THE GEOMETRIC THETA CORRESPONDENCE FOR HILBERT MODULAR SURFACES

JENS FUNKE* AND JOHN MILLSON**

Abstract. We give a new proof and an extension of the celebrated theorem of Hirzebruch and Zagier [17] that the generating function for the intersection numbers of the Hirzebruch-Zagier cycles in (certain) Hilbert modular surfaces is a classical modular form of weight 2. In our approach we replace Hirzebuch's smooth complex analytic compactification of the Hilbert modular surface with the (real) Borel-Serre compactification. The various algebro-geometric quantities that occur in [17] are replaced by topological quantities associated to 4-manifolds with boundary. In particular, the "boundary contribution" in [17] is replaced by sums of linking numbers of circles (the boundaries of the cycles) in 3-manifolds of type *Sol* (torus bundle over a circle) which comprise the Borel-Serre boundary.

1. Introduction

In a series of papers [12, 13, 14, 15] we have been studying the geometric theta correspondence (see below) for non-compact arithmetic quotients of symmetric spaces associated to orthogonal groups. It is our overall goal to develop a general theory of geometric theta liftings in the context of the real differential geometry/topology of non-compact locally symmetric spaces of orthogonal and unitary groups which generalizes the theory of Kudla-Millson in the compact case, see [25].

In this paper, we study in detail the geometric theta correspondence for Hilbert modular surfaces.

The geometric theta correspondence. We first explain the term "geometric theta correspondence". The key point is that the Weil (or oscillator) representation gives a method to construct *closed* differential forms on locally symmetric spaces associated to groups which belong to dual pairs. Let V be (for simplicity) an even-dimensional rational quadratic space of signature (p,q). Then the Weil representation induces an action of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{O}(V_\mathbb{R})$ on $\mathcal{S}(V_\mathbb{R})$, the Schwartz functions on $V_\mathbb{R}$. Let $G = \mathrm{SO}_0(V_\mathbb{R})$ with maximal compact subgroup K. We let $\mathfrak{g}$ and $\mathfrak{k}$ be their respective Lie algebras and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the associated Cartan decomposition. Suppose

$$\varphi \in [\mathcal{S}(V_\mathbb{R}) \otimes \bigwedge^r \mathfrak{p}^*]^K$$

is a cocycle in the relative Lie algebra complex for G with values in $\mathcal{S}(V_\mathbb{R})$. Then φ corresponds to a closed differential r-form $\tilde{\varphi}$ on the symmetric space $D = G/K$ of

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dimension pq with values in $\mathcal{S}(V_{\mathbb{R}})$. For $\mathcal{L}$ a coset of a lattice in V, we define the theta distribution $\Theta = \Theta_{\mathcal{L}}$ by $\Theta = \sum_{\ell \in \mathcal{L}} \delta_\ell$, where δ_ℓ is the delta measure concentrated at ℓ. Then Θ is invariant under $\Gamma = \mathrm{Stab}(\mathcal{L}) \subset G$, and there exists a congruence subgroup Γ' of $\mathrm{SL}_2(\mathbb{Z})$ such that Θ is also invariant under Γ'. Hence we can apply the theta distribution to $\tilde{\varphi}$ to obtain a closed r-form θ_φ on $X = \Gamma \backslash D$ given by

$$\theta_\varphi(\mathcal{L}) = \langle \Theta_{\mathcal{L}}, \tilde{\varphi} \rangle.$$

Assume now in addition that φ has weight k under the Weil representation action of the maximal compact subgroup $\mathrm{SO}(2)$ in $\mathrm{SL}_2(\mathbb{R})$. Then θ_φ also gives rise to a (in general) non-holomorphic function on the upper half plane $\mathbb{H}$ which is modular of weight k for Γ'. We may then use θ_φ as the kernel of a pairing of modular forms f with (closed) differential $(pq-r)$-forms η or r-chains (cycles) C in X. We call the resulting pairing in f, η (or C), and φ the **geometric theta correspondence**.

The cocycles of Kudla-Millson. The point of the work of Kudla and Millson [22, 23] is that they found (in greater generality) a family of cocycles φ_q^V in $[\mathcal{S}(V_{\mathbb{R}}) \otimes \bigwedge^q \mathfrak{p}^*]^K$ with weight $\frac{p+q}{2}$ for SL_2. Moreover, these cocycles give rise to Poincaré dual forms of certain totally geodesic, "special" cycles in X. Recently it has been shown, first [18] for $\mathrm{SO}(3,2)$ in the rationally split case, and then [1] for all $\mathrm{SO}(p,q)$ in "low degree" in the cocompact case that the geometric theta correspondence specialized to φ_q^V induces on the adelic level an *isomorphism* from the appropriate space of classical modular forms to the summand of the space of automorphic forms with infinite component a certain fixed unitary representation with cohomology. For example, for $q=1$, when X is a hyperbolic p-manifold, $H^1(X)$ for $p>3$ is spanned by the Poincaré duals of immersed totally geodesic hypersurfaces, while for $q=2$, the Hermitian case, $H^{1,1}(X)$ for $p>2$ is spanned by the Poincaré duals of complex algebraic hypersurfaces corresponding to embedded rational $\mathrm{SO}(p-1,2)$'s. These results give further justification to the term "geometric theta correspondence" and highlight the significance of the above cocycles. In [13] we generalized φ_q^V to allow suitable non-trivial coefficient systems (and one has analogous results for cycles with coefficients in [1]).

The main results. In the present paper, we consider the case when V has signature $(2,2)$ with $\mathbb{Q}$-rank 1. Then $D \simeq \mathbb{H} \times \mathbb{H}$, and X is a Hilbert modular surface. Here by "Hilbert modular surface" we mean *any congruence cover* of $\mathrm{SL}_2(\mathcal{O}) \backslash D$ where $\mathcal{O}$ is the ring of integers in a real quadratic field.

We let $\overline{X}$ be the Borel-Serre compactification of X which is obtained by replacing each isolated cusp associated to a rational parabolic P by a boundary face $e'(P)$ which in this case is a torus bundle over a circle, a 3-manifold of type *Sol*, one of the eight 3-dimensional homogeneous Riemannian geometries, see [29], Section 3.8. This makes $\overline{X}$ a 4-manifold with boundary, and we write $\iota : \partial\overline{X} \hookrightarrow \overline{X}$ for the inclusion. We assume for the introduction that X has only one cusp. The special cycles C_n[1] are parameterized by non-negative integers and for $n>0$ are given by embedded modular and Shimura curves. They define relative homology classes in $H_2(X, \partial X, \mathbb{Q})$.

[1]We distinguish the relative cycles C_n in X from the Hirzebruch-Zagier cycles T_n in $\tilde{X}$, see below.

The geometric theta correspondence of Kudla-Millson [25] for the cocycle φ_2^V in this situation takes the following shape. For a *compact* cycle C in X, we have that

$$\langle \theta_{\varphi_2^V}, C\rangle = \int_C \theta_{\varphi_2^V} = \sum_{n\geq 0} (C_n \cdot C) q^n \tag{1.1}$$

is a holomorphic modular form of weight 2 and is equal to the generating series of the intersection numbers with C_n. Here $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$. Our first result is

Theorem 1.1. *(Theorem 7.3) The differential form $\theta_{\varphi_2^V}$ on X extends to a form on $\overline{X}$, and the restriction ι^* of $\theta_{\varphi_2^V}$ to $\partial\overline{X}$ gives an exact differential form on $\partial\overline{X}$. Moreover, there exists a theta series $\theta_{\phi_1^W}$ of weight 2 for a space W of signature $(1,1)$ with values in the 1-forms on $\partial\overline{X}$ such that $\theta_{\phi_1^W}$ is a primitive for $\iota^*\theta_{\varphi_2^V}$:*

$$d(\theta_{\phi_1^W}) = \iota^*\theta_{\varphi_2^V}.$$

Considering the mapping cone for the inclusion $\iota : \partial\overline{X} \hookrightarrow \overline{X}$ (see Section 3.3) we then view the pair $[\theta_{\varphi_2^V}, \theta_{\phi_1^W}]$ as an element of the compactly supported cohomology $H_c^2(X)$. Explicitly, let C be a *relative* cycle in $\overline{X}$ representing a class in $H_2(X, \partial X, \mathbb{Q})$. Then the Kronecker pairing between $[\theta_{\varphi_2^V}, \theta_{\phi_1^W}]$ and C is given by

$$\langle [\theta_{\varphi_2^V}, \theta_{\phi_1^W}], C\rangle = \int_C \theta_{\varphi_2^V} - \int_{\partial C} \theta_{\phi_1^W}. \tag{1.2}$$

In this way, we obtain an extension of the geometric theta lift (see (1.1)) which captures the non-compact situation.

To describe the geometric interpretation of this extension, we study the cycle C_n at the boundary $\partial\overline{X}$ (Section 4). The intersection of C_n with $\partial\overline{X}$ is a union of circles contained in the torus fibers of *Sol*. But rationally such cycles are homologically trivial. Hence we can find a (suitably normalized) rational 2-chain A_n in $\partial\overline{X}$ whose boundary is the boundary of C_n in $\partial\overline{X}$. "Capping" off C_n by A_n, we obtain a *closed* cycle C_n^c in $\overline{X}$ defining a class in $H_2(X, \mathbb{Q})$. Our main result is the extension of (1.1):

Theorem 1.2. *(Theorem 7.7) Let C be a relative cycle in $\overline{X}$. Then*

$$\langle [\theta_{\varphi_2^V}, \theta_{\phi_1^W}], C\rangle = \sum_{n\geq 0} (C_n^c \cdot C) q^n$$

is a holomorphic modular form of weight 2 and is equal to the generating series of the intersection numbers with the capped cycles C_n^c. (Similarly for the pairing with an arbitrary closed 2-form on $\overline{X}$ representing a class in $H^2(X)$).

Note that in view of (1.2) the lift of classes of $H_2(X, \partial X)$ (resp. $H^2(X)$) is the sum of two (non-holomorphic) modular forms of weight 2.

In [14] we systematically study for $\mathrm{O}(p,q)$ the restriction of the classes $\theta_{\varphi_q^V}$ (also with non-trivial coefficients) to the Borel-Serre boundary. Whenever the restriction vanishes cohomologically, we can expect that a similar analysis to the one given in this paper will give analogous extensions of the geometric theta correspondence. In fact, aside from this paper we have at present managed to do this for several other cases, namely for modular curves with non-trivial coefficients [15] generalizing work of Shintani [28] and for Picard modular surfaces [16] generalizing work of Cogdell [7].

Linking numbers in 3*-manifolds of type Sol.* The theta series $\theta_{\phi_1^W}$ at the boundary is of independent interest and has geometric meaning in its own right. Recall that for two disjoint (rationally) homologically trivial 1-cycles a and b in a 3-manifold M we can define the *linking number* of a and b as the intersection number

$$\mathrm{Lk}(a,b) = A \cdot b$$

of (rational) chains in M. Here A is a rational 2-chain in M with boundary a. In Section 4.4, we extend the notion of linking number to two closed (not necessarily disjoint) geodesics lying in torus fibers of the same boundary component *Sol* of the Borel-Serre boundary. By a slight abuse of notation we use the same symbol $\mathrm{Lk}(\cdot,\cdot)$ for this extension. In particular, we obtain a 'self-linking number' for such geodesics and also a linking number $\mathrm{Lk}(\partial C_n, \partial C_m)$ for the boundary of two special cycles C_n and C_m.

In Theorem 4.12, we give a simple formula for the (extended) linking number of two such 'vertical' circles in terms of the glueing homeomorphism for the torus bundle.

For the theta series $\theta_{\phi_1^W}$ we show

Theorem 1.3. *(Theorem 6.3) Let c be a* 1*-cycle in the Borel-Serre boundary contained in a torus fiber of $\partial\overline{X}$. Then the holomorphic part of the weight* 2 *non-holomorphic modular form $\int_c \theta_{\phi_1^W}$ is given by the generating series $\sum_{n>0} \mathrm{Lk}(\partial C_n, c) q^n$ of the (extended) linking numbers.*

Theorem 1.3 (and its analogues for the Borel-Serre boundary of modular curves with non-trivial coefficients and Picard modular surfaces) suggest that there is a more general connection between modular forms and linking numbers of nilmanifold subbundles over special cycles in nilmanifold bundles over locally symmetric spaces.

Relation to the work of Hirzebruch and Zagier. In their seminal paper [17], Hirzebruch-Zagier provided a map from the second homology of the smooth compactification of certain Hilbert modular surfaces $j : X \hookrightarrow \tilde{X}$ to modular forms. They introduced the Hirzebruch-Zagier curves T_n in $\tilde{X}$, which are given by the closure of the cycles C_n in $\tilde{X}$. They then defined "truncated" cycles T_n^c as the projections of T_n orthogonal to the subspace of $H_2(\tilde{X}, \mathbb{Q})$ spanned by the compactifying divisors of $\tilde{X}$. The principal result of [17] was that $\sum_{n\geq 0}[T_n^c]q^n$ defines a holomorphic modular form of weight 2 with values in $H_2(\tilde{X}, \mathbb{Q})$. We show $j_* C_n^c = T_n^c$ (Proposition 4.9), and hence the Hirzebruch-Zagier theorem follows easily from Theorem 1.2 above, see Theorem 7.9.

The main work in [17] was to show that the generating function

$$F(\tau) = \sum_{n=0}^{\infty} (T_n^c \cdot T_m) q^n$$

for the intersection numbers in $\tilde{X}$ of T_n^c with a fixed T_m is a modular form of weight 2. The Hirzebruch-Zagier proof of the modularity of F was a remarkable synthesis of algebraic geometry, combinatorics, and modular forms. They explicitly computed the intersection numbers $T_n^c \cdot T_m$ as the sum of two terms, $T_n^c \cdot T_m = (T_n \cdot T_m)_X + (T_n \cdot T_m)_\infty$, where $(T_n \cdot T_m)_X$ is the geometric intersection number of T_n and T_m in the interior of X and $(T_n \cdot T_m)_\infty$, which they called the "contribution from infinity". They then proved that both generating functions $\sum_{n=0}^{\infty}(T_n \cdot T_m)_X q^n$ and $\sum_{n=0}^{\infty}(T_n \cdot T_m)_\infty q^n$ are the holomorphic parts of two non-holomorphic forms F_X and F_∞ with the *same* non-holomorphic part (with opposite signs). Hence combining these two forms gives $F(\tau)$.

We recover this feature of the original Hirzebruch-Zagier proof via (1.2) with $C = C_m$. The first term on the right hand side of (1.2) was studied in the thesis of the first author of this paper [10] and gives the interior intersections $(T_n \cdot T_m)_X$ encoded in F_X. So via Theorem 1.2 the second term on the right hand side of (1.2) must match the boundary contribution F_∞ in [17], that is, we obtain

Theorem 1.4.

$$(T_n \cdot T_m)_\infty = \mathrm{Lk}(\partial C_n, \partial C_m).$$

Hence we give an interpretation for the boundary contribution in [17] in terms of (extended) linking numbers in $\partial\overline{X}$. In fact, the construction of $\theta_{\phi_1^W}$ owes a great deal to Section 2.3 in [17], where a scalar-valued version of $\theta_{\phi_1^W}$ is introduced, see also Example 6.4. Using Theorem 4.13 one can also make the connection between our linking numbers and the formulas of the boundary contribution in [17] explicit.

To summarize, we start with the difference of theta integrals (1.2) (which we know a priori is a holomorphic modular form, see Theorem 7.6), then by functorial differential topological computations we relate its Fourier coefficients to intersection/linking numbers, and by direct computation of the integrals involved we obtain the explicit intersection-number formulas of Hirzebruch-Zagier and a "closed form" for their generating function.

Note that Bruinier [5] and Oda [27] use related theta series to consider [17], but their overall approach is different.

Currents. One of the key properties of the cocycle φ_2^V is that the n-th Fourier coefficients of $\theta_{\varphi_2^V}$ represents the Poincaré dual class for the cycle C_n. Kudla-Millson establish this by showing that φ_2^V gives rise to a Thom form for the normal bundle of each of the components of C_n. To prove our main result, Theorem 1.2, we follow a different approach using currents which is implicit in [6] and is closely related to the Green's function $\Xi(n)$ for the divisors C_n constructed by Kudla [19, 20]. This function plays an important role in the Kudla program (see eg [21]) which considers the analogous generating series for the special cycles in arithmetic geometry. In the non-compact situation however, one needs to modify $\Xi(n)$ to obtain a Green's function for the cycle T_n^c in $\tilde{X}$. Discussions with U. Kühn suggest that the constructions in this paper indeed give rise to such a modification of $\Xi(n)$, see Remark 8.5.

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We dedicate this paper to the memory of Gretchen Taylor Millson, beloved wife of the second author.

2. The Hilbert modular surface and its Borel-Serre compactification

2.1. **The symmetric space and its arithmetic quotient.**

2.1.1. *The orthogonal group and its symmetric space.* Let V be a rational vector space of dimension 4 with a non-degenerate symmetric bilinear form $(\ ,\ )$ of signature $(2,2)$. We will use q to denote the quadratic form associated to $(\ ,\ )$ whence $q(x) = \frac{(x,x)}{2}$.

