_{t\to\infty}\int_{S(\mathcal{C})} \varphi^0_{KM}(tx) = I(S(\mathcal{C}),D_x).$$

Now using Corollary 6.2, we have the inductive formula

$$\begin{aligned}\int_{\partial S(\mathcal{C})} \Psi^0_{KM}(x) &= \sum_{j=0}^{q}(-1)^j \int_{S(\mathcal{C}_{\perp j})} \kappa_j^*\Psi^0_{KM}(x) \\ &= \sum_{j=0}^{q} 2^{\frac{1}{2}}\,(x,\underline{C}_j)\int_1^\infty e^{-2\pi t^2 (x,\underline{C}_j)^2}\, I^0(tx_{\perp j};\mathcal{C}_{\perp j})\,dt.\end{aligned} \tag{9.10}$$

Using this, we obtain the following explicit formula.

Theorem 9.3. *For a subset $I\subset\{0,1,\dots,q\}$, let $\mathcal{C}^{(I)}$ be the collection of $q+1-|I|$ elements where the C_i with $i\in I$ have been omitted.*

$$I^0(x/\sqrt{2};\mathcal{C}) = (-1)^q 2^{-q}\sum_{r=0}^{[q/2]}\sum_{\substack{I\\ |I|=2r+1}} E_{q-2r}(\mathcal{C}^{(I)};x).$$

Here $E_0(\dots)=1$.

Remark 9.4. (i) Note that if this formula is proved for x regular, then it holds for all x by continuity.
(ii) Substituting tx for x and letting t go to infinity, we obtain the 'holomorphic' part:

$$(-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \\ |I|=2r+1}} \operatorname{sgn}(\mathcal{C}^I, x), \tag{9.11}$$

where $\operatorname{sgn}(\emptyset, x) = 1$. In the case of x regular, (9.9) implies that this must coincide with $I(S(\mathcal{C}), D_x)$. In fact, it is easily checked that (9.11) is equal to $\Phi_q^{\triangle}(x, \mathcal{C})$ for all x. Thus our theta integral is the non-holomorphic completion of the series

$$\sum_{x \in \mu + L} \Phi_q^{\triangle}(x, \mathcal{C})\, \mathbf{q}^{Q(x)}.$$

Proof. The case $q = 1$ is (9.8). In the induction, we use the notation

$$\mathcal{C}[j] = [C_{0\perp j}, \dots, C_{j-1\perp j}, C_{j+1\perp j}, \dots, C_{q\perp j}].$$

Let $A = \{0, 1, \dots, q\}$ and for a subset $I \subset A$, let $\mathcal{C}^I$ be the collection of $q + 1 - |I|$ vectors obtained by omitting the C_i with $i \in I$. Also denote by $I[j]$ a subset of $A[j] := \{0, 1, \dots, \hat{j}, \dots, q\}$.

We have

$$I^0(x/\sqrt{2}; \mathcal{C}) - I(S(\mathcal{C}), D_x) = \int_{\partial S(\mathcal{C})} \Psi^0_{KM}(x/\sqrt{2})$$

$$= \sum_{j=0}^{q} (x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2}\, I^0(t x_{\perp j}/\sqrt{2}; \mathcal{C}[j])\, dt$$

$$= (-1)^q 2^{-q} \sum_{j=0}^{q} -2(x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I[j] \subset A[j] \\ |I[j]|=2r+1}} E_{q-1-2r}(\mathcal{C}[j]^{I[j]}, t x_{\perp j})\, dt$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{j \in A} \sum_{\substack{I \subset A \\ |I|=2r+1 \\ j \notin I}} -2(x, \underline{C}_j) \int_1^{\infty} e^{-\pi t^2 (x, \underline{C}_j)^2}\, E_{q-1-2r}(\mathcal{C}[j]^I; t x_{\perp j})\, dt$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \sum_{\substack{j \in A \\ j \notin I}} -2(x, \underline{C}_j) \int_1^\infty e^{-\pi t^2 (x,\underline{C}_j)^2}\, E_{q-1-2r}((\mathcal{C}^I)[j]; t x_{\perp j})\, dt$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \big(\ E_{q-2r}(\mathcal{C}^I; x) - \operatorname{sgn}(x, \mathcal{C}^I)\ \big)$$

$$= (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} E_{q-2r}(\mathcal{C}^I; x)$$

$$- (-1)^q 2^{-q} \sum_{r=0}^{[(q-1)/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \operatorname{sgn}(x, \mathcal{C}^I) - (-1)^q 2^{-q}\, \delta_{q,\text{even}}.$$