We let $\underline{G} = \mathrm{SO}(V)$, viewed as an algebraic group over $\mathbb{Q}$. We let $G = \underline{G}_0(\mathbb{R}) \simeq \mathrm{SO}_0(2,2)$ be the connected component of the identity of the real points of $\underline{G}$. It is most convenient to identify the associated symmetric space $D = D_V$ with the Grassmannian of negative 2-planes in $V(\mathbb{R})$ on which the bilinear form $(\ ,\ )$ is negative definite:

$$D = \{z \subset V_{\mathbb{R}};\ \dim z = 2 \text{ and } (\ ,\ )|_z < 0\}.$$

We pick an orthogonal basis $\{e_1, e_2, e_3, e_4\}$ of $V_{\mathbb{R}}$ with $(e_1,e_1) = (e_2,e_2) = 1$ and $(e_3,e_3) = (e_4,e_4) = -1$. We denote the coordinates of a vector x with respect to this basis by x_i. We pick the plane $z_0 = [e_3, e_4]$ spanned by e_3 and e_4 as base point of D, and we let $K \simeq \mathrm{SO}(2) \times \mathrm{SO}(2)$ be the maximal compact subgroup of G stabilizing z_0. Thus $D \simeq G/K$. It is well known that $D \simeq \mathbb{H} \times \mathbb{H}$.

We let $\underline{P}$ be a rational parabolic subgroup stabilizing a rational isotropic line ℓ and define $P = \underline{P}_0(\mathbb{R})$ as before. We let $\underline{N}$ be its unipotent subgroup and $N = \underline{N}(\mathbb{R})$. We pick another rational isotropic line ℓ' not in $\ell^\perp$ and obtain a rational Witt decomposition

$$V = \ell \oplus W \oplus \ell'$$

with $W = \ell^\perp \cap \ell'^\perp$. The choice of ℓ' gives a Levi splitting of $\underline{P}$, and we write

$$P = NAM$$

for the Langlands decomposition of P. Note that M is the subgroup of P that acts trivially on ℓ and ℓ'. Conversely, the choice of a Levi splitting of $\underline{P}$ gives a reductive subgroup M in P whose fixed set in V is a hyperbolic plane spanned by ℓ and another rational isotropic line ℓ'. Explicitly, we assume that $u = (e_1 + e_4)/\sqrt{2}$ and $u' = (e_1 - e_4)/\sqrt{2}$ are defined over $\mathbb{Q}$ and set $\ell = \mathbb{Q}u$ and $\ell' = \mathbb{Q}u'$. Hence $W_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(e_2, e_3)$. Then with respect to the basis u, e_2, e_3, u', we have

$$\underline{N}(\mathbb{Q}) = \left\{ n(w) = \begin{pmatrix} 1 & (\cdot,w) & -(w,w)/2 \\ & 1_W & -w \\ & & 1 \end{pmatrix};\ w \in W \right\},$$

$$A = \left\{ a(t) = \begin{pmatrix} t & & \\ & 1_W & \\ & & t^{-1} \end{pmatrix};\ t \in \mathbb{R}_+ \right\},$$

$$M = \left\{ m(s) = \begin{pmatrix} 1 & & & \\ & \cosh(s) & \sinh(s) & \\ & \sinh(s) & \cosh(s) & \\ & & & 1 \end{pmatrix};\ s \in \mathbb{R} \right\}.$$

We obtain coordinates $z = z(t,s,w)$ for D where z is the negative two-plane in $V_{\mathbb{R}}$ given by $z = n(w)a(t)m(s)[e_3, e_4]$.

2.1.2. *Arithmetic Quotient.* We let L be an even lattice in V of level N, that is $L \subseteq L^\#$, the dual lattice, $(x,x) \in 2\mathbb{Z}$ for $x \in L$, and $q(L^\#)\mathbb{Z} = \frac{1}{N}\mathbb{Z}$. Here $q(L^\#) = \{q(x), x \in L^\#\}$ whence N is the smallest positive integer such that $Nq(x) \in \mathbb{Z}$, for all $x \in L^\#$. We fix $h \in L^\#$ and let $\Gamma \subseteq \mathrm{Stab}\, L$ be a subgroup of finite index of the stabilizer of $\mathcal{L} := L + h$ in G. For each isotropic line ℓ in V, we write $\ell = \mathbb{Q}u_\ell$ where $u = u_\ell$ is a primitive vector in L. We will throughout assume that the $\mathbb{Q}$-rank of $\underline{G}$ is 1, that is, V splits exactly one hyperbolic plane over $\mathbb{Q}$. Then we define the Hilbert modular surface

$$X = \Gamma \backslash D.$$

Example 2.1. An important example is the following. Let $d > 0$ be the discriminant of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ over $\mathbb{Q}$, $\mathcal{O}_K$ its ring of integers. We denote the Galois involution on K by $x \mapsto x'$. We let $V = \{x \in M_2(K);\, {}^t x' = -x\}$ be the space of skew-hermitian matrices in $M_2(K)$. Then the determinant on $M_2(K)$ gives V the structure of a non-degenerate rational quadratic space of signature $(2,2)$ with $\mathbb{Q}$-rank 1, and $\mathrm{SL}_2(\mathcal{O}_K)$ acts on V by $\gamma.x = \gamma x {}^t\gamma'$ as isometries. We define the integral skew-hermitian matrices by

$$L = \left\{ \left(\begin{smallmatrix} a\sqrt{d} & \lambda \\ -\lambda' & b\sqrt{d} \end{smallmatrix}\right);\ a, b \in \mathbb{Z},\ \lambda \in \mathcal{O}_K \right\}.$$

Then L is a lattice of level d stabilized by $\mathrm{SL}_2(\mathcal{O}_K)$.

Hirzebruch and Zagier actually considered this case for $d \equiv 1 \pmod 4$ a prime.

The quotient space X is in general an oriented uniformizable orbifold with isolated singularities. We will treat X as a manifold - we will use Stokes' Theorem and Poincaré duality over $\mathbb{Q}$ on X. This is justified because in each instance we can pass to a finite normal cover Y of X with Y a manifold. Hence, the formulas we want hold on Y. We then then go back to the quotient by taking invariants or summing over the group Φ of covering transformations. The point is that the de Rham complex of X is the algebra of Φ-invariants in the one of Y and the *rational* homology (cohomology) groups of X are the groups of Φ-coinvariants (invariants) of those of Y.

2.2. **Compactifications.**

2.2.1. *Admissible Levi decompositions of P.* Let ℓ be a rational isotropic line in V and P be its associated parabolic as in Subsection 2.1.1. We let $\Gamma_P = \Gamma \cap P$ and $\Gamma_N = \Gamma_P \cap N$. We will say a Levi decomposition $P = NAM$ (or equivalently, the corresponding Witt splitting $V = \ell \oplus W \oplus \ell'$) is admissible if

$$\Gamma_P = (M \cap \Gamma_P) \ltimes \Gamma_N.$$

We now construct an admissible Levi decomposition.The quotient Γ_P/Γ_N is a non-trivial arithmetic subgroup of $\underline{P}/\underline{N}$ and lies inside the connected component of the identity of the real points of $\underline{P}/\underline{N}$. Furthermore, Γ_P/Γ_N acts as isometries of spinor norm 1 on the anisotropic quadratic space $\ell^\perp/\ell$ of signature $(1,1)$. Hence $\Gamma_P/\Gamma_N \simeq \mathbb{Z}$ is infinite cyclic. Therefore the exact sequence

$$1 \to \Gamma_N \to \Gamma_P \to \Gamma_P/\Gamma_N \to 1$$

splits. We fix $f \in \Gamma_P$ such that its image $\bar{f}$ generates Γ_P/Γ_N. Then f defines a Levi subgroup M and hence as explained in Subsection 2.1.1 an isotropic line ℓ' dually paired with ℓ. Moreover, the element f generates $\Gamma_M := \Gamma_P \cap M$ and

$$\Gamma_P = \Gamma_M \ltimes \Gamma_N.$$

In the following we assume that we have picked an admissible Levi decomposition for each rational parabolic.

2.2.2. *Borel-Serre compactification.* We let $\overline{D}$ be the (rational) Borel-Serre enlargement of D, see [3] or [2], III.9. For any parabolic $\underline{P}$ with admissible Levi decomposition $P = NAM$, we define the boundary component

$$e(P) = MN \simeq D_W \times W_{\mathbb{R}}.$$

Here $D_W \simeq M \simeq \mathbb{R}$ is the symmetric space associated to the orthogonal group of W. Then $\overline{D}$ is given by

$$\overline{D} = D \cup \coprod_{\underline{P}} e(P),$$

where $\underline{P}$ varies over all rational parabolics. The action of Γ on D extends to $\overline{D}$ in a natural way, and we let

$$\overline{X} := \Gamma \backslash \overline{D}$$

be the Borel-Serre compactification of $X = \Gamma \backslash D$. This makes $\overline{X}$ a manifold with boundary such that

$$\partial \overline{X} = \coprod_{[\underline{P}]} e'(P),$$

where for each cusp, the corresponding boundary component is given by

$$e'(P) = \Gamma_P \backslash e(P).$$

Here $[\underline{P}]$ runs over all Γ-conjugacy classes. The space $X_W := \Gamma_M \backslash D_W$ is a circle. Hence $e'(P)$ is a torus bundle over the circle, where the torus T^2 is given by $\Gamma_N \backslash N$. That is, $e'(P) = X_W \times T^2$, and we have the natural map $\kappa : e'(P) \to X_W$. For T sufficiently large, the set $\{z(t,s,w); t > T\}$ defines a natural product neighborhood of $e(P)$ in $\overline{D}$ and hence a product neighborhood for $e'(P)$ in $\overline{X}$ given by $[(T, \infty] \times e'(P)]$. We let $i : X \hookrightarrow \overline{X}$ and $i_P : e'(P) \hookrightarrow \overline{X}$ be the natural inclusions.

It is one of the fundamental properties of the Borel-Serre compactification $\overline{X}$ that it is homotopically equivalent to X itself. Hence their (co)homology groups coincide.

2.2.3. *Hirzebruch's smooth compactification.* We let X' be the Baily-Borel compactficiation of X, which is obtained by collapsing each boundary component $e'(P)$ of $\overline{X}$ to a single point. It is well known that X' is a projective algebraic variety. We let $\tilde{X}$ be Hirzebruch's smooth resolution of the cusp singularities and $\pi : \tilde{X} \to X'$ be the natural map collapsing the compactifying divisors for each cusp. We let $j : X \hookrightarrow \tilde{X}$ be the natural embedding. Note that the Borel-Serre boundary separates $\tilde{X}$ into two pieces, the (connected) inside X^{in}, which is isomorphic to X and the (possibly disconnected) outside X^{out}, which for each cusp is a neighborhood of the compactifying divisors. Note that we can view $e'(P)$ as lying in both X^{in} and X^{out} since the intersection $X^{in} \cap X^{out}$ is equal to $\coprod_{\underline{P}} e'(P)$.

3. (Co)homology

In this section we describe the relationship between the (co)homology of the various compactifications.

3.1. **The homology of the boundary components.** Every element of $\Gamma_N = \pi_1(T^2)$ is a rational multiple of a commutator in Γ_P and accordingly the image of $H_1(T^2,\mathbb{Q})$ in $H_1(e'(P),\mathbb{Q})$ is trivial. Let $a_P \in H_1(e'(P),\mathbb{Z})$ be the class of the identity section of $\kappa : e'(P) \to X_W$ and $b_P \in H_2(e'(P),\mathbb{Z})$ be the class of the torus fiber of κ. It is clear that the intersection number of a_P and b_P is 1 (up to sign) whence a_P and b_P are nontrivial primitive classes. Furthermore, a_P generates $H_1(e'(P),\mathbb{Q})$ and $H_2(e'(P),\mathbb{Z}) \cong \mathbb{Z}$, generated by b_P. So

Lemma 3.1. (i) *The first rational homology group of* $e'(P)$ *is generated by* a_P.
(ii) *The second homology group of* $e'(P)$ *is generated by* b_P.

Remark 3.2. To compute the homology over $\mathbb{Z}$ one has only to use the Wang sequence for a fiber bundle over a circle, see [26], page 67.

Let Ω_P be the unique P-invariant 2-form on $e'(P)$ such that

$$\int_{b_P} \Omega_P = 1. \tag{3.1}$$

Since b_P is the image of the fundamental class of T^2 inside $H_2(e'(P),\mathbb{Z})$, we see that the restriction of Ω_P to T^2 lifts to the area form on $W_{\mathbb{R}} \simeq N$ normalized such that $T^2 = \Gamma_N\backslash N$ has area 1.

3.2. **Homology and cohomology of $\overline{X}$ and $\tilde{X}$.** In what follows we will identify the groups $H_2(X)$ and $H_2(\overline{X})$ using the isomorphism induced by the inclusion $i : X \to \overline{X}$. We then have an induced map $\oplus_{[\underline{P}]} H_2(e'(P)) = H_2(\partial\overline{X}) \to H_2(X)$. The discussion in Section 2.2.3 then gives rise to the Mayer-Vietoris sequence

$$0 \to \oplus_{[\underline{P}]} H_2(e'(P)) \to H_2(X) \oplus (\oplus_{[\underline{P}]} S_P) \to H_2(\tilde{X}) \to 0.$$

Here S_P denotes the span of the classes defined by compactifying divisors at the cusp associated to P. The zero on the left comes from $H_3(\tilde{X}) = 0$ and the zero on the right comes from the fact that for each P the class a_P injects into $H_1(X^{out})$, see [30], II.3. Since the generator b_P has trivial intersection with each of the compactifying divisors, b_P bounds on the outside (by the non-degeneracy of the intersection pairing on $H_2(X^{out})$). Hence b_P a fortiori bounds in $\tilde{X}$. Thus the above short exact sequence is the sum of the two short exact sequences $\oplus_{[\underline{P}]} H_2(e'(P)) \to H_2(X) \to j_* H_2(X)$ and $0 \to \oplus_{[\underline{P}]} S_P \to \oplus_{[\underline{P}]} S_P$. By adding the third terms of the two sequences and equating them to $H_2(\tilde{X})$ we obtain the orthogonal splittings (for the intersection pairing) - see also [30], p.123,

$$H_2(\tilde{X}) = j_* H_2(X) \oplus \left(\oplus_{[\underline{P}]} S_P\right) \qquad \text{and} \qquad H^2(\tilde{X}) = j_\# H^2_c(X) \oplus \left(\oplus_{[\underline{P}]} S_P^\vee\right).$$

Here $j_\#$ is the push-forward map. Furthermore, the pairings on each summand are non-degenerate. Considering $\oplus_{[\underline{P}]} H_2(e'(P)) \to H_2(X) \to j_* H_2(X)$ we also obtain

Proposition 3.3. $H_2(\partial\overline{X})$ *is the kernel of* j_* *so that*

$$j_* H_2(X) \simeq H_2(X)/H_2(\partial\overline{X}).$$

3.3. **Compactly supported cohomology and the cohomology of the mapping cone.** We briefly review the mapping-cone-complex realization of the cohomology of compact supports of X. For a more detailed discussion, see the appendix of [15]. In what follows we will let $k : \partial\overline{X} \to \overline{X}$ be the inclusion. We remind the reader that $i^* : A^\bullet(\overline{X}) \to A^\bullet(X)$ is a quasi-isomorphism where $i : X \to \overline{X}$ is the inclusion. We will use the induced isomorphism on cohomology to identify the cohomology groups of the de Rham algebras.

We let $A_c^\bullet(X)$ be the complex of compactly supported differential forms on X which gives rise to $H_c^\bullet(X)$, the cohomology of compact supports. We now represent the compactly-supported cohomology of X by the cohomology of the mapping cone $C^\bullet$ of k^*, see [31], p.19, where as before $k^* : A^\bullet(\overline{X}) \to A^\bullet(\partial\overline{X})$. However, we will change the sign of the differential on $C^\bullet$ and shift the grading down by one. Thus

$$C^i = \{(a,b), a \in A^i(\overline{X}), b \in A^{i-1}(\partial\overline{X})\}$$

with $d(a,b) = (da, i^*a - db)$. If (a,b) is a cocycle in $C^\bullet$ we will use $[[a,b]]$ to denote its cohomology class. We have

Proposition 3.4. *The cochain map $A_c^\bullet(\overline{X}) \to C^\bullet$ given by $c \mapsto (c,0)$ is a quasi-isomorphism.*

We now give a cochain map from $C^\bullet$ to $A_c^\bullet(X)$ which induces the inverse to the above isomorphism. For simplicity assume that X has only one cusp. We let V be a product neighborhood of $\partial\overline{X}$ as in Section 2.2.2, and we let $\pi : V \to \partial\overline{X}$ be the projection. If b is a form on $\partial\overline{X}$ we obtain a form π^*b on V. Let f be a smooth function of the geodesic flow coordinate t which is 1 near $t = \infty$ and zero for $t \leq T$ for some sufficiently large T. We may regard f as a function on V by making it constant on the $\partial\overline{X}$ factor. We extend f to all of $\overline{X}$ by making it zero off of V. Let (a,b) be a cocycle in C^i. Then there exist a compactly supported closed form α and a form μ which vanishes on $\partial\overline{X}$ such that

$$a - d(f\pi^*b) = \alpha + d\mu.$$

We define the cohomology class $[a,b]$ in the compactly supported cohomology $H_c^i(X)$ to be the class of α, and the assignment $[[a,b]] \mapsto [a,b]$ gives the desired inverse. From this we obtain the following integral formulas for the Kronecker pairings with $[a,b]$.

Lemma 3.5. *Let η be a closed form on $\overline{X}$ and C a relative cycle in $\overline{X}$ of appropriate degree. Then*

$$\langle [a,b], [\eta] \rangle = \int_{\overline{X}} a \wedge \eta - \int_{\partial\overline{X}} b \wedge i^*\eta \qquad \text{and} \qquad \langle [a,b], C \rangle = \int_C a - \int_{\partial C} b.$$

4. Capped special cycles and linking numbers in three manifolds of type *Sol*

For $x \in V$ such that $(x,x) > 0$, we define

$$D_x = \{z \in D;\ z \perp x\}.$$

Then D_x is an embedded upper half plane in D. We let $\Gamma_x \subset \Gamma$ be the stabilizer of x. We define the special (or Hirzebruch-Zagier) cycle by

$$C_x = \Gamma_x \backslash D_x,$$

and by slight abuse we identify C_x with its image in X. These cycles are modular or Shimura curves. For positive $n \in \mathbb{Q}$, we write $\mathcal{L}_n = \{x \in \mathcal{L};\ \frac{1}{2}(x,x) = n\}$. Then the composite cycles C_n are given by

$$C_n = \sum_{x \in \Gamma \backslash \mathcal{L}_n} C_x.$$

Since the divisors define in general relative cycles, we take the sum in $H_2(X, \partial X, \mathbb{Q})$.

4.1. **The closure of special cycles in the Borel-Serre boundary and the capped cycle C_x^c.** We now study the closure of C_x in $\partial \overline{X}$, which is the same as the intersection of $\overline{C}_x$ or ∂C_x with the union of the hypersurfaces $e'(P)$. Given a rational null vector u, primitive in L as before, we let P be the associated parabolic subgroup. A straightforward calculation gives

Proposition 4.1. *If $(x,u) \neq 0$ then there exists a neighborhood U_∞ of $e(P)$ such that*

$$D_x \cap U_\infty = \emptyset.$$

If $(x,u) = 0$, then $\overline{D}_x \cap e(P)$ is contained in the fiber of p over $s(x)$, where $s(x)$ is the unique element of $\mathbb{R}$ satifying

$$(x, m(s(x))e_3) = 0.$$

At $s(x)$ the intersection $\overline{D}_x \cap e(P)$ is the affine line in $W_{\mathbb{R}}$ given by

$$\{w \in W_{\mathbb{R}} : (x,w) = (u',x)\}.$$

We define $c_x \subset \partial C_x$ to be the closed geodesic in the fiber over $s(x)$ which is the image of $\overline{D}_x \cap e(P)$ under the covering $e(P) \to e'(P)$. We easily see

Proposition 4.2. (i) *The 1-cycle ∂C_x is a finite union of circles.*
(ii) *At a cusp associated to P, each circle is contained in a fiber of the map $\kappa : e'(P) \to X_W$ and hence is a rational boundary (by Lemma 3.1).*
(iii) *Two boundary circles c_x and c_y are parallel if they are contained in the same fiber. In particular, $c_x \cap c_y \neq \emptyset \iff c_x = c_y$.*

We now describe the intersection of $\overline{C}_n$ or ∂C_n with $e'(P)$. For $\mathcal{L}_V = \mathcal{L}$, we set

$$\mathcal{L}_W = \mathcal{L}_{W_P} = (\mathcal{L}_V \cap \ell^\perp)/(\mathcal{L}_V \cap \ell),$$

which using the Levi splitting we can view as a finite union of cosets of lattices in the subspace W of V. Via the isomorphism $W \simeq N$, we can identify $\Gamma_N = N \cap \Gamma$ with a lattice Λ_W in W. Since u is primitive in L and $n(w)x = x + (w,x)u$ for a vector $x \in \ell^\perp$ we see that $\mathcal{L}_W$ is contained in the dual lattice of Λ_W. We set

$$\mathcal{L}_{n,u} = \{x \in \mathcal{L} \cap u^\perp;\ (x,x) = 2n\}.$$

We thank the referee for the simple proof of the following lemma.

Lemma 4.3. *The intersection* $(\partial C_n)_P := \partial C_n \cap e'(P)$ *is given by*

$$(\partial C_n)_P = \coprod_{y \in \Gamma_P \backslash \mathcal{L}_{n,u}} c_y.$$

Proof. We define the infinite, but locally finite cycle $D_n = \sum_{x \in \mathcal{L}_n} D_x$ in $\overline{D}$. Hence D_n is the preimage of C_n under $\overline{D} \to \overline{X}$. If we define d_x as the lift of c_x to $e(P)$, we have $\partial D_n \cap e(P) = \coprod_{x \in \mathcal{L}_{n,u}} d_x$. Taking the quotient by Γ_P proves the lemma. □

Remark 4.4. The intersection $(\partial C_x)_P := \partial C_x \cap e'(P)$ of a single component in C_n with $e'(P)$ consists of those c_y where y runs through a set of representatives of Γ_P-equivalence classes in $\mathcal{L}_{n,u}$ which are Γ-equivalent to x. We denote the set of those y by $[x]_P$. Hence

$$(\partial C_x)_P = \coprod_{y \in [x]_P} c_y.$$

A simple calculation yields

Lemma 4.5. *A complete set of representatives of* Γ_P*-equivalence classes in* $\mathcal{L}_{n,u}$ *is given by*

$$\coprod_{\substack{x \in \Gamma_M \backslash \mathcal{L}_W \\ (x,x)=2n}} \coprod_{0 \leq k < \min'_{\lambda \in \Lambda_W} |(\lambda,x)|} \{x + ku\}.$$

Here min′ *denotes the minimum over the nonzero elements. Hence*

$$(\partial C_n)_P = \coprod_{\substack{x \in \Gamma_M \backslash \mathcal{L}_W \\ (x,x)=2n}} \coprod_{0 \leq k < \min'_{\lambda \in \Lambda_W} |(\lambda,x)|} c_{x+ku}.$$

Proposition 4.6. *Let* $x \in \mathcal{L}_{n,u}$ *with* $n > 0$. *Then there exists a rational* 2*-chain* a_x *in* $e'(P)$ *such that*

1. $\partial a_x = c_x$;
2. $\int_{a_x} \Omega_P = 0$, *here* Ω_P *is the area form for the fibers (see* (3.1)*).*

Proof. The existence of a rational cap for c_x was established in Proposition 4.2. The issue is to find a cap a'_x such that $\int_{a'_x} \Omega_P \in \mathbb{Q}$, which we then can normalize by adding a rational multiple of b_P. We will prove this in Section 4.3 below. □

We define the rational 2-chain $(A_x)_P$ in $e'(P)$ by $(A_x)_P = \sum_{y \in [x]_P} a_y$ and sum over the boundary components $e'(P)$ to obtain the 2-chain A_x in ∂X with

$$\partial A_x = \partial C_x.$$

Summing over $x \in \Gamma \backslash \mathcal{L}_n$ we obtain the 2-chain A_n with $\partial A_n = \partial C_n$ and also the 2-chain $(A_n)_P = A_n \cap e'(P)$ so that $(\partial C_n)_P = (A_n)_P$.

Definition 4.7. We define the rational absolute 2-cycle in $\overline{X}$ by

$$C_x^c = C_x \cup (-A_x).$$

In particular, C_x^c defines a class in $H_2(\overline{X}) = H_2(X)$. In the same way we obtain $C_n^c = C_n \cup (-A_n)$.

4.2. **The Hirzebruch-Zagier cycles T_n and T_n^c.** Following Hirzebruch-Zagier we let T_n be the cycle in $\tilde{X}$ given by the closure of the cycle C_n in $\tilde{X}$. Hence T_n defines a class in $H_2(\tilde{X})$.

Definition 4.8. Consider the decomposition $H_2(\tilde{X}) = j_* H_2(X) \oplus \left(\oplus_{[P]} S_p\right)$, which is orthogonal with respect to the intersection pairing on $\tilde{X}$. We let T_n^c be the image of T_n under orthogonal projection onto the summand $j_* H_2(X)$.

Proposition 4.9. *We have*

$$j_* C_n^c = T_n^c.$$

Proof. For simplicity, we assume that X has only one cusp. The 3-manifold $e'(P)$ separates T_n, and we can write $T_n = T_n \cap X^{in} + T_n \cap X^{out}$ as (appropriately oriented) 2-chains in $\tilde{X}$. It is obvious that we have $j_* \overline{C}_n = T_n \cap X^{in}$ as 2-chains. We write $B_n = T_n \cap X^{out}$. We have $\partial C_n = -\partial B_n$. Hence we can write $T_n = j_* C_n^c + B_n^c$, the sum of two 2-cycles in $\tilde{X}$. Here B_n^c is obtained by 'capping' B_n in $e'(P)$ with the 2-chain A_n. Since $j_* C_n^c$ is clearly orthogonal to S_P (since it lies in X^{in}) and $B_n^c \in S_P$ (since it lies in X^{out}), the decomposition $T_n = j_* C_n^c + B_n^c$ is just the decomposition of T_n relative to the splitting $H_2(\tilde{X}) = j_* H_2(X) \oplus S_P$. Hence $T_n^c = j_* C_n^c$, as claimed. □

4.3. **Rationality of the cap and the monodromy two-chain $\mathcal{M}(c)$.** We will now prove Proposition 4.6. In fact, we will show that it holds for any circle α contained in a torus fiber of $e(P)$ and passing through a rational point. For convenience we will assume α is in the fiber torus over the base-point $s = 0$. We would like to thank Misha Kapovich for simplifying our original argument. The idea is to construct a 2-chain A with α as boundary so that A is a sum $P + T + \mathcal{M}(\gamma_0)$ of three simplicial 2-chains such that "parallelogram" P and the "triangle" T have rational area and the period of Ω over the "monodromy chain" $\mathcal{M}(\gamma_0)$ is zero.

In what follows we will pass from pictures in the plane involving directed line segments, triangles and parallelograms to identities in the space of simplicial 1-chains $C_1(T^2)$ on T^2. The principle behind this is that any k-dimensional subcomplex S of a simplicial complex Y which is the fundamental cycle of an oriented k-submanifold $|S|$ (possibly with boundary) of Y corresponds in a *unique* way to a sum of oriented k-simplices in $C_k(Y)$.

In this subsection we will work with a general 3-manifold S with *Sol* geometry. Of course this includes all the manifolds $e'(P)$ that occur in this paper. Let $f \in \mathrm{SL}(2, \mathbb{Z})$ be a hyperbolic element. We will then consider the 3-manifold S obtained from $\mathbb{R} \times T^2$ (with the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$) given by the relation

$$(s, w) \sim (s + 1, f(w)). \tag{4.1}$$

We let $\pi : \mathbb{R} \times T^2 \to S$ be the resulting infinite cyclic covering. (For us, $\mathbb{R} \simeq M$, the Levi subgroup of P, $T^2 = \Gamma_N/N$, and the gluing map f is the element $f \in \Gamma_P$ which generates $\Gamma_P/\Gamma_N \simeq \mathbb{Z}$, see Subsection 2.2.1.) We will abuse notation and use $\mathbb{R}^2$ to denote the universal cover of T^2.

We now define notation we will use below. We will use Greek letters to denote closed geodesics on T^2 and lower case Roman letters to denote points on T^2. A subscript 0 will indicate that the geodesic starts at the point (identity) 0 on T^2.

We will use the analogous notation for geodesic arcs on $\mathbb{R}^2$. We will often identify closed geodesics with the 1-cycles (e.g. simplicial cycles for a suitable triangulation) they represent.We will use $[\alpha]$ to denote the corresponding homology class of a closed geodesic α on T^2. If x and y are points on $\mathbb{R}^2$ we will use $\overline{xy}$ to denote the oriented line segment joining x to y and $\overrightarrow{xy}$ to denote the corresponding (free) vector i.e. the equivalence class of $\overline{xy}$ modulo parallel translation.

We first take care of the fact that α does not necessarily pass through the origin. Let α_0 be the parallel translate of α to the origin. Then we find a cylinder P, which is the image of an oriented parallelogram $\widetilde{P}$ with rational vertices under the universal cover $\mathbb{R}^2 \to T^2$, such that in $Z_1(T^2, \mathbb{Q})$, the group of rational 1-cycles, we have

$$\partial P = \alpha - \alpha_0. \tag{4.2}$$

Since $\widetilde{P}$ has rational vertices we find $\int_P \Omega = \int_{\widetilde{P}} \Omega \in \mathbb{Q}$.

The key to find A is the construction of "monodromy 2-chains". For any closed geodesic $\gamma_0 \subset T^2$ starting at 0 we define the monodromy 2-chain $\mathcal{M}(\gamma_0)$ to be the image of the cylinder $\gamma_0 \times [0,1] \subset T^2 \times \mathbb{R}$ in S. The reader will verify using (4.1) that in $Z_1(T^2, \mathbb{Q})$ we have

$$\partial \mathcal{M}(\gamma_0) = f^{-1}(\gamma_0) - \gamma_0. \tag{4.3}$$

Here we have identified the geodesic γ_0 (and will continue to do so) with the simplicial one-cycle it corresponds to (the sum of the one-simplices it contains) in a suitable triangulation of T^2. Since f preserves the origin, the geodesic $f^{-1}(\gamma_0)$ is also a closed geodesic starting at the origin. However, we now pass from cycles to their homology classes so more care must be taken.

Since f^{-1} is hyperbolic we have $|\operatorname{tr}(f^{-1})| > 2$ and hence $\det(f^{-1} - I) = \det(I - f) = \operatorname{tr}(f) - 2 \neq 0$. Put $N = \det(f^{-1} - I)$ and define $[\gamma_0] \in H_1(T^2, \mathbb{Z})$ by

$$f^{-1}([\gamma_0]) - [\gamma_0] = N[\alpha_0]. \tag{4.4}$$

with α_0 as above. Note that $[\gamma_0] = N\{(f^{-1} - I)^{-1}([\alpha_0])\}$ is necessarily an integer homology class. Also note that is an equation in the first *homology*, it is not an equation in the group of 1-cycles $Z_1(T^2, \mathbb{Q})$. Since any integral homology class contains a unique closed geodesic starting at the origin we obtain a closed geodesic $\gamma_0 \in [\gamma_0]$ and a corresponding monodromy 2-chain $\mathcal{M}(\gamma_0)$ so that (4.3) holds in $Z_1(T^2, \mathbb{Q})$. We now solve

Problem 4.10. Find a simplicial 2-chain that realizes the homology between the geodesic 1-cycles $f^{-1}(\gamma_0) - \gamma_0$, resp. $N\alpha_0$, giving rise to the left, resp. right-hand, sides of Equation (4.4).

Let h_1 (resp. h_2) denote the covering transformation of π corresponding to the element α_0 (resp. γ_0) of the fundamental group of T^2. Define c_1 and c_2 in $\mathbb{R}^2$ by $c_1 = Nh_1(0)$ and $c_2 = h_2(0)$. Define $d \in \mathbb{R}^2$ by $d = f^{-1}(c_2)$. Let $\widetilde{T}$ be the oriented triangle with vertices $0, c_2, d$. Then

$$(i) \quad \pi(\overline{0c_1}) = N\alpha_0 \qquad (ii) \quad \pi(\overline{0c_2}) = \gamma_0 \qquad (iii) \quad \pi(\overline{0d}) = \pi(f^{-1}(\overline{0c_2})) = f^{-1}(\gamma_0).$$

We now leave it to the reader to combine the homology equation (4.4) and the three equations to show the equality of directed line segments

$$h_2(\overline{0c_1}) = \overline{c_2 d}. \tag{4.5}$$

With this we can solve the problem. We see that if we consider $\widetilde{T}$ as an oriented 2-simplex we have the following equality of one chains

$$\partial\widetilde{T} = \overline{0c_2} + \overline{c_2 d} - \overline{0d}.$$

Let T be the image of $\widetilde{T}$ under π. Take the direct image of the previous equation under π and use equation (4.5) which implies that the second edge $\overline{c_2 d}$ is equivalent under h_2 in the covering group to the directed line segment $\overline{0c_1}$ which maps to $N\alpha_0$. Hence $\overline{c_2 d}$ also maps to $N\alpha_0$. We obtain the following equation in $Z_1(T^2, \mathbb{Z})$

$$\partial T = \gamma_0 + N\alpha_0 - f^{-1}(\gamma_0), \tag{4.6}$$

and we have solved the above problem. Combining (4.3) and (4.6) we have

$$\partial(\mathcal{M}(\gamma_0) + T) = f^{-1}(\gamma_0) - \gamma_0 + \gamma_0 + N\alpha_0 - f^{-1}(\gamma_0) = N\alpha_0.$$

Combining this with (4.2) and setting $A_0 = \mathcal{M}(\gamma_0) + T$ we obtain

$$\partial(NP + A_0) = N\alpha, \tag{4.7}$$

in $Z_1(S, \mathbb{Z})$. Hence if we define A to be the *rational* chain $A = \frac{1}{N}(NP + A_0) = P + \frac{1}{N}T + \frac{1}{N}\mathcal{M}(\gamma_0)$ in S we have the following equation in $Z_1(S, \mathbb{Q})$:

$$\partial A = \alpha.$$

Finally, the integral of Ω over A is rational. Indeed, the integral over P is rational. Since all vertices of $\widetilde{T}$ are integral the area of $\widetilde{T}$ is integral and the integral of Ω over T is integral. Thus it suffices to observe that the restriction of Ω to $\mathcal{M}(c)$ is zero. With this we have completed the proof of Proposition 4.6.

4.4. **Extended Linking Numbers in manifolds of type** *Sol***.** The linking number of two disjoint homologically trivial 1-cycles a and b in a closed 3-manifold S is given by $\mathrm{Lk}(a, b) = A \cdot b$, where A is any rational 2-chain in S with boundary a. Since b defines a trivial homology class in S, the link is well-defined, ie, does not depend on the choice of A.