Thus, to finish the proof, we note that

$$I(S(\mathcal{C}), D_x) = (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \operatorname{sgn}(x, \mathcal{C}^I), \tag{9.12}$$

where we use the convention that $\operatorname{sgn}(x, \emptyset) = 1$. Here recall that we are assuming that x is regular with respect to $\mathcal{C}$. To check this, observe that

$$(-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \subset A \\ |I|=2r+1}} \operatorname{sgn}(x, \mathcal{C}^I) = (-1)^q 2^{-q} \sum_{\substack{J \subset A \\ |J| \equiv q (2)}} \prod_{j \in J} \sigma_j$$

$$= (-1)^q 2^{-q-1} \bigg(\prod_{j \in A} (1 + \sigma_j) + (-1)^q \prod_{j \in A} (1 - \sigma_j)\ \bigg).$$

□

10. An example

In this section, we write out a very simple example, which illustrates the relation between the (degenerate) cubical formula and the simplicial formula in the case $q = 2$.

Let $\mathcal{A} = \{A_0, A_1, A_2\}$ be the data for a 2-simplex. The vertices are:

$$z_0 = \operatorname{span}\{A_1, A_2\}_{\text{p.o.}}, \quad z_1 = \operatorname{span}\{-A_0, A_2\}_{\text{p.o.}}, \qquad z_2 = \operatorname{span}\{A_0, A_1\}_{\text{p.o.}},$$

and the theta integral is

$$\frac{1}{4}\big(\ E_2(A_1, A_2) + E_2(A_0, A_2) + E_2(A_0, A_1) + 1\ \big).$$

We can consider the related cubical data $\mathcal{C} = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}\}$, where

$$C_1 = A_0,\ C_2 = A_1,\ C_{2'} = -A_2,\ C_{1'} = C_{2'} - C_2 = -A_1 - A_2,$$

so that the associated (degenerate) 2-cube has vertices

$$z_2 = \{C_1, C_2\},\ z_1 = \{C_1, C_{2'}\},\ z_0 = \{C_{1'}, C_{2'}\} = \{C_{1'}, C_2\},$$

and theta integral

$$\begin{aligned}&\frac{1}{4}\big(\ E_2(C_1,C_2) - E_2(C_1,C_{2'}) - E_2(C_{1'},C_2) + E_2(C_{1'},C_{2'})\ \big)\\ =&\frac{1}{4}\big(\ E_2(A_0,A_1) + E_2(A_0,A_2) + E_2(A_1+A_2,A_1) + E_2(A_1+A_2,A_2)\ \big).\end{aligned}$$

Coincidence of the two theta integrals is the equivalent to the identity

$$E_2(A_1+A_2,A_1) + E_2(A_1+A_2,A_2) = E_2(A_1,A_2) + 1,$$

where all terms are given by integrals over the negative 2-plane z_0. Writing $y \in z_0$ as $y = aA_1^\vee + bA_2^\vee$, with respect to the dual basis, and noting that

$$\operatorname{sgn}(a+b)(\operatorname{sgn}(a) + \operatorname{sgn}(b)) = \operatorname{sgn}(a)\operatorname{sgn}(b) + 1,$$

for a and b not both 0, the identity follows.