We extend the notion of linking number to two closed geodesics lying in torus fibers of the same boundary component S of the Borel-Serre boundary. It is important to recall the elementary result that all such geodesics (closed geodesics in a flat torus) are multiples of embedded geodesics; that is, as mappings from the circle to the torus the image of the mapping is an embedded geodesic perhaps traversed several times. Now, suppose that a and b are two closed geodesics in a torus fiber T^2 of *Sol*. We may assume that T^2 is the fiber over the identity element of the circle. For $\epsilon > 0$ sufficiently small, we let $b(\epsilon)$ be the parallel translate of b (obtained by choosing a trivialization of the torus bundle) into the fiber over ϵ. Since a and $b(\epsilon)$ lie in different fibers and are rational boundaries we may then define our extension $\mathrm{Lk}(\cdot, \cdot)$ of the linking pairing by

$$\mathrm{Lk}(a, b) = \mathrm{Lk}(a, b(\epsilon)).$$

Here the Lk on the right is the usual linking pairing. Note that this is well-defined: $\mathrm{Lk}(a, b(\epsilon))$ is independent of (small) ϵ and depends only on the homology classes of a and b. Also note that with this definition, we find that the extended self-linking number of a circle c in a fiber torus of *Sol* is defined by $\mathrm{Lk}(c,c) = \mathrm{Lk}(c, c(\epsilon))$. However, note that the extended linking pairing is no longer symmetric.

We let S be the manifold of type *Sol* realized as in Section 4.3 via (4.1) and consider as above the case when a and b are contained in two torus fibers. We take $a, b \in H_1(T^2, \mathbb{Z})$, and in the following we are allowed to confuse a and b with their representatives in the lattice $\mathbb{Z}^2$ and the unique closed geodesic in T^2 passing through the origin that represents them. We will denote the images of a and b in $\mathbb{R} \times T^2$ and S by $a = a(0) = 0 \times a$ and $b = b(\varepsilon) = \varepsilon \times b$. Our goal is to compute the linking number $\mathrm{Lk}(a, b(\epsilon))$. By the explicit construction of the cap A in Section 4.3 we obtain

Lemma 4.11. *Let S be realized as in* (4.1)*. Then for a and b in $H_1(T^2, \mathbb{Z})$, we have*

$$\mathrm{Lk}(a, b(\epsilon)) = \mathcal{M}(c) \cdot b(\epsilon) = c(\epsilon) \cdot b(\epsilon) = c \cdot b.$$

Here c is the rational 1*-cycle obtained by solving $(f^{-1} - I)(c) = a$ and $\mathcal{M}(c)$ is the (rational) monodromy* 2*-chain associated to c (see above) with boundary $\partial \mathcal{M}(c) = (f^{-1} - I)(c) = a$. Here the first $\cdot$ is the intersection of chains in S, the next $\cdot$ is the intersection number of* 1*-cycles in the fiber $\epsilon \times T^2$ and the last $\cdot$ is the intersection number of* 1*-cycles in $0 \times T^2$.*

Noting that this last intersection number coincides with the intersection number of the underlying homology classes which in turn coincides with the symplectic (intersection) form $\langle \cdot, \cdot \rangle$ on $H_1(T^2, \mathbb{Q})$ we have found our desired formula for the linking number.

Theorem 4.12. $\mathrm{Lk}(a, b(\epsilon)) = \langle (f^{-1} - I)^{-1}(a), b \rangle$.

It is a remarkable fact that there is a simple formula involving only the action of the glueing homeomorphism $f \in \mathrm{SL}(2, \mathbb{Z})$ on $H_1(T^2, \mathbb{Z})$ for linking numbers for 1-cycles contained in fiber tori T^2 of *Sol* (unlike the case of linking numbers in $\mathbb{R}^3$).

This immediately leads to an explicit formula for the numbers $\mathrm{Lk}(\partial C_n, \partial C_m)$. Using Lemma 4.5 we obtain

Theorem 4.13. *Let $g = (f^{-1} - I)^{-1}$. Then*

$$\mathrm{Lk}((\partial C_n)_P, (\partial C_m)_P) = \sum_{\substack{x \in \Gamma_M \backslash \mathcal{L}_W \\ (x,x) = 2n}} \sum_{\substack{x' \in \Gamma_M \backslash \mathcal{L}_W \\ (x,x) = 2m}} (\min_{\lambda \in \Lambda_W}{}' |(\lambda, x)|)(\min_{\mu \in \Lambda_W}{}' |(\mu, x')|) \langle g(Jx), Jx' \rangle.$$

Here Jx is the properly oriented primitive vector in Λ_W such that $(Jx, x) = 0$.

Example 4.14. We consider the integral skew Hermitian matrices in Example 2.1 with $d \equiv 1 \pmod 4$. Pick $u = \left(\begin{smallmatrix} \sqrt{d} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $u' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sqrt{d} \end{smallmatrix}\right)$ so that $W = \{\left(\begin{smallmatrix} 0 & \lambda \\ -\lambda' & 0 \end{smallmatrix}\right); \lambda \in K\} \simeq K$ and $\mathcal{L}_W = \mathcal{O}_K$. Then for $\lambda, \mu \in W$, we have $(\lambda, \mu) = \lambda \mu' + \mu \lambda'$. Furthermore, we have $n(\mu)\lambda = \lambda + (\lambda, \mu)u$ for $n(\mu) := \left(\begin{smallmatrix} 1 & -\sqrt{d}\mu \\ 0 & 1 \end{smallmatrix}\right) \in N$. Hence with the conventions above we have $\Lambda_W = \frac{1}{\sqrt{d}} \mathcal{O}_K$, the inverse different (which is the dual lattice for $\mathcal{O}_K$ wrt the trace form). Then one easily sees that for $x \in \mathcal{O}_K$, we have

$Jx = \frac{1}{(\min'_{\lambda\in\Lambda_W}|(\lambda,x)|)\sqrt{d}}x$. The symplectic form on $\mathbb{R}^2$ which gives rise to the intersection form for $T^2 = W_{\mathbb{R}}/\Lambda_W = K_{\mathbb{R}}/(\frac{1}{\sqrt{d}}\mathcal{O}_K)$ is given by $\langle\lambda,\mu\rangle = \sqrt{d}(\lambda'\mu - \lambda\mu')$. We can assume that the glueing map f for the bundle is realized by multiplication with ε', where ε is a generator of the totally positive units in $\mathcal{O}_K$, so that $g = (f^{-1} - I)^{-1} = 1/(\varepsilon - 1)$.

For $d = p$ a prime and $m = 1$, the cycle C_1 has only component arising from $x = 1 \in K$ and $C_1 \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then Theorem 4.13 becomes

$$\mathrm{Lk}((\partial C_n)_P, (\partial C_1)_P) = 2 \sum_{\substack{\mu\in U_+\backslash\mathcal{O}_K \\ \mu\mu'=n,\mu\gg 0}} \left\langle \tfrac{\mu}{\sqrt{p}(\varepsilon-1)}, \tfrac{1}{\sqrt{p}} \right\rangle = \frac{2}{\sqrt{p}} \sum_{\substack{\mu\in U_+\backslash\mathcal{O}_K \\ \mu\mu'=n,\mu\gg 0}} \frac{\mu+\mu'\varepsilon}{\varepsilon-1}.$$

Here U_+ is the subgroup of totally positive units in $\mathcal{O}_K$ whence, for this example, $\Gamma_M \cong U_+$. The formula on the right-hand side of the equation immediately above is (twice) the "boundary contribution" in [17], Section 1.4, see also Section 7.5.

5. Schwartz function and distribution valued relative Lie-algebra cochains and Schwartz function and distribution valued forms

In this section, V and its subspace W obtained by a Witt splitting will denote real *quadratic* spaces.

Let U be a non-degenerate real quadratic space of signature (p,q) and even dimension m. We will later apply the following to $U = V$ and $U = W$. Changing notation, we write $G = \mathrm{SO}_0(U)$ with maximal compact subgroup K and $D = G/K$ for the associated symmetric space. We let $\mathcal{S}(U)$ be the space of Schwartz functions on U on which $\mathrm{SL}_2(\mathbb{R})$ acts via the Weil representation ω, see eg. [28]. We identify the maximal compact subgroup $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$ with $U(1)$ by mapping $k'(\theta) := \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ to $\exp \imath\theta$. Then an eigenfunction $\varphi \in \mathcal{S}(U_{\mathbb{R}})$ of weight r under the action of $\mathrm{SO}(2)$ satisfies

$$\omega(k')\varphi = \chi^r(k')\varphi,$$

where χ is the standard (identity) character of $\mathrm{SO}(2) \simeq U(1)$.

5.1. Extending Schwartz function-valued relative cochains to $\tau \in \mathbb{H}$ and $z \in D$.

Let $\varphi \in \mathcal{S}(U)$ be any Schwartz function. In what follows we will often write

$$\varphi^0(x) = \varphi(x)e^{\pi(x,x)}.$$

For $\tau = u + iv \in \mathbb{H}$, we let $g'_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then $(\omega(g'_\tau)\varphi)(x) = v^{m/4}\varphi(\sqrt{v}x)e^{\pi i(x,x)u}$, and for a fixed r, we define

$$\varphi(x,\tau) := v^{-r/2}\omega(g'_\tau)\varphi(x) = v^{-r/2+m/4}\varphi^0(\sqrt{v}x)e^{\pi i(x,x)\tau}. \tag{5.1}$$

Note that typically (5.1) is applied only to eigenfunctions of weight r, but we will also consider (5.1) for other functions as well.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ associated to K. Let $\pi : G \to D$ be the quotient by K. Note that we can identify $\mathfrak{p}$ with the tangent space of D at the base point $z_0 \in D$. Let $\rho_E : G \to \mathrm{Aut}(E)$ be a finite dimensional representation and let

$$\varphi \in [\mathcal{S}(U) \otimes \bigwedge^{\ell} \mathfrak{p}^* \otimes E]^K.$$

Evaluation at the basepoint $z_0 = \mathrm{span}(e_3, e_4)$ of D gives an $\mathrm{SL}_2(\mathbb{R})$-invariant isomorphism (here $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{S}(U)$ on both sides by the oscillator representation)

$$[\mathcal{S}(U) \otimes \mathcal{A}^{\ell}(D) \otimes E]^G \to [\mathcal{S}(U) \otimes \bigwedge^{\ell} \mathfrak{p}^* \otimes E]^K,$$

where G, resp. K, acts diagonally on all factors on the left, resp. right. Here $\mathcal{A}^{\ell}(D) \otimes E$ denotes the differential ℓ-forms on D with values in the vector space E. In this isomorphism we have used the identification (induced by π^*) of forms on D with horizontal, right K-invariant forms on G. By a slight abuse of notation, we again denote the inverse image of φ under this isomorphism by φ. Let $x \in U$ and $z \in D$. Then $\varphi(x, z) \in \bigwedge^{\ell} T_z^*(D) \otimes E$. Precisely, if $g_z \in G$ is any element carrying z_0 to z and L_g denotes left-translation by g, then we have

$$\pi^*\big(\varphi\big)(x, z) = \Big(\omega(g_z) \otimes dL^*_{g_z^{-1}} \otimes \rho_E(g_z)\Big)\,(\varphi)\,(x) = dL^*_{g_z^{-1}}\big(\rho_E(g_z)(\varphi(g_z^{-1}x))\big). \tag{5.2}$$

Here the operator $\omega(g_z) \otimes dL^*_{g_z^{-1}} \otimes \rho_E(g_z)$ acts on $\mathcal{S}(U) \otimes \bigwedge^{\ell} T^*(G) \otimes E$.

Applying (5.1) and (5.2) to φ gives

$$\pi^*\big(\varphi\big)(x, \tau, z) = dL^*_{g_z^{-1}}\big(\rho_E(g_z)\big(\varphi(g_z^{-1}x, \tau)\big)\big). \tag{5.3}$$

We often view $x \mapsto \varphi(x, \bullet)$ as a map from U to the E-valued differential ℓ-forms on D.

We can extend the formulas (5.1), (5.2) and (5.3) to singular functions on U, that is, to functions which are not everywhere defined on U. More precisely, we consider functions which are locally integrable on U and which give rise to cochains with values in the tempered distributions on U, that is, to elements in $[\mathcal{S}'(U) \otimes \bigwedge^{\ell} \mathfrak{p}^* \otimes E]^K$ where $\mathcal{S}'(U)$ denotes the dual space of $\mathcal{S}(U)$. Note that the second condition on our singular functions is global. For such φ, the actions of $SL_2(\mathbb{R})$ and the orthogonal group *considered as acting on distribution-valued relative Lie algebra cochains* are again given by the formulas above. This justifies the application of (5.1), (5.2) and (5.3) to such singular functions. In the following we usually take the singular functions/forms point of view rather the one of distributions. We will call the resulting relative Lie algebra cochains "cochains with singularities" and the forms (currents) resulting by applying (5.2) "singular forms".

5.2. **Schwartz forms and singular forms for V.** Let V be the real quadratic space of signature $(2, 2)$ introduced in Section 2.1.1. As above, we let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ associated to K. We identify $\mathfrak{g} \simeq \bigwedge^2 V_{\mathbb{R}}$ via $(v_1 \wedge v_2)(v) = (v_1, v)v_2 - (v_2, v)v_1$. We write $X_{ij} = e_i \wedge e_j \in \mathfrak{g}$ and note that $\mathfrak{p}$ is spanned by X_{ij} with $1 \leq i \leq 2$ and $3 \leq j \leq 4$. We write ω_{ij} for the dual basis elements in $\mathfrak{p}^*$. We orient D such that $\omega_{13} \wedge \omega_{14} \wedge \omega_{23} \wedge \omega_{24}$ gives rise to the G-invariant volume element on D.

5.2.1. *Special forms for V.* The 2-form φ_2 of Kudla-Millson is an element in

$$[\mathcal{S}(V) \otimes \mathcal{A}^2(D)]^G \simeq [\mathcal{S}(V) \otimes \bigwedge^2 \mathfrak{p}^*]^K,$$

which at the base point is given by

$$\begin{aligned}\varphi_2 &= \frac{1}{2}\prod_{\mu=3}^{4}\sum_{\alpha=1}^{2}\left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right)\varphi_0 \otimes \omega_{\alpha\mu} \\ &= (2x_1x_2)\ \varphi_0 \otimes (\omega_{13}\wedge\omega_{34} + \omega_{23}\wedge\omega_{14}) \\ &\quad + (2x_1^2 - \tfrac{1}{2\pi})\ \varphi_0 \otimes \omega_{13}\wedge\omega_{14} + (2x_2^2 - \tfrac{1}{2\pi})\varphi_0 \otimes \omega_{13}\wedge\omega_{14}.\end{aligned}$$

Here $\varphi_0(x) := e^{-\pi(x,x)_0}$, where $(x,x)_0 = \sum_{i=1}^4 x_i^2$ is the minimal majorant associated to $z_0 \in D$. Note that φ_2 has weight 2, see [22]. There is another Schwartz form

$$\psi_1 \in [\mathcal{S}(V) \otimes \mathcal{A}^1(D)]^G \simeq [\mathcal{S}(V) \otimes \mathfrak{p}^*]^K$$

of weight 0 given by

$$\psi_1 = -x_1x_3\varphi_0(x)\otimes\omega_{14} + x_1x_4\varphi_0(x)\otimes\omega_{13} - x_2x_3\varphi_0(x)\otimes\omega_{24} + x_2x_4\varphi_0(x)\otimes\omega_{23}. \tag{5.4}$$

The key relationship between φ_2 and ψ_1 is (see [25], §8)

Theorem 5.1.

$$\omega(L)\varphi_2 = d\psi_1.$$

Here $\omega(L)$ is the Weil representation action of the SL_2-lowering operator $L = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1\end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{R})$ on $\mathcal{S}(V)$, while d denotes the exterior differentiation on D.

On the upper half plane $\mathbb{H}$, the action of L corresponds to the action of the classical Maass lowering operator which we also denote by L. For a function f on $\mathbb{H}$, we have

$$Lf = -2iv^2\frac{\partial}{\partial\bar\tau}f.$$

When made explicit using (5.1), Theorem 5.1 translates to

$$v\frac{\partial}{\partial v}\varphi_2^0(\sqrt{v}x) = d\left(\psi_1^0(\sqrt{v}x)\right). \tag{5.5}$$

5.2.2. *The singular 1-form $\tilde\psi_1(x,z)$ for V.* We now define our first example of a 1-cochain with singularities and the associated singular 1-form, starting from the Schwartz function valued 1-cochain $\psi_1(x)$ above. Assume that the coordinates x_3, x_4 are not both zero, that is, $x \notin z_0^\perp$, in particular, $x \neq 0$. For such an x, we define a singular 1-cochain $\tilde\psi_1 \in [\mathcal{S}'(V)\otimes\mathfrak{p}^*]^K$ by

$$\tilde\psi_1(x) = -\left(\int_1^\infty \psi_1^0(\sqrt{r}x)\frac{dr}{r}\right)e^{-\pi(x,x)} = -\frac{1}{2\pi(x_3^2+x_4^2)}\psi_1(x). \tag{5.6}$$

As before we set $\tilde\psi_1^0(x) = \tilde\psi_1(x)e^{\pi(x,x)}$. We leave it to the reader to check that the cochain $\tilde\psi_1(x)$ takes values in the space of locally integrable tempered functions (hence in the tempered distributions). We define $\tilde\psi_1(x,z)$ by (5.2) and see that $\tilde\psi_1(x,z)$ is smooth for the pairs x,z such that $x \notin z^\perp$. Hence, for fixed x, the form $\tilde\psi_1(x,z)$ is smooth for $z \notin D_x$. Furthermore, as in (5.1) we may use the action of $\mathrm{SL}_2(\mathbb{R})$ to pull-back $\tilde{\psi}_1(x,z)$ to a function $\tilde{\psi}_1(x,\tau,z)$ on $\mathbb{H}$ with values in the singular 1-forms on D and we obtain the formula

$$\tilde{\psi}_1(x,\tau,z) = \tilde{\psi}_1^0(\sqrt{v}x,z)e^{\pi i(x,x)\tau} = -\left(\int_v^\infty \psi_1^0(\sqrt{r}x,z)\frac{dr}{r}\right)e^{\pi i(x,x)\tau}. \tag{5.7}$$

We emphasize that $\tilde{\psi}_1(x,\tau,z)$ is not defined for $x=0$.