11. Appendix: Some proofs and details

11.1. **Proof of part (iii) of Lemma 3.3.** Suppose that $\mathcal{C}$ is in good position and that $x \in V$ with $\Phi_q(x;\mathcal{C}) \neq 0$. Let $s_0 = s(x)$ be the unique point of $[0,1]^q$ such that $\rho_{\mathcal{C}}(s_0) = D_x \cap S(\mathcal{C})$. Note that the map $\rho_{\mathcal{C}}$ extends to an open neighborhood of $[0,1]^q$ so that, even if s_0 lies on the boundary, we can define $\rho_{\mathcal{C}}$ on an open set $\mathcal{U}$ around s_0. We lift $\rho_{\mathcal{C}}$ to a map $\tilde\rho_{\mathcal{C}} : \mathcal{U} \to \mathrm{OFD}$, defined by

$$\tilde\rho_{\mathcal{C}} : s \mapsto \zeta(s) = B(s)P^{-1}, \qquad P \in \mathrm{Sym}_q(\mathbb{R})_{>0}, \quad P^2 = -(B(s), B(s)).$$

For convenience, we write $B = [B_1, \dots, B_q] = B(s)$. Then

$$(\tilde\rho_{\mathcal{C}})_*(\frac{\partial}{\partial s_j}) = \dot B_j P^{-1} - \zeta \dot P_j P^{-1}, \qquad \dot B_j := \frac{\partial}{\partial s_j}B = [0,\dots,-C_j + C_{j'},\dots,0], \quad \dot P_j := \frac{\partial}{\partial s_j}P.$$

The components in the connection subspace $U(z)^q$ of $T_\zeta(\mathrm{OFD})$ are then

$$(\rho_{\mathcal{C}})_*(\frac{\partial}{\partial s_j}) = \tau_j\, P^{-1}, \qquad \tau_j = [0,\dots,\mathrm{pr}_{U(z)}(-C_j + C_{j'}),\dots,0]$$

and these are linearly independent provided $\mathrm{pr}_{U(z)}(-C_j + C_{j'}) \neq 0$ for all j. But at the point $z_0 = \rho_{\mathcal{C}}(s_0)$, we have $x \in U(z_0)$, and the q vectors

$$\eta(x,j) = [0,\dots,0,,x,0,\dots,0],$$

with x in the jth component, span the normal to $T_{z_0}(D_x)$. Note that the metric g on $T_z(D) \simeq U(z)^q$ is given by

$$g(\eta,\eta') = \mathrm{tr}(\,(\eta_i,\eta'_j)\,).$$

Then we have

$$g\big(\eta(x,i),\tau_j\big) = [\ (x,C_{j'}) - (x,C_j)\ ]\,\delta_{ij}.$$

This shows that $\tau_j \neq 0$ for all j and hence $\rho_{\mathcal{C}}$ is immersive at $s(x)$. We can choose the open neighborhood $\mathcal{U}$ of $s(x)$ in $\mathbb{R}^q$ so that the restriction of $\rho_{\mathcal{C}}$ to $\mathcal{U}$ is an embedding. The orientation of the codimension q cycle D_x is defined by an element of $\nu_{z,x} \in \bigwedge^{(p-1)q} T_z(D_x)$ such that

$$\nu_x \wedge \nu_{z,x} \in \textstyle\bigwedge^{pq}(T_z(D))$$

is properly oriented, where

$$\nu_x = \eta(x,1) \wedge \cdots \wedge \eta(x,q).$$

Here we have fixed an orientation of D. Thus the intersection number at z_0 of D_x with $\rho_{\mathcal{C}}(\mathcal{U})$ is

$$I(D_x, \rho_{\mathcal{C}}(\mathcal{U})) = \operatorname{sgn}\det(g(\eta(x,i),\tau_j)) = \prod_j \operatorname{sgn}\big(\ (x, C_{j'}) - (x, C_j)\ \big).$$

If x is regular with respect to $\mathcal{C}$, then this quantity is

$$2^{-q} \prod_{j=1}^{q} \big(\ \operatorname{sgn}(x, C_{j'}) - \operatorname{sgn}(x, C_j)\ \big) = (-1)^q \Phi_q(x; \mathcal{C}).$$

In general, we have

$$(-1)^q \Phi_q(x; \mathcal{C}) = 2^{-r}\, I(D_x, \rho_{\mathcal{C}}(\mathcal{U})), \tag{11.1}$$

where r, $0 \le r \le q$, is the number of walls passing through $s(x)$. Thus, $\Phi_q(x;\mathcal{C})$ is a 'weighted' intersection number.