Remark 5.2. In the above formula we used (5.1) (with $r=2$ and $m=4$) even though $\tilde{\psi}_1(x)$ is not a weight-2 vector of $\mathrm{SO}(2)$. The idea (behind $r=2$) is that $\tilde{\psi}_1(x,\tau)$ is obtained by applying to 1-cochain $\psi_1(x,\tau)$ of weight 0 an 'inverse' of the lowering operator L and this inverse operator should raise the weight by 2 (it doesn't because it is not $\mathrm{SL}_2(\mathbb{R})$-invariant). See Proposition 5.3 below.

We will proceed analogously with the 0-cochains $\tilde{\psi}_{0,1}(x)$ with singularities and $\tilde{\psi}'_{0,1}(x)$ for W (which are not weight vectors for $\mathrm{SO}(2)$ either) in the next section.

Proposition 5.3. *For $x\in V$ fixed, $\tilde{\psi}_1(x,z)$ is a differential 1-form with singularities along D_x. Outside D_x, we have*

$$d\tilde{\psi}_1(x,\tau,z) = \varphi_2(x,\tau,z).$$

Here d denotes the exterior differentiation on D. In particular, for $(x,x)\leq 0$ and $x\neq 0$, we see that $\varphi_2(x,z)$ is exact. Furthermore, for triples x,τ,z such that $z\notin D_x$ we have

$$L\tilde{\psi}_1(x,\tau,z) = \psi_1(x,\tau,z).$$

Proof. Using (5.6) and (5.5), we see

$$d\tilde{\psi}_1^0(x,z) = -\int_1^\infty d\left(\psi_1^0(\sqrt{r}x,z)\right)\frac{dr}{r} = -\int_1^\infty \frac{\partial}{\partial r}\left(\varphi_2^0(\sqrt{r}x,z)\right)dr = \varphi_2^0(x,z),$$

as claimed. The last formula follows easily from (5.7). □

Remark 5.4. The construction of the singular form $\tilde{\psi}_1$ works in much greater generality for $\mathrm{O}(p,q)$ whenever we have two Schwartz forms ψ and φ (of weights $r-2$ and r and degrees k and $k-1$ resp.) such that

$$d\psi = L\varphi.$$

Then the analogous construction of $\tilde{\psi}$ immediately yields $d\tilde{\psi}=\varphi$ outside a singular set. The main example for this are the forms of Kudla-Millson φ_q and ψ_{q-1}, see [25], for which this construction is already implicit in [6]. In particular, the proof of Theorem 7.2 in [6] shows that $\tilde{\psi}$ gives rise to a differential character for the analogous cycle C_x, see also Section 8 of this paper. The unitary case will be considered in [11].

5.3. **W-valued Schwartz forms and W-valued singular functions for W.** Let $W\subset V$ be the real quadratic space of signature $(1,1)$ obtained from the Witt decomposition of V given in Section 2.1.1. We write $\mathfrak{m}\simeq\mathbb{R}$ for the Lie algebra of $M=\mathrm{SO}_0(W)$. Then $X_{23}=e_2\wedge e_3$ is its natural generator with dual ω_{23}. We identify the associated symmetric space D_W with the lines in W on which the bilinear form $(\ ,\ )$ is negative definite:

$$D_W = \{z_W\subset W;\ \dim z_W = 1 \text{ and } (\ ,\ )|_{z_W}<0\}.$$

We pick the line $z_{W,0}$ spanned by e_3 as base point of D_W. For $s \in \mathbb{R}$, we set

$$z(s) := m(s)e_3 = \sinh(s)e_2 + \cosh(s)e_3.$$

and define $z_W(s)$ to be the span of $z(s)$. Then the isomorphism $\mathbb{R} \simeq D_W$ is realized by $s \mapsto z_W(s)$. Accordingly, we often write s for $z_W(s)$ and vice versa. A vector $x \in W$ of positive length defines a point

$$D_{W,x} = \{z_W \in D_W;\ z_W \perp x\}$$

in D_W. So $z_W(s) = D_{W,x}$ if and only if $(x, z(s)) = 0$. We also let $s(x) \in \mathbb{R}$ be the parameter value corresponding to $D_{W,x}$ under the above isomorphism.

5.3.1. *Special forms for W.* We first consider the Schwartz form $\varphi_{1,1}$ on W constructed in [13] (in greater generality) with values in $\mathcal{A}^1(D_W) \otimes W$. More precisely,

$$\varphi_{1,1} \in [\mathcal{S}(W) \otimes \mathcal{A}^1(D_W) \otimes W]^M \simeq [\mathcal{S}(W) \otimes \mathfrak{m}^* \otimes W].$$

(Note that $(K_M)_0$ is the trivial group). Explicitly at the base point, we have

$$\varphi_{1,1}(x) = \frac{1}{2^{3/2}} \left(4x_2^2 - \frac{1}{\pi}\right) e^{-\pi(x_2^2+x_3^2)} \otimes \omega_{23} \otimes e_2.$$

Note that $\varphi_{1,1}$ has weight 2, see [13], Theorem 6.2. We define $\varphi_{1,1}^0$ and $\varphi_{1,1}(x,s) = m(s)[\varphi_{1,1}(m(s)^{-1}x)]$ as in Section 5.1. There is another Schwartz function $\psi_{0,1}$ of weight 0 given by

$$\psi_{0,1}(x) = -\frac{1}{\sqrt{2}} x_2 x_3 e^{-\pi(x_2^2+x_3^2)} \otimes 1 \otimes e_2 + \frac{1}{4\sqrt{2}\pi} e^{-\pi(x_2^2+x_3^2)} \otimes 1 \otimes e_3$$
$$\in [\mathcal{S}(W) \otimes \bigwedge\nolimits^0 \mathfrak{m}^* \otimes W],$$

to which we associate functions $\psi_{0,1}(x,s)$ and $\psi_{0,1}^0$ as in Section 5.1 as well. Note that the notation differs from [13], section 6.5. The function $\psi_{0,1}$ defined here is the term $-\psi_{1,1} - \frac{1}{2}\Lambda_{1,1}$ given in Theorem 6.11 in [13]. The key relation between $\varphi_{1,1}$ and $\psi_{0,1}$ (correcting a sign mistake in [13]) is given by

Theorem 5.5. *([13], Theorem 6.2)*

$$\omega(L)\varphi_{1,1} = d\psi_{0,1}.$$

When made explicit, we have, again using (5.1),

$$v^{3/2} \frac{\partial}{\partial v} \left(v^{-1/2} \varphi_{1,1}^0(\sqrt{v}x, s)\right) = d\left(\psi_{0,1}^0(\sqrt{v}x, s)\right). \tag{5.8}$$

5.3.2. *The W-valued singular functions $\tilde{\psi}_{0,1}$ and $\tilde{\psi}'_{0,1}$.* We will now construct certain W-valued functions on W, considered as elements of the relative Lie algebra complex $\mathcal{S}'(W) \otimes \bigwedge^0 \mathfrak{m}^*$. The functions $\tilde{\psi}_{0,1}(x)$ and $\tilde{\psi}'_{0,1}(x)$ are continuous except for jump singularities along a positive line (the x_2-axis) in W. Hence they correspond to 0-cochains with singularities . For each fixed s, the corresponding W-valued functions $\tilde{\psi}_{0,1}(x,s)$ and $\tilde{\psi}'_{0,1}(x,s)$ (defined by (5.2)) will again have jump singularities but along the transform of the x_2-axis by $m(s)$, while for each fixed x with $(x,x) > 0$ the resulting function of s will have a jump singularity at $s(x)$.

5.3.3. *The W-valued singular function $\tilde{\psi}_{0,1}$.* We define $\tilde{\psi}_{0,1}$ (in the same way as $\tilde{\psi}_1$ for V) by

$$\tilde{\psi}_{0,1}(x) = -\left(\int_1^\infty \psi^0_{0,1}(\sqrt{r}x) r^{-3/2} dr\right) e^{-\pi(x,x)}, \tag{5.9}$$

but now for *all* $x \in W$, including $x = 0$. Set $\tilde{\psi}^0_{0,1}(x) = \tilde{\psi}_{0,1}(x)e^{\pi(x,x)}$ as before. We define functions A and B on W by

$$\tilde{\psi}_{0,1}(x) = A(x) \otimes 1 \otimes e_2 + B(x) \otimes 1 \otimes e_3,$$

From the definition of $\psi_{0,1}$ we easily see

$$-X_{23}B(x) = A(x). \tag{5.10}$$

We see by integrating by parts

Lemma 5.6.

$$A(x) = \frac{1}{2\sqrt{\pi}} x_2 \frac{x_3}{|x_3|} \Gamma(\tfrac{1}{2}, 2\pi x_3^2) e^{-\pi(x,x)},$$

$$B(x) = -\frac{1}{2\sqrt{2}\pi} e^{-\pi(x_2^2+x_3^2)} + \frac{1}{2\sqrt{\pi}} |x_3| \Gamma(\tfrac{1}{2}, 2\pi x_3^2) e^{-\pi(x,x)}.$$

Here $\Gamma(\frac{1}{2}, a) = \int_a^\infty e^{-u}u^{-1/2}du$ is the incomplete Γ-function at $s = 1/2$.

Note that both A and B are integrable and square-integrable on W. The singularities of A and B are given as follows.

Lemma 5.7. (i) *$B(x) - \frac{1}{2}|x_3|e^{-\pi(x,x)}$ is C^2 on W.*
(ii) *$A(x) - \frac{1}{2}x_2\frac{x_3}{|x_3|}e^{-\pi(x,x)}$ is C^1 on W.*

Proof. Use Lemma 5.6, expand the incomplete gamma function around $x_3 = 0$, and observe that $|x|x^n$ is C^n for $n > 0$. □

We define $\tilde{\psi}_{0,1}(x, s)$ according to (5.2) in Section 5.1, and we obtain functions $A(x, s)$ and $B(x, s)$ as well. We have $\big($noting $\operatorname{sgn}(x_3) = -\operatorname{sgn}((x, e_3))$ $\big)$

Lemma 5.8.

$$\tilde{\psi}_{0,1}(x, s) = -\frac{1}{2\sqrt{\pi}} e^{-\pi(x,x)} \left(\operatorname{sgn}(x, z(s))\Gamma\left(\tfrac{1}{2}, 2\pi(x, z(s))^2\right) \otimes x + \frac{1}{\sqrt{2\pi}} e^{-2\pi(x,z(s))^2} \otimes z(s) \right).$$

In particular, for fixed $x \in W$, the function $\tilde{\psi}_{0,1}(x, s)$ is continuous on D_W, except for a discontinuity at $s = s(x)$ when $(x, x) > 0$. Note that $B(x, s)$ is actually continuous (and bounded) on D_W.

We define using (5.1) (analogous to (5.7), see Remark 5.2, with $r = m = 2$)

(5.11)

$$\tilde{\psi}_{0,1}(x, \tau, s) = v^{-1/2}\ \tilde{\psi}_{0,1}(\sqrt{v}x, s) e^{i(x,x)u} = -\left(\int_v^\infty \psi^0_{0,1}(\sqrt{r}x, z) r^{-3/2} dr\right) e^{\pi i(x,x)\tau}.$$

The key properties of $\tilde{\psi}_{0,1}(x, \tau, s)$ analogous to Proposition 5.3 (proved in the same way using (5.8) and (5.11)) are given by

Proposition 5.9. *For $x \in W$ fixed and $s \neq s(x)$, the singular function $\tilde{\psi}_{0,1}(x,\tau,s)$ satisfies*

$$d\tilde{\psi}_{0,1}(x,\tau,s) = \varphi_{1,1}(x,\tau,s).$$

Here d denotes the exterior differentiation on D_W. Furthermore, $\tilde{\psi}_{0,1}(x,\tau,s)$ is smooth in τ, and we have

$$L\tilde{\psi}_{0,1}(x,\tau,s) = \psi_{0,1}(x,\tau,s).$$

Remark 5.10. It is natural to consider $\tilde{\psi}_{0,1}(x,s)$ (for fixed x) as a current $[\tilde{\psi}_{0,1}(x,s)]$ on D_W. From the explicit nature of the singularity it is quite straightforward to strengthen the first part of Proposition 5.9 and to show

$$d[\tilde{\psi}_{0,1}(x,\tau,s)] = \operatorname{sgn}(x,z(s))(\delta_{D_{W,x}} \otimes x)e^{\pi i(x,x)\tau} + [\varphi_{1,1}(x,\tau,s)].$$

Here $(\delta_{D_{W,x}} \otimes x)(f) = (f(D_{W,x}),x)$ for a W-valued test function f on D_W.

5.3.4. *The W-valued singular function $\tilde{\psi}'_{0,1}$.* Inspired by [17], section 2.3, we define functions $A'(x)$ and $B'(x)$ on W by

$$B'(x) = \begin{cases} \frac{1}{2}\min(|x_2 - x_3|, |x_2 + x_3|)e^{-\pi(x,x)} & \text{if } (x,x) = x_2^2 - x_3^2 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.12}$$

$$A'(x) = -X_{23}B'(x) = -\operatorname{sgn}(x_2x_3)B'(x).$$

We then define $\tilde{\psi}'_{0,1}(x)$ by

$$\tilde{\psi}'_{0,1}(x) = A'(x) \otimes 1 \otimes e_2 + B'(x) \otimes 1 \otimes e_3.$$

The singularities of $\tilde{\psi}'_{0,1}(x)$ are given as follows.

Lemma 5.11. (i) *$B'(x) + \frac{1}{2}|x_3|e^{-\pi(x,x)}$ is C^2 on the complement of the null-cone in W and C^2 on nonzero M-orbits.*

(ii) *$A'(x) + \frac{1}{2}x_2\frac{x_3}{|x_3|}e^{-\pi(x,x)}$ is C^1 on the complement of the null-cone in W and C^1 on nonzero M-orbits.*

Note that A' and B' are integrable and square-integrable on W. We define $\tilde{\psi}'_{0,1}(x,s)$ as in (5.3) and also again as in (5.7) we set

$$\tilde{\psi}'_{0,1}(x,\tau,s) = v^{-1/2}\tilde{\psi}'_{0,1}(\sqrt{v}x,s)e^{\pi i(x,x)u}. \tag{5.13}$$

We define isotropic vectors in W by $u_W = \frac{1}{\sqrt{2}}(e_2+e_3)$ and $u'_W = \frac{1}{\sqrt{2}}(e_2-e_3)$. A little calculation gives the following formula for $\tilde{\psi}'_{0,1}(x,\tau,s)$ which immediately implies the key properties analogous to Proposition 5.3 and Proposition 5.9.

Proposition 5.12. *For x with $(x,x) > 0$, we have*

$$\tilde{\psi}'_{0,1}(x,\tau,s) = \begin{cases} |(x,u'_W)|e^{\pi i(x,x)\tau} \otimes u_w & \text{if } s > s(x), \\ -|(x,u_W)|e^{\pi i(x,x)\tau} \otimes u'_w & \text{if } s < s(x). \end{cases}$$

In particular, the function $\tilde{\psi}'_{0,1}(x,\tau,s)$ is a W-valued function that is constant in the complement of the point $D_{W,x}$ in D_W and has a jump discontinuity at $D_{W,x}$. Moreover, $\tilde{\psi}'_{0,1}(x,\tau,s)$ is holomorphic in τ for all x,s for which it is defined.

Similarly as in Remark 5.10 we have

Remark 5.13.

$$d[\tilde{\psi}'_{0,1}(x,\tau,s)] = -\operatorname{sgn}(x,z(s))(\delta_{D_{W,x}} \otimes x)e^{\pi i \tau(x,x)}.$$

5.3.5. *The W-valued singular function $\phi_{0,1}$ on W.* We now combine $\tilde{\psi}_{0,1}$ and $\tilde{\psi}'_{0,1}$ to obtain an integrable and also square-integrable W-valued function

$$\phi_{0,1} \in [L^2(W) \otimes \bigwedge^0 \mathfrak{m}^* \otimes W]$$

given by

$$\begin{aligned}\phi_{0,1}(x) &= \tilde{\psi}_{0,1}(x) + \tilde{\psi}'_{0,1}(x)\\ &= (A(x)+A'(x)) \otimes 1 \otimes e_2 + (B(x)+B'(x)) \otimes 1 \otimes e_3\end{aligned}$$

Combining Lemmas 5.7, 5.11 and (5.10), (5.12) we obtain

Proposition 5.14. *The function $\phi_{0,1}$ is continuous on all of W. Moreover,*
- (i) *$B(x)+B'(x)$ is C^2 on the complement of the null-cone in W and C^2 on nonzero M-orbits.*
- (ii) *$A(x)+A'(x)$ is C^1 on the complement of the null-cone in W and C^1 on nonzero M orbits.*
- (iii) *$X_{23}(B+B') = -(A+A')$ on all of W.*

We define the associated W-valued function $\phi_{0,1}(x,s) = m(s)[\phi_{0,1}(m(s)^{-1}x)] = \tilde{\psi}_{0,1}(x,s) + \tilde{\psi}'_{0,1}(x,s)$ as in (5.3). Proposition 5.14 immediately implies that for given x, the function $\phi_{0,1}(x,s)$ is a C^1-function on D_W. Furthermore, we also set

$$\phi_{0,1}(x,\tau,s) = \tilde{\psi}_{0,1}(x,\tau,s) + \tilde{\psi}'_{0,1}(x,\tau,s).$$

The following theorem is fundamental for us. It is an immediate consequence of Proposition 5.9/Remark 5.10 and Proposition 5.12/Remark 5.13.

Theorem 5.15. *The form $\varphi_{1,1}$ on D_W is exact. Namely,*

$$d\phi_{0,1} = \varphi_{1,1}.$$

Furthermore,

$$L\phi_{0,1} = \psi_{0,1}.$$

Finally, we have

Proposition 5.16. *The function $\phi_{0,1} = \tilde{\psi}_{0,1} + \tilde{\psi}'_{0,1}$ is an eigenfunction of* $\mathrm{SO}(2)$ *of weight* 2 *under the Weil representation.*

Proof. It suffices to show this for one component of $\phi_{0,1}$, i.e., the function $B(x) + B'(x)$. Then the assertion has been already proved in §2.3 of [17] by showing that $B(x) + B'(x)$ is an eigenfunction under the Fourier transform. We give here an infinitesimal proof using the interpretation as distributions, see Section 5.1. Since $\omega(k')$ for $k' \in \mathrm{SO}(2)$ acts essentially as Fourier transform and $B + B'$ is L^1, we see that $\omega(k')(B + B')$ is continuous. Hence it suffices to establish the corresponding current equality $[\omega(k')(B+B')] = \chi^2(k')[B+B']$, since continuous functions coincide when they induce the same current. The infinitesimal generator of K' acts by $H := \frac{-i}{4\pi}\left(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right) + \pi i(x_2^2 - x_3^2)$, and a straightforward calculation gives

$$HB' = 2iB' \qquad \text{and} \qquad HB = 2iB,$$

outside the singularity $x_2^2 - x_3^2 = 0$. Now we consider the currents $H[B]$ and $H[B']$. An easy calculation using the fact that B and B' are C^2 up to $|x_3|e^{-\pi(x_2^2-x_3^2)}$ shows that for a test function f on W we have

$$H[B](f) = [HB](f) + \int_0^\infty e^{-\pi x_2^2} f(x_2,0)dx_2,$$
$$H[B'](f) = [HB'](f) - \int_0^\infty e^{-\pi x_2^2} f(x_2,0)dx_2.$$

Thus $H[B+B'] = [H(B+B')] = 2i[B+B']$ as claimed. □

Remark 5.17. Note that Proposition 5.16 also justifies the definitions of $\tilde{\psi}_{0,1}(x,\tau)$ and $\tilde{\psi}'_{0,1}(x,\tau)$ as functions of τ using (5.1), see (5.11), (5.13), and Remark 5.2. While the 0-cochains with singularities $\tilde{\psi}_{0,1}$ and $\tilde{\psi}'_{0,1}$ do not have weight 2, their sum is the 0-cochain $\phi_{0,1}$ of weight 2.

5.3.6. *The map ι_P.* We define a map

$$\iota_P : \mathcal{S}(W) \otimes \bigwedge^i \mathfrak{m}^* \otimes W \to \mathcal{S}(W) \otimes \bigwedge^{i+1} (\mathfrak{m}^* \oplus \mathfrak{n}^*)$$

by

$$\iota_P(\varphi \otimes \omega \otimes w) = \varphi \otimes (\omega \wedge (w \wedge u')).$$

Here we used the isomorphism $\mathfrak{n} \simeq W \wedge \mathbb{R}u \in \bigwedge^2 V_{\mathbb{R}} \simeq \mathfrak{g}$ and identify W with its dual via the bilinear form $(\ ,\ )$ so that $\mathfrak{n}^* \simeq W \wedge \mathbb{R}u'$. In [14], Section 6.2 we explain that ι_P is a map of Lie algebra complexes. Hence we obtain a map of complexes

$$[\mathcal{S}(W) \otimes \mathcal{A}^i(D_W) \otimes W]^M \to [\mathcal{S}(W) \otimes \mathcal{A}^{i+1}(e(P))]^{NM},$$

which we also denote by ι_P. Here N acts trivially on $\mathcal{S}(W)$. Explicitly, the vectors e_2 and e_3 in W map under ι_P to the left-invariant 1-forms

$$e_2 \mapsto \cosh(s)dw_2 - \sinh(s)dw_3 \qquad e_3 \mapsto \sinh(s)dw_2 - \cosh(s)dw_3$$

with the coordinate functions w_2, w_3 on W defined by $w = w_2e_2 + w_3e_3$. We apply ι_P to the forms on W of this section, and we obtain $\varphi_{1,1}^P$, $\phi_{0,1}^P$, $\psi_{0,1}^P$, and $\psi'^P_{0,1}$. Note that for the last three forms one must replace $\mathcal{S}(W)$ by the space of singular functions.

6. The boundary theta lift and linking numbers in *Sol*

For the rest of the paper, V and W denote again rational quadratic spaces.

6.1. **Global theta functions for W.** We let $\mathcal{L}_W$ be a Γ_P-invariant (coset of a) lattice in W, where Γ_N acts trivially on W. We define its theta function associated to $\varphi_{1,1}$ by

$$\theta_{\varphi_{1,1}}(\tau, \mathcal{L}_W) = \sum_{x \in \mathcal{L}_W} \varphi_{1,1}(x, \tau)$$

and similarly $\theta_{\psi_{0,1}}$ and $\theta_{\phi_{0,1}}$. One needs to be careful in forming the naive theta series associated to $\tilde{\psi}_{0,1}$ and $\tilde{\psi}'_{0,1}$ by summing over all lattice elements. This would give functions on D_W with singularities on a dense subset. Instead, we define the partial theta function

$$\tilde{\psi}_{0,1}(n) = \tilde{\psi}_{0,1}(n, s) = \sum_{x \in \mathcal{L}_W, (x,x)=2n} \tilde{\psi}_{0,1}(x, s),$$

which descends to a function on D_W with a locally finite singularity set. Note that for $n = 0$, the term for $x = 0$ is included. We define $\tilde{\psi}'_{0,1}(n)$ and also $\varphi_{1,1}(n)$, $\psi_{0,1}(n)$, and $\phi_{0,1}(n)$ in the same way. Here, to make our formulas less complicated, we have systematically left out the dependence on s and we will continue to do so. The fundamental invariance property of the theta distribution (on W) gives that $\theta_{\varphi_{1,1}}(\tau, \mathcal{L}_W)$ and $\theta_{\phi_{0,1}}(\tau, \mathcal{L}_W)$ both transform like (non)-holomorphic modular forms of weight 2 for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Remark 6.1. The claim is not obvious for $\theta_{\phi_{0,1}}$, since $\phi_{0,1}$ is not a Schwartz function. In that case, we use Proposition 5.14. The component $B + B'$ of $\phi_{0,1}$ is C^2 outside the null-cone. Since W is anisotropic we can then apply Possion summation, and this component transforms like a modular form. Then apply the differential operator X_{23} to obtain the same for the other component $A + A'$ of $\phi_{0,1}$.

In fact, if W is isotropic and $\mathcal{L}_W$ intersects non-trivially with the null-cone, then $\theta_{\phi_{0,1}}$ is not quite a modular form. The case, in which the $\mathbb{Q}$-rank of V is 2, is interesting in its own right. We will discuss this elsewhere.

Via the map ι_P from Section 5.3.6 we can view all theta functions for W as differential forms on $e'(P)$. We set $\theta^P_{\varphi_{1,1}} = \theta_{\varphi^P_{1,1}}$, and similarly $\theta^P_{\psi_{0,1}}$ and $\theta^P_{\phi_{0,1}}$. Since ι_P is a map of complexes we immediately see by Theorem 5.5 and Theorem 5.15

Proposition 6.2.

$$\theta^P_{\varphi_{1,1}} = d\theta^P_{\phi_{0,1}} \qquad \textit{and} \qquad L\theta^P_{\phi_{0,1}} = \theta^P_{\psi_{0,1}}.$$

We now interpret the (holomorphic) Fourier coefficients of the boundary theta lift associated to $\theta^P_{\phi_{0,1}}(\tau, \mathcal{L}_W)$. They are given by linking numbers. We have

Theorem 6.3. *Let c be a 1-cycle contained in a torus fiber in $e'(P)$. Then*

$$\int_c \theta^P_{\phi_{0,1}}(\tau, \mathcal{L}_W) = \sum_{n \in \mathbb{Q}_{>0}} \mathrm{Lk}((\partial C_n)_P, c) q^n \; + \sum_{n \in \mathbb{Q}} \int_c \tilde{\psi}^P_{0,1}(n)(\tau).$$

So the Fourier coefficients of the holomorphic part of the weight 2 modular form $\int_c \theta^P_{\phi_{0,1}}(\tau, \mathcal{L}_W)$ are the extended linking numbers of the cycles c and $(\partial C_n)_P$ at the *boundary component $e'(P)$. Here we define for $n > 0$ and c not disjoint to the locally finite cycle $(\partial C_n)_P$, the integral $\int_c \tilde{\psi}^P_{0,1}(n)(\tau)$ by right continuity:*

$$\int_c \tilde{\psi}^P_{0,1}(n)(\tau) := \lim_{\varepsilon \to 0^+} \int_{c(\varepsilon)} \tilde{\psi}^P_{0,1}(n)(\tau),$$

where $c(\varepsilon)$ is the parallel translate of c by $\varepsilon > 0$, see Section 4.4 and also Lemma 5.8.

Theorem 6.3 follows from $\phi_{0,1} = \tilde{\psi}_{0,1} + \tilde{\psi}'_{0,1}$ combined with Theorem 6.5 below.

Example 6.4. In the situation of Examples 2.1 and 4.14, we obtain

$$\int_{\partial C_1} \theta^P_{\phi_{0,1}}(\tau, \mathcal{L}_W) = \frac{1}{\sqrt{2d}} \sum_{\substack{\lambda \in \mathcal{O}_K \\ \lambda\lambda' > 0}} \min(|\lambda|, |\lambda'|) e^{-2\pi\lambda\lambda'\tau} - \frac{\sqrt{2}}{\sqrt{dv}} \sum_{\lambda \in \mathcal{O}_K} \beta(\pi v(\lambda - \lambda')^2) e^{-2\pi\lambda\lambda'\tau},$$

where $\beta(s) = \frac{1}{16\pi} \int_1^\infty e^{-st} t^{-3/2} dt$. This is (up to a constant) exactly Zagier's function $\mathcal{W}(\tau)$ in [17], §2.3.

6.2. **The 1-form $e^{2\pi n} \tilde{\psi}'^P_{0,1}(n)$ is an extended linking dual of $(\partial C_n)_P$.** In this subsection, we will give an integral formula for the extended linking numbers of vertical closed geodesics. Formulas such as (6.1) below go back to the classical Gauss-Ampère formula for $\mathbb{R}^3$, see [9], p.79-81, and [8] for its generalization to S^3 and H^3.

In what follows, we drop subscript and superscript P's since we are fixing a boundary component $e'(P)$. We let F_n be the union of the fibers containing components of ∂C_n, and we let F_x be the fiber containing c_x. Recall that c_x is the image of $D_x \cap e(P)$ in $e'(P)$. In what follows recall that $A_n \subset \partial \overline{X}$ is the cap for ∂C_n.

Theorem 6.5. *Let $n > 0$. Let c be a vertical closed geodesic c in $e'(P)$. Then*

$$\int_c \tilde{\psi}'_{0,1}(n) = \operatorname{Lk}(\partial C_n, c) e^{-2\pi n}. \tag{6.1}$$

Here if c is not disjoint from ∂C_n then $\operatorname{Lk}(\partial C_n, c)$ denotes the extended linking number as defined in Section 4.4. In that case the integral in (6.1) might not be defined. To rectify this, we define $\int_c \tilde{\psi}'_{0,1}(n)$ in the same way as $\int_c \tilde{\psi}_{0,1}(n)$ in Theorem 6.3. Since $\tilde{\psi}'_{0,1}(n)$ is locally constant, we actually do not have to take the limit.

It is natural to say that (6.1) means that $e^{2\pi n} \tilde{\psi}'_{0,1}(n)$ is an **(extended) linking dual** of ∂C_n. The key step for the proof of Theorem 6.5 is the following

Proposition 6.6. *Let $n > 0$ and let η be a closed 2-form in $e'(P)$ which is compactly supported in the complement of F_n. Then*

$$\int_{e'(P)} \eta \wedge \tilde{\psi}'_{0,1}(n) = \left(\int_{A_n} \eta \right) e^{-2\pi n}. \tag{6.2}$$

6.3. **Proof of Proposition 6.6.** Proposition 6.6 will follow from the next two lemmas. First, note that any η satisfying the hypotheses of Proposition 6.6 is exact. Indeed, choose a torus fiber $F \subset F_n$. Since η is supported away from F_n its restriction to F is zero, hence its integral over F is zero. But F generates the homology group $H_2(e'(P), \mathbb{R})$.

In the next two lemmas, we will show each side of (6.2) is equal to

$$\sum_{\substack{x\in\Gamma_M\backslash\mathcal{L}_W\\(x,x)=2n}} \min_{\lambda\in\Lambda_W}{}'|(\lambda,x)| \left(\int_{a_x}\eta\right) e^{-2\pi n}.$$

Lemma 6.7. *Under the hypothesis on η in Proposition 6.6 we have*

$$\int_{A_n}\eta = \sum_{\substack{x\in\Gamma_M\backslash\mathcal{L}_W\\(x,x)=2n}} \min_{\lambda\in\Lambda_W}{}'|(\lambda,x)| \int_{a_x}\eta$$

Proof. Using Lemma 4.5 we see

$$\int_{A_n}\eta = \sum_{\substack{x\in\Gamma_M\backslash\mathcal{L}_W\\(x,x)=2n}} \sum_{0\le k<\min'_{\lambda\in\Lambda_W}|(\lambda,x)|} \int_{a_{x+ku}}\eta.$$

Write $\eta = d\omega$ for some 1-form ω which by the support condition on η is closed in F_n. Since c_{x+ku} and c_x are parallel hence homologous circles in F_x, we see $\int_{a_{x+ku}}\eta = \int_{c_{x+ku}}\omega = \int_{c_x}\omega = \int_{a_x}\eta$. □

Since

$$\tilde{\psi}'_{0,1}(n) = \sum_{\substack{x\in\Gamma_M\backslash\mathcal{L}_W\\(x,x)=2n}} \sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x),$$

Proposition 6.6 will now follow from

Lemma 6.8. *Under the hypothesis on η in Proposition 6.6, we have for any positive length vector $x\in\mathcal{L}_W$*

$$\int_{e'(P)}\eta\wedge\sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x) = \min_{\lambda\in\Lambda_W}{}'|(\lambda,x)| \left(\int_{a_x}\eta\right) e^{-\pi(x,x)}.$$

Proof. By choosing coordinates in D_W appropriately, we can assume that $x=\mu e_2$ with $\mu=\pm\sqrt{2n}$, so that the singularity of $\sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x)$ in $e'(P)$ occurs at $s=0$. We pick a tubular neighborhood $U_\varepsilon = (-\varepsilon,\varepsilon)\times T^2$ in $e'(P)$ around F_x. Then we have

$$\int_{e'(P)}\eta\wedge\sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x) = \lim_{\epsilon\to 0}\int_{e'(P)-U_\varepsilon}\eta\wedge\sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x).$$

Since $\eta\wedge\tilde{\psi}'_{0,1}(x) = d(\omega\wedge\tilde{\psi}'_{0,1}(x))$ outside U_ε and $\partial(e'(P)-U_\varepsilon) = -\partial U_\varepsilon$ we see by Stokes' theorem

$$\begin{aligned}(6.3)\qquad \int_{e'(P)-U_\varepsilon}\eta\wedge\sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x) &= -\int_{\partial U_\varepsilon}\omega\wedge\sum_{\gamma\in\Gamma_M}\gamma^*\tilde{\psi}'_{0,1}(x)\\ &= \sum_{\gamma\in\Gamma_M}\int_{T^2}\left[\omega(-\varepsilon,w)\wedge\tilde{\psi}'_{0,1}(\gamma^{-1}x,-\epsilon,w) - \omega(\varepsilon,w)\wedge\tilde{\psi}'_{0,1}(\gamma^{-1}x,\epsilon,w)\right].\end{aligned}$$

For $\gamma \neq 1$ we note that $\omega(s,w) \wedge \tilde{\psi'}_{0,1}(\gamma^{-1}x, s, w)$ is continuous at $s = 0$, while for $\gamma = 1$, we have

$$\tilde{\psi'}_{0,1}(\mu e_2, s, w) = \frac{1}{2}|\mu|(\mathrm{sgn}(s)dw_2 - dw_3)e^{-\pi\mu^2}. \tag{6.4}$$

Hence taking the limit in the last term of (6.3), using that the jump in $\tilde{\psi'}_{0,1}$ at $s = 0$ is a multiple of dw_2 we obtain

$$|\mu|e^{-\pi\mu^2} \int_{T^2} \omega_3(0,w)dw_2dw_3 = |\mu|e^{-\pi\mu^2} \int_{T^2/c_{e_2}} \left(\int_{c_{e_2}(w_2)} \omega_3(0,w_2,w_3)dw_3 \right) dw_2.$$

Here ω_3 is the dw_3 component of ω and $c_{e_2}(w_2)$ is the (horizontal) parallel translate of the cycle c_{e_2} to w_2. Also, in the expression T^2/c_{e_2} (and for the rest of this proof) we have abused notation and identified the cycle c_{e_2} with the subgroup $0 \times S^1$ of T^2.

Since the restriction of ω to F_x is closed, $\int_{c_{e_2}(w_2)} \omega(0,w_2,w_3)$ is independent of w_2, and the last integral becomes $\left(\int_{T^2/c_{e_2}} dw_2\right)\left(\int_{c_{e_2}} \omega\right) e^{-\pi\mu^2}$. But $\int_{c_{e_2}} \omega = \int_{a_{e_2}} \eta$. The proposition is then a consequence of

$$|\mu| \int_{T^2/c_{e_2}} dw_2 = |\mu| \min_{\lambda\in\Lambda_W}{}' |(\lambda, e_2)| = \min_{\lambda\in\Lambda_W}{}' |(\lambda, x)|,$$

which follows from the fact that the map $W \to \mathbb{R}$ given by $w \mapsto (w, e_2)$ induces an isomorphism $T^2/c_{e_2} \simeq \mathbb{R}/\min'_{\lambda\in\Lambda_W} |(\lambda, e_2)|\mathbb{Z}$. □

6.4. **Proof of Theorem 6.5.** We now prove Theorem 6.5. First we will assume that c is disjoint from F_n. Choose an (open) tubular neighborhood $N(c)$ of c such that $N(c)$ is disjoint from F_n. Let η_c be a closed 2-form which is supported inside $N(c)$ and has integral 1 on the disk fibers of $N(c)$ (a representative for the Thom class of the normal disk bundle $N(c)$ of c in $e'(P)$). Extend η_c to $e'(P)$ by zero. Let U_n be a neighborhood of F_n which is a union of inverse images under the fiber bundle projection of small open intervals in the base circle of $e'(P)$. We may assume these intervals do not intersect whence U_n is a disjoint union of neighborhoods of the components of F_n. We may also assume that $e'(P) - U_n$ contains $N(c)$. Hence, $\overline{U}_n$, the closure of U_n, is contained in the complement of $N(c)$. Hence $\partial\overline{U}_n \subset e'(P) - N(c)$ and

$$\eta_c|\partial\overline{U}_n = 0. \tag{6.5}$$

Note that $Y_n := e'(P) - U_n$ is a compact manifold with boundary a disjoint union of torus fibers. Since $\partial Y_n = \partial\overline{U}_n$ the closed 2-form η_c vanishes on ∂Y_n and consequently represents a relative cohomology class for the pair $(Y_n, \partial Y_n)$. We note that this class is the (relative) Poincaré dual of the absolute 1-cycle c in Y_n . This is because η_c is the extension by zero of the Thom class of c in $N(c)$ to Y_n.

Note further that the cap A_n meets $\partial\overline{U}_n = \partial Y_n$ transversally in a union of vertical geodesics since this intersection will see only the part of A_n consisting of monodromy 2-chains $\{\mathcal{M}(c_j)\}$ and each $\mathcal{M}(c_j)$ is locally the product of the vertical geodesic c_j and an interval in the base circle. Then $A'_n := A_n \cap Y_n$ is a relative 2-cycle on the manifold with boundary Y_n and the 1-form $\tilde{\psi'}_{0,1}(n)$ is smooth and closed on Y_n. Hence, by the cycle (resp. form) version of Poincaré/Lefschetz duality on Y_n we have

Lemma 6.9.

(1) $\int_{A_n} \eta_c = \int_{A'_n} \eta_c = \int_{A'_n} PD(c) = c \cdot A'_n = A'_n \cdot c = A_n \cdot c = \mathrm{Lk}(\partial C_n, c)$.

(2) $\int_{e'(P)} \eta_c \wedge \tilde{\psi'}_{0,1}(n) = \int_{Y_n} \tilde{\psi'}_{0,1}(n) \wedge \eta_c = \int_{Y_n} \tilde{\psi'}_{0,1}(n) \wedge PD(c) = \int_c \tilde{\psi'}_{0,1}(n)$.

Now we apply Proposition 6.6 for the special case $\eta = \eta_c$ and obtain

$$\int_{e'(P)} \eta_c \wedge \tilde{\psi'}_{0,1}(n) = \left(\int_{A_n} \eta_c \right) e^{-2\pi n}. \tag{6.6}$$

Combining this with the two equations from Lemma 6.9 we have

$$\int_c \tilde{\psi'}_{0,1}(n) = \int_{e'(P)} \eta_c \wedge \tilde{\psi'}_{0,1}(n) = \left(\int_{A_n} \eta_c \right) e^{-2\pi n} = \mathrm{Lk}(\partial C_n, c) e^{-2\pi n}.$$

Here the first equality is the second equation from Lemma 6.9, the second equality is (6.6), and the third equality is the first equation from Lemma 6.9.

We now drop the assumption that c is disjoint from F_n. We have

$$\int_c \widetilde{\psi}'_{0,1}(n) = \int_{c(\epsilon)} \widetilde{\psi}'_{0,1}(n) = \mathrm{Lk}(\partial C_n, c(\epsilon)).$$

The first equality holds by definition, and the second equality follows from what we have just proved because $c(\epsilon)$ is disjoint from F_n. But by definition of the extended linking number we have $\mathrm{Lk}(\partial C_n, c) = \mathrm{Lk}(\partial C_n, c(\epsilon))$.

With this Theorem 6.5 is proved.

7. The generating series of the capped cycles

In this section, we show that the generating series of the 'capped' cycles C_n^c gives rise to a modular form, extending Theorem 7.1 to a lift of the full cohomology $H^2(X)$ of X. In particular, we give our new proof of the theorem of Hirzebruch and Zagier.

7.1. **The theta series associated to φ_2.** We define the theta series

$$\theta_{\varphi_2}(\tau, \mathcal{L}) = \sum_{x \in \mathcal{L}} \varphi_2(x, \tau). \tag{7.1}$$

In the following we will often drop the argument $\mathcal{L} = L + h$. For $n \in \mathbb{Q}$, we also set

$$\varphi_2(n) = \sum_{n \in \mathcal{L}_n, x \neq 0} \varphi_2(x). \tag{7.2}$$

Clearly, $\theta_{\varphi_2}(\tau, \mathcal{L})$ and $\varphi_2(n)$ descend to closed differential 2-forms on X. Furthermore, $\theta_{\varphi_2}(\tau, \mathcal{L})$ is a non-holomorphic modular form in τ of weight 2 for the principal congruence subgroup $\Gamma(N)$. In fact, for $\mathcal{L} = L$ as in Example 2.1, $\theta_{\varphi_2}(\tau, \mathcal{L})$ transforms like a form for $\Gamma_0(d)$ of nebentypus. We define $\theta_{\psi_1}(\tau, \mathcal{L})$ and $\psi_1(n)$ in the same way.

Theorem 7.1 (Kudla-Millson [25]). *We have*

$$[\theta_{\varphi_2}(\tau)] = -\frac{1}{2\pi}\delta_{h0}[\omega] + \sum_{n>0} \mathrm{PD}[C_n]q^n \in H^2(X,\mathbb{Q}) \otimes M_2(\Gamma(N)).$$

That is, for any closed 2-form η on X with compact support, we have

$$\Lambda(\eta,\tau) := \int_X \eta \wedge \theta_{\varphi_2}(\tau,\mathcal{L}) = -\frac{1}{2\pi}\delta_{h0}\int_X \eta\wedge\omega + \sum_{n>0}\left(\int_{C_n}\eta\right)q^n.$$

Here δ_{h0} is Kronecker delta, and ω is the Kähler form on D normalized such that its restriction to the base point is given by $\omega_{13}\wedge\omega_{14}+\omega_{23}\wedge\omega_{24}$. We obtain a map

$$\Lambda : H_c^2(X,\mathbb{C}) \to M_2(\Gamma(N)) \tag{7.3}$$

from the cohomology with compact supports to the space of holomorphic modular forms of weight 2 for the principal congruence subgroup $\Gamma(N)\subset \mathrm{SL}_2(\mathbb{Z})$). Alternatively, for C an absolute 2-cycle in X defining a class in $H_2(X,\mathbb{Z})$, the lift $\Lambda(C,\tau)$ is given by (1.1) with C_0 the class given by $-\frac{1}{2\pi}\delta_{h0}[\omega]$.

The key steps for the proof of Theorem 7.1 are as follows. The holomorphicity of the lift $\Lambda(\eta,\tau)$ follows from

$$L\theta_{\varphi_2} = d\theta_{\psi_1} \tag{7.4}$$

together with Stokes Theorem. On the other hand, the crucial fact of the Fourier expansion is that for $n>0$, the form $\varphi_2(n)$ is a Poincaré dual form of C_n, while $\varphi_2(n)$ is exact for $n\le 0$, see also Section 8.

7.2. **The restrictions of the global theta functions.**

Theorem 7.2. *The differential forms $\theta_{\varphi_2}(\mathcal{L}_V)$ and $\theta_{\psi_1}(\mathcal{L}_V)$ on X extend to the Borel-Serre compactification $\overline{X}$. More precisely, for the restriction i_P^* to the boundary face $e'(P)$ of $\overline{X}$, we have*

$$i_P^*\theta_{\varphi_2}(\mathcal{L}_V) = \theta^P_{\varphi_{1,1}}(\mathcal{L}_{W_P}) \qquad \text{and} \qquad i_P^*\theta_{\psi_1}(\mathcal{L}_V) = \theta^P_{\psi_{0,1}}(\mathcal{L}_{W_P}).$$

Proof. The restriction of $\theta_{\varphi_2}(\mathcal{L}_V)$ is the theme (in much greater generality) of [14]. For $\theta_{\psi_1}(\mathcal{L}_V)$ one proceeds in the same way. In short, one detects the boundary behavior of the theta functions by switching to a mixed model of the Weil representation. For a model calculation see the proof of Proposition 7.4 below. □

We conclude by Proposition 6.2

Theorem 7.3. *The restriction of $\theta_{\varphi_2}(\mathcal{L}_V)$ to the boundary of $\overline{X}$ is exact and*

$$i_P^*\theta_{\varphi_2}(\mathcal{L}_V) = d\left(\theta^P_{\phi_{0,1}}(\mathcal{L}_{W_P})\right).$$

We also have a crucial restriction result for the singular form $\tilde{\psi}_1$. As in the case for the singular forms $\tilde{\psi}_{0,1}$ and $\tilde{\psi}'_{0,1}$ for W, we only define

$$\tilde{\psi}_1(n) = \sum_{n\in\mathcal{L}_n, x\neq 0} \tilde{\psi}_1(x),$$

to avoid defining a form on X with singularities on a dense subset of X. Now $\tilde{\psi}_1(n)$ is a 1-form on X with singularities for $n > 0$ along the locally finite cycle C_n. Note

$$d\tilde{\psi}_1(n) = \varphi_2(n) \tag{7.5}$$

as differential 2-forms away from the singular set C_n. (For the relationship between $\tilde{\psi}_1(n)$ and $\varphi_2(n)$ as currents, see Theorem 8.2 below.) We have

Proposition 7.4. *Away from the cycle C_n, the restriction of the differential 1-form $\tilde{\psi}_1(n)$ to $e'(P)$ is given by*

$$i_P^*\tilde{\psi}_1(n) = \tilde{\psi}^P_{0,1}(n).$$

Proof. We assume that P is the stabilizer of the isotropic line $\ell = \mathbb{Q}u$. For $x = au + x_W + bu'$, we have for the majorant at $z = (w,t,s)$ the formula

$$(x,x)_z = \frac{1}{t^2}(a - (x_W,w) - b\frac{(w,w)}{2})^2 + (x_w + bw, x_w + bw)_s + b^2t^2.$$

Here $(\ ,\ )_s$ is the majorant associated to $s \in D_W$. Hence by (5.6) and (5.4) we see that the sum of all $x \in \mathcal{L}_V$ with $b \neq 0$ in $\tilde{\psi}_1(n)$ is uniformly rapidly decreasing as $t \to \infty$. Now fix an element $x_W \in \mathcal{L}_W$. Then there exists $h \in \mathbb{Q}/\mathbb{Z}$ such that $x_W + (a+h)u \in \mathcal{L}_V$ for all $a \in \mathbb{Z}$; in fact all elements in $\mathcal{L}_V \cap u^\perp$ are of this form. We consider $\sum_{a\in\mathbb{Z}} \tilde{\psi}_1(x_W + (a+h)u, z)$ as $t \to \infty$. We can assume $w = 0$ and $s = 0$. We apply Poisson summation for the sum on $a \in \mathbb{Z}$ and obtain

$$\sum_{a\in\mathbb{Z}} \tilde{\psi}_1(x_W + (a+h)u, z) = \sum_{k\in\mathbb{Z}} \left(\int_1^\infty P(x,t,r) e^{-2\pi x_3^2 r - \pi t^2k^2/r} \frac{dr}{r} \right) e^{-2\pi i k h} e^{-\pi(x_W,x_W)},$$

where

$$P(x,t,r) = \frac{x_2x_3\sqrt{r}}{\sqrt{2}}dw_2 + \frac{1}{2\sqrt{2}}\left(\frac{1}{2\pi} - \frac{t^2k^2}{r}\right)dw_3 - \frac{ix_3k}{\sqrt{2}}dt + \frac{ix_2kt}{\sqrt{2}}ds.$$

Note that $x_3 \neq 0$. (Otherwise, z would lie in the singular set). By Lebesgue dominant convergence the sum over all $k \neq 0$ vanishes as $t \to \infty$, while for $k = 0$ we obtain $\tilde{\psi}_{0,1}(x_W)$. If $x_W = 0$, i.e., for $n = 0$ one needs to argue slightly differently. Then

$$\sum_{a\neq 0} \tilde{\psi}_1(au, z) = \frac{1}{2\sqrt{2}\pi} \sum_{a\neq 0} e^{-\pi a^2/t^2} \frac{dw_3}{t} = \frac{1}{2\sqrt{2}\pi} \left(\sum_{k\in\mathbb{Z}} e^{-\pi t^2k^2} \right) dw_3 - \frac{1}{2\sqrt{2}\pi}\frac{dw_3}{t},$$

which goes to $\frac{1}{2\sqrt{2}\pi}dw_3 = \tilde{\psi}_{0,1}(0)$. This proves the proposition. □

7.3. **Main result.** In the previous sections, we constructed a closed 2-form θ_{φ_2} on $\overline{X}$ such that the restriction of θ_{φ_2} to the boundary $\partial\overline{X}$ was exact with primitive $\sum_{[\underline{P}]} \theta^P_{\phi_{0,1}}$. From now on we usually write φ for φ_2 and ϕ for $\phi_{0,1}$ if it does not cause any confusion. By the definition of the differential for the mapping cone complex $C^\bullet$ we immediately obtain by Theorem 7.2 and Theorem 7.3

Proposition 7.5. *The pair $(\theta_{\varphi_2}(\mathcal{L}_V), \sum_{[P]} \theta^P_{\phi_{0,1}}(\mathcal{L}_{W_P}))$ is a 2-cocycle in $C^\bullet$.*

We write for short $(\theta_\varphi, \theta_\phi)$. We obtain a class $[[\theta_\varphi, \theta_\phi]]$ in $H^2(C^\bullet)$ and hence a class $[\theta_\varphi, \theta_\phi]$ in $H^2_c(X)$. The pairing with $[\theta_\varphi, \theta_\phi]$ then defines a lift Λ^c on differential 2-forms on $\overline{X}$, which factors through $H^2(\overline{X}) = H^2(X)$. By Lemma 3.5 it is given by

$$\Lambda^c(\eta, \tau) = \int_{\overline{X}} \eta \wedge \theta_{\varphi_2} - \sum_{[P]} \int_{e'(P)} i^*\eta \wedge \theta^P_{\phi_{0,1}}.$$

The extension of (7.4) is

Theorem 7.6. *The class $[[\theta_\varphi, \theta_\phi]]$ is holomorphic, that is,*

$$L\,(\theta_\varphi, \theta_\phi) = d(\theta_{\psi_1}, 0).$$

Hence $[\theta_\varphi, \theta_\phi]$ is a holomorphic modular form with values in the compactly supported cohomology of X, so that the lift Λ^c takes values in the holomorphic modular forms.

Proof. Using (7.4), Theorem 7.2, and Proposition 6.2 we calculate

$$d(\theta_{\psi_1}, 0) = (d\theta_{\psi_1}, i^*\theta_{\psi_1}) = \left(L\theta_{\varphi_2}, \sum_{[P]} \theta^P_{\psi_{0,1}} \right) = L\left(\theta_{\varphi_2}, \sum_{[P]} \theta^P_{\phi_{0,1}} \right). \qquad \square$$

It remains to compute the Fourier expansion in τ of $[\theta_\varphi, \theta_\phi](\tau)$. We will carry this out in Section 8. The main result of the paper is

Theorem 7.7. *We have*

$$[\theta_\varphi, \theta_\phi](\tau) = -\frac{1}{2\pi}\delta_{h0}[\omega] + \sum_{n>0} \mathrm{PD}[C^c_n] q^n \in H^2_c(X, \mathbb{Q}) \otimes M_2(\Gamma(N)).$$

That is, for any closed 2-form η on $\overline{X}$

$$\Lambda^c(\eta, \tau) = -\frac{1}{2\pi}\delta_{h0} \int_X \eta \wedge \omega + \sum_{n>0} \left(\int_{C^c_n} \eta \right) q^n,$$

In particular, the map takes values in the holomorphic modular forms and factors through cohomology. We obtain a map

$$\Lambda^c : H^2(X) \to M_2(\Gamma(N)) \tag{7.6}$$

from the cohomology of X to the space of holomorphic modular forms of weight 2 for the principal congruence subgroup $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$). Alternatively, for C any relative 2-cycle in X defining a class in $H_2(\overline{X}, \partial\overline{X}, \mathbb{Z})$, we have

$$\Lambda^c(C, \tau) = -\frac{1}{2\pi}\delta_{h0}\,\mathrm{vol}(C) + \sum_{n>0} (C^c_n \cdot C) q^n \in M_2(\Gamma(N)).$$

Remark 7.8. In the theorem we now consider the Kähler form ω as representing a class in the compactly supported cohomology. In fact, our mapping cone construction gives an explicit coboundary by which ω is modified to become rapidly decreasing.

7.4. **The Hirzebruch-Zagier Theorem.** We now view $[\theta_\varphi, \theta_\phi]$ as a class in $H^2(\tilde{X})$ via the map $j_\# : H^2_c(X) \to H^2(\tilde{X})$. We recover the Hirzebruch-Zagier Theorem.

Theorem 7.9. *We have*

$$j_\#[\theta_\varphi, \theta_\phi](\tau) = -\frac{1}{2\pi}\delta_{h0}[\omega] + \sum_{n>0}[T^c_n]q^n \in H^2(\tilde{X}, \mathbb{Q}) \otimes M_2(\Gamma(N)).$$

In particular,

$$-\frac{1}{2\pi}\delta_{h0}\operatorname{vol}(T_m) + \sum_{n>0}(T^c_n \cdot T_m)_{\tilde{X}} q^n \in M_2(\Gamma(N)).$$

This is the result Hirzebruch-Zagier proved for certain Hilbert modular surfaces (Example 2.1) by explicitly computing the intersection numbers $T_m \cdot T^c_n$.

Proof. This follows by combining Theorem 7.7 and the equation $j_* C^c_n = T^c_n$ of Proposition 4.9 with the following general principle. Suppose ω is a compactly supported form on X such that the cohomology class of ω is the Poincaré dual of the homology class of a cycle C: $[\omega] = \operatorname{PD}(C)$. Then we have $j_\#[\omega] = \operatorname{PD}(j_* C)$. To see this we only have to replace ω by a cohomologous 'Thom representative' of $\operatorname{PD}(C)$, namely a closed form $\tilde{\omega}$ supported in a tubular neighborhood $N(C)$ of C in X such that the integral of $\tilde{\omega}$ over any disk of $N(C)$ is 1. Then it is a general fact from algebraic topology (extension by zero of a Thom class) that $\tilde{\omega}$ represents the Poincaré dual of C in any manifold M containing $N(C)$, in particular for $M = \tilde{X}$. □

Remark 7.10. If one is only interested in recovering the statement of this theorem, then there is also a different way of deriving this from the Kudla-Millson theory. Namely, the lift Λ on $H_2(X)$ (Theorem 7.1) factors through the quotient of $H_2(X)$ by $H_2(\partial X)$ since the restriction of θ_{φ_2} is exact (Theorem 7.3). But by Proposition 3.3 we have $j_* H_2(X) \simeq H_2(X)/H_2(\partial X)$, and the Hirzebruch-Zagier result exactly stipulates the modularity of the lift of classes in $j_* H_2(X)$. However, in that way one misses the remarkable extra structure coming from ∂X as we will explain in the next subsection.

7.5. **The lift of special cycles.** We now consider the lift of a special cycle C_y. By Theorem 7.7 and Lemma 3.5 we see

$$\begin{aligned}
\Lambda^c(C_y, \tau, \mathcal{L}_V) &= -\frac{1}{2\pi}\delta_{h0}\operatorname{vol}(C_y) + \sum_{n>0}(C^c_n \cdot C_y)q^n \\
&= \int_{C_y}\theta_{\varphi_2}(\tau, \mathcal{L}_V) - \sum_{[P]}\int_{(\partial C_y)_P}\theta^P_{\phi_{0,1}}(\tau, \mathcal{L}_{W_P}).
\end{aligned} \tag{7.7}$$

The integrals over C_y and ∂C_y, are both non-holomorphic modular forms of weight 2 (see below) whose difference is holomorphic (by Theorem 7.6). So the generating series series of $(C^c_n \cdot C_y)$ is the sum of two non-holomorphic modular forms. We now give geometric interpretations for the two individual non-holomorphic forms.

Following [17] we define the *interior* intersection number of two special cycles by

$$(C_n \cdot C_y)_X = (C_n \cdot C_y)^{tr} + \operatorname{vol}(C_n \cap C_y),$$

the sum of the transversal intersections and the volume of the 1-dimensional (complex) intersection of C_n and C_y which occur if one of the components of C_n is equal to C_y.

Theorem 7.11. *We have*

$$\int_{C_y} \theta_{\varphi_2}(\tau, \mathcal{L}_V) = -\frac{1}{2\pi}\delta_{h0}\operatorname{vol}(C_y) \;+\; \sum_{n=1}^{\infty}(C_n \cdot C_y)_X q^n \;+\; \sum_{n\in\mathbb{Q}}\sum_{[P]}\int_{(\partial C_y)_P} \tilde{\psi}^P_{0,1}(n)(\tau).$$

So the Fourier coefficients of the holomorphic part of the non-holomorphic modular form $\int_{C_y}\theta_{\varphi_2}$ are the interior intersection numbers of the cycles C_y and C_n.

Proof. This is essentially [10], section 5, where more generally $\mathrm{O}(p,2)$ is considered. There the interpretation of the holomorphic Fourier coefficients as interior intersection number is given. (For more details of an analogous calculation see [15], section 8). A little calculation using the formulas in [10] gives the non-holomorphic contribution. A more conceptual proof would use the relationship between φ_2 and $\tilde{\psi}_1$ (see Proposition 5.3 and Section 8) and the restriction formula for $\tilde{\psi}_1(n)$ (Proposition 7.4). $\square$

By slight abuse of notation we write $\mathrm{Lk}(C_n, C_y) = \sum_{[P]}\mathrm{Lk}((\partial C_n)_P, (\partial C_y)_P)$ for the total linking number of ∂C_n and ∂C_y. Then by Theorem 6.3 we obtain

Theorem 7.12.

$$\sum_{[P]}\int_{(\partial C_y)_P} \theta^P_{\phi_{0,1}}(\tau, \mathcal{L}_{W_P}) = \sum_{n>0}\mathrm{Lk}(C_n, C_y)q^n \;+\; \sum_{n\in\mathbb{Q}}\sum_{[P]}\int_{(\partial C_y)_P} \tilde{\psi}^P_{0,1}(n)(\tau).$$

So the Fourier coefficients of the holomorphic part of $\int_{(\partial C_y)_P}\theta^P_{\phi}(\tau, \mathcal{L}_{W_P})$ are the linking numbers of the cycles ∂C_y and ∂C_n at the boundary component $e'(P)$.

Remark 7.13. There is also another “global” proof for Theorem 7.12. The cycle C_y intersects $e'(P)$ transversally (when pushed inside) and hence also the cap A_n. From this it is not hard to see that we can split the intersection number $C^c_n \cdot C_y$ as

$$C^c_n \cdot C_y = (C_n \cdot C_y)_X - \mathrm{Lk}(C_n, C_y).$$

Hence Theorem 7.12 also follows from combining (7.7) and Theorem 7.11.

Hirzebruch-Zagier also obtain the modularity of the functions given in Theorems 7.11 and 7.12, but by quite different methods. In particular, they explicitly calculate the intersection number $T^c_n \cdot T_m$. They split the intersection number into the interior part $(T_n \cdot T_m)_X$ and a ‘boundary contribution’ $(T_n \cdot T_m)_\infty$ given by

$$(T_n \cdot T_m)_\infty = (T_n \cdot T_m)_{\tilde{X}-X} - (T_m - T^c_m)\cdot(T_n - T^c_n).$$

Now by Theorem 7.9 and its proof we have

$$T^c_n \cdot T_m = C^c_n \cdot C_m.$$

We have (by definition) $(T_n \cdot T_m)_X = (C_n \cdot C_m)_X$, so Theorem 7.11 gives the generating series for $(T_n \cdot T_m)_X$. Note that Theorem 5.4 in [10] also compares the explicit formulas in [17] for $(T_n \cdot T_m)_X$ with the ones obtained via $\int_{C_y}\theta_{\varphi_2}(\tau, \mathcal{L}_V)$. All this implies

$$(T_n \cdot T_m)_\infty = \mathrm{Lk}(C_n \cdot C_m).$$

Independently, we also obtain this from comparing the explicit formulas for the boundary contribution in [17], Section 1.4 with our formulas for the linking numbers, Theorem 4.13 and Example 4.14.

8. A current approach for the special cycles

We now prove Theorem 7.7, the crucial Fourier coefficient formula for our lift Λ^c.

8.1. **A differential character for C_n^c.** The key step for the theory of Kudla and Millson is that for $n > 0$ the form $\varphi_2(n)$ is a Poincaré dual form for the cycle C_n, i.e.,

Theorem 8.1 ([23, 24])**.** *Let η be a closed rapidly decreasing 2-form. Then*

$$\int_X \eta \wedge \varphi_2(n) = \left(\int_{C_n} \eta \right) e^{-2\pi n}.$$

To show this they employ at some point a homotopy argument which requires η to be rapidly decaying. Since we require η to be any closed 2-form on the compactification $\overline{X}$, their approach is not applicable in our case. Instead, we use a differential character argument already implicit in [6], Section 7 for general signature (p, q).

Theorem 8.2. *([6], Section 7) Let $n > 0$. The singular form $\tilde{\psi}_1(n)$ is a differential character in the sense of Cheeger-Simons for the cycle C_n. More precisely, $\tilde{\psi}_1(n)$ is a locally integrable 1-form on X, and for any compactly supported 2-form η we have*

$$\int_X \eta \wedge \varphi_2(n) = \left(\int_{C_n} \eta \right) e^{-2\pi n} - \int_X d\eta \wedge \tilde{\psi}_1(n).$$

Proof. This is the content of the proofs of Theorem 7.1 and Theorem 7.2 in [6]. There the analogous properties for a singular theta lift associated to ψ of Borcherds type is established. However, the proofs boil down to establish the claims for $\tilde{\psi}_1$. The form $\tilde{\psi}$ defined in [6] is indeed the form $\tilde{\psi}_1$ of this paper. □

Remark 8.3. The form $\tilde{\psi}_1$ is closely related to Kudla's Green function ξ [19, 20] (more generally for $\mathrm{O}(p, 2)$) which is given by

$$\xi(x) = \left(\int_1^\infty \varphi_0^0(\sqrt{r}x) \frac{dr}{r} \right) e^{-\pi(x,x)}.$$

Then $\Xi(n) = \sum_{x \in \mathcal{L}_n} \xi(x)$ gives rise to a Green's function for the divisor C_n and moreover $dd^c\xi = \varphi_2$. Here $d^c = \frac{1}{4\pi i}(\partial - \overline{\partial})$. This suggests $d^c\xi = \tilde{\psi}_1$, which indeed follows from $d^c\varphi_0 = -\psi_1$, see [6], Remark 4.5.

For $n \in \mathbb{Q}$, we define

$$\varphi_2^c(n) := \varphi_2(n) - \sum_{[P]} d(f\pi^*\phi_{0,1}^P(n))$$

and follow the current approach to show that for $n > 0$ the form $\varphi_2^c(n)$ is a Poincaré dual form for the cycle C_n^c. Here we follow the notation of subsection 3.3. That is, $\pi^*\phi_{0,1}^P(n)$ is the pullback to a product neighborhood V of $\partial\overline{X}$, and f is a smooth function on V of the geodesic flow coordinate t which is 1 near $t = \infty$ and 0 elsewhere. Note that $\varphi_2^c(n)$ is for $n > 0$ exactly the n-th Fourier coefficient of the mapping cone element $[\theta_\varphi, \theta_\phi]$, when realized as a rapidly decreasing form on X. We also define

$$\tilde{\psi}_1^c(n) = \tilde{\psi}_1(n) - f\pi^*\phi_{0,1}^P(n).$$

We call a differential form η on $\overline{X}$ special if in a neighborhood of each boundary component $e'(P)$ it is the pullback of a form η_P on $e'(P)$ under the geodesic retraction and the pullback of the form η_P to the universal cover $e(P)$ is N-left-invariant. The significance of the forms lies in the fact that the complex of special forms also computes the cohomology of $\overline{X}$. Hence we only need to prove Theorem 7.7 for η special. Note that the proof of Theorem 7.2 (given in [14]) shows that θ_{φ_2} is 'almost' special; it only differs from a special form by a rapidly decreasing form.

Theorem 8.4. *Let $n > 0$. The form $\tilde{\psi}_1^c(n)$ is a differential character for the cycle C_n^c. More precisely, $\tilde{\psi}_1^c(n)$ is a locally integrable 1-form on X and satisfies the following current equation on special 2-forms on $\overline{X}$:*

$$d[\tilde{\psi}_1^c(n)] + \delta_{C_n^c} e^{-2\pi n} = [\varphi_2^c(n)].$$

That is, for any special 2-form η on $\overline{X}$ we have

$$\int_X \eta \wedge \varphi_2^c(n) = \left(\int_{C_n^c} \eta \right) e^{-2\pi n} - \int_X d\eta \wedge \tilde{\psi}_1^c(n).$$

This implies Theorem 7.7 for the positive Fourier coefficients. For $n \leq 0$, Proposition 5.3 implies that the form $\varphi_2^c(n)$ is exact with primitive $\tilde{\psi}_1^c(n)$ which by Proposition 7.4 is decaying. (Note $\tilde{\psi}'_{0,1}(n) = 0$ and hence $\phi_{0,1}(n) = \tilde{\psi}_{0,1}(n)$.) So Theorem 8.4 holds for $n \leq 0$ with $C_n^c = \emptyset$. Hence for these coefficients only the term $x = 0$ contributes, which gives the integral of η against Ω. Note that for $n = 0$ the vector $x = 0$ is excluded in the defining sum for $\varphi_2(n)$, see (7.2), and hence also for $\varphi_2^c(n)$.

Remark 8.5. In view of Remark 8.3 it is very natural question to ask how one can modify Kudla's Green's function $\Xi(n)$ to obtain a Green's function for the cycle T_n^c in $\tilde{X}$. Discussions with Kühn suggest that (if X has only one cusp)

$$\Xi(n) - t \sum_{\substack{x \in \mathcal{L}_W \\ (x,x)=2n}} f\pi^*(B(x) + B'(x))$$

is such a Green's function, but we have not checked all details.

8.2. **Proof of Theorem 8.4.** For simplicity assume that X has only one cusp and drop the superscript P. We let ρ_T be a family of smooth functions on a standard fundamental domain $\mathcal{F}$ of Γ in D only depending on t which are 1 for $t \leq T$ and 0 for $t \geq T + 1$. We then have

$$\int_X \eta \wedge \varphi_2^c(n) = \lim_{T \to \infty} \int_X \rho_T \eta \wedge (\varphi_2(n) - d(f\pi^* \phi_{0,1}(n))).$$

We apply Theorem 8.2 for the compactly supported form $\rho_T \eta$ and obtain

$$\begin{aligned} \int_X \eta \wedge \varphi_2^c(n) = \lim_{T \to \infty} \Bigg[\left(\int_{C_n} \rho_T \eta \right) e^{-2\pi n} - \int_X d(\rho_T \eta) \wedge \tilde{\psi}_1(n) \\ - \int_X d\left(\rho_T \eta \wedge (f\pi^* \phi_{0,1}(n))\right) - d(\rho_T \eta) \wedge f\pi^* \phi_{0,1}(n) \Bigg] \end{aligned} \tag{8.1}$$

The first term on the right hand side of (8.1) goes to $\left(\int_{C_n}\eta\right)e^{-2\pi n}$ as $T\to\infty$, while the third vanishes for any T by Stokes' theorem. For the two remaining terms of (8.1) we first note $d(\rho_T\eta)=\rho'_T(t)dt\wedge\eta+\rho_T d\eta$ and $\rho'_T(t)=0$ outside $[T,T+1]$. We obtain for these two terms

$$-\int_X (d\eta)\wedge\left(\tilde\psi_1(n)-f\pi^*\phi_{0,1}(n)\right) \\ -\lim_{T\to\infty}\int_T^{T+1}\int_{e'(P)}\rho'_T(t)dt\wedge\eta\wedge\left(\tilde\psi_1(n)-f\pi^*\phi_{0,1}(n)\right). \tag{8.2}$$

It remains to compute the second term in (8.2). For T sufficiently large we have $f\equiv 1$. By Proposition 7.4 we can replace $\tilde\psi_1(n)$ by $\pi^*\tilde\psi_{0,1}(n)$. As $\phi_{0,1}(n)=\tilde\psi_{0,1}(n)+\tilde\psi'_{0,1}(n)$, we can replace $\tilde\psi_1(n)-f\pi^*\phi_{0,1}(n)$ by $-\pi^*\tilde\psi'_{0,1}(n)$. For the second term in (8.2) we finally claim

$$\lim_{T\to\infty}\int_T^{T+1}\rho'_T(t)dt\int_{e'(P)}\eta\wedge\pi^*\tilde\psi'_{0,1}(n)=-\int_{e'(P)}\eta\wedge\tilde\psi'_{0,1}(n)=-\left(\int_{A_n}\eta\right)e^{-2\pi n}. \tag{8.3}$$

The first equality of (8.3) holds since η does not depend on t near the boundary as η is special. For the second equality of (8.3), we first note that it holds for $\eta=\Omega$. Indeed, we have $\int_{A_n}\Omega=0$ by the normalization of A_n. But also $\Omega\wedge\tilde\psi'_{0,1}(n)=0$ since Ω has bidegree $(0,2)$ and $\tilde\psi'_{0,1}(n)$ has bidegree $(0,1)$ (in the obvious base/fiber bigrading on the de Rham algebra of $e'(P)$). We are left to consider the second equality in (8.3) for η exact with an N-left-invariant primitive. But then the proof of Proposition 6.6 carries over to the present situation.

Since $C_n^c=C_n\coprod(-A_n)$ collecting all terms completes the proof of Theorem 8.4.

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Department of Mathematical Sciences, University of Durham, Science Laboratories, South Rd, Durham DH1 3LE, United Kingdom
E-mail address: jens.funke@durham.ac.uk

Department of Mathematics, University of Maryland, College Park, MD 20742, USA
E-mail address: jjm@math.umd.edu