11.2. **Proof of (7.5).** For y, $y' \in Z = \operatorname{span}\{C\}$, we write $(\!(y, y')\!) = -(y, y')$, and we assume that $x \in Z$. We let

$$C^\vee = [C_1^\vee, \ldots, C_q^\vee] = C(\!(C, C)\!)^{-1}$$

be the dual basis. We write

$$x = \sum_i x_i\, C_i^\vee, \qquad x_i = (\!(x, C_i)\!).$$

For a fixed index j, we write

$$x = x_{\perp j} + x'\, C_j, \qquad x_{\perp j} = \sum_{i \neq j} x_i C_i^\vee, \qquad x_j = (\!(x, C_j)\!) = x'(\!(C_j, C_j)\!),$$

and similarly for our variable of integration $y \in Z$. Note that, in particular,

$$\operatorname{sgn}(\!(y, C_j)\!) = \operatorname{sgn}(y').$$

We can write

$$dy = dy_{\perp j}\, dy'$$

where

$$1 = \int_Z e^{-\pi(\!(y,y)\!)}\, dy = \int_{Z_{\perp j}} \int_{\mathbb{R}} e^{-\pi(\!(y_{\perp j}, y_{\perp j})\!)}\, e^{-\pi(y')^2(\!(C_j, C_j)\!)}\, dy_{\perp j}\, dy',$$

where dy' is $(\!(C_j, C_j)\!)^{\frac{1}{2}}$ times Lebesque measure, so that

$$\int_{\mathbb{R}} e^{-\pi(y')^2(\!(C_j, C_j)\!)}\, dy' = 1.$$

We write[3]

$$\begin{aligned}(-1)^q E_q(C;x) &= \int_Z e^{-\pi(\!(y-x,y-x)\!)} \prod_i \operatorname{sgn}(\!(y,C_i)\!)\, dy \\ &= \int_{Z_{\perp j}} \int_{\mathbb{R}} e^{-\pi(\!(y_{\perp j}-x_{\perp j},y_{\perp j}-x_{\perp j})\!)}\, e^{-\pi(y'-x')^2(\!(C_j,C_j)\!)} \prod_{i\neq j} \operatorname{sgn}(\!(y,C_i)\!) \operatorname{sgn}(y')\, dy_{\perp j}\, dy' \\ &= (-1)^{q-1} E_{q-1}(C[j];x_{\perp j}) \int_{\mathbb{R}} e^{-\pi(y'-x')^2(\!(C_j,C_j)\!)} \operatorname{sgn}(y')\, dy' \\ &= (-1)^{q-1} E_{q-1}(C[j];x_{\perp j}) \int_{\mathbb{R}} e^{-\pi(y')^2(\!(C_j,C_j)\!)} \operatorname{sgn}(y_j+x_j)\, dy'.\end{aligned}$$

But then, taking into account that $dy' = (\!(C_j,C_j)\!)^{-\frac{1}{2}}\, d_{\mathrm{Leb}} y_j$, we have

$$\begin{aligned}x_j \frac{\partial}{\partial x_j}\big\{\, (-1)^q E_q(C;x) \,\big\} &= (\!(x,C_j)\!)\, (-1)^{q-1} E_{q-1}(C[j];x_{\perp j}) \\ &\qquad \times \int_{\mathbb{R}} e^{-\pi y_j^2 (\!(C_j,C_j)\!)^{-1}}\, 2\delta(y_j+x_j)\, (\!(C_j,C_j)\!)^{-\frac{1}{2}}\, d_{\mathrm{Leb}} y_j \\ &= 2(\!(x,\underline{C}_j)\!)(-1)^{q-1}\, E_{q-1}(C[j];x_{\perp j})\, e^{-\pi(\!(x,\underline{C}_j)\!)^2}.\end{aligned}$$

Here recall that $\underline{C}_j = C_j(\!(C_j,C_j)\!)^{-\frac{1}{2}}$. Summing on j, we obtain (7.5).

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[3]Note that the extra factor of $(-1)^q$ etc. is due to our temporary change in the sign of the inner product on Z, so that our E_q differs from that in [18] by this sign.

